1 Elementi trigonometrije

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha},\ \alpha \neq \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}, \quad \cot\alpha = \frac{\cos\alpha}{\sin\alpha},\ \alpha \neq k\pi,\ k \in \mathbb{Z}, \tag{1}$$

$$\sin^2\alpha + \cos^2\alpha = 1, \quad \tan\alpha \cdot \cot\alpha = 1,\ \alpha \neq \frac{k\pi}{2},\ k \in \mathbb{Z}, \tag{2}$$

$$\begin{aligned} \sin(-\alpha) = -\sin\alpha, &\quad \cos(-\alpha) = \cos\alpha, \\ \tan(-\alpha) = -\tan\alpha, &\quad \cot(-\alpha) = -\cot\alpha, \end{aligned} \tag{3}$$

$$\sec\alpha = \frac{1}{\cos\alpha},\ \alpha \neq \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}, \quad \operatorname{cosec}\alpha = \frac{1}{\sin\alpha},\ \alpha \neq k\pi,\ k \in \mathbb{Z}. \tag{4}$$

$$\sin\alpha > 0,\ \alpha \in (0,\pi),\ \sin\alpha < 0,\ \alpha \in (\pi, 2\pi),$$

$$\cos\alpha > 0,\ \alpha \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, 2\pi\right),\ \cos\alpha < 0,\ \alpha \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right),$$

$$\tan\alpha > 0,\ \alpha \in \left(0, \frac{\pi}{2}\right) \cup \left(\pi, \frac{3\pi}{2}\right),\ \tan\alpha < 0,\ \alpha \in \left(\frac{\pi}{2}, \pi\right) \cup \left(\frac{3\pi}{2}, 2\pi\right),$$

$$\cot\alpha > 0,\ \alpha \in \left(0, \frac{\pi}{2}\right) \cup \left(\pi, \frac{3\pi}{2}\right),\ \cot\alpha < 0,\ \alpha \in \left(\frac{\pi}{2}, \pi\right) \cup \left(\frac{3\pi}{2}, 2\pi\right),$$

$$\begin{aligned} \sin(\alpha + 2k\pi) = \sin\alpha,\ k \in \mathbb{Z}, &\quad \cos(\alpha + 2k\pi) = \cos\alpha,\ k \in \mathbb{Z}, \\ \tan(\alpha + k\pi) = \tan\alpha,\ k \in \mathbb{Z}, &\quad \cot(\alpha + k\pi) = \cot\alpha,\ k \in \mathbb{Z}, \end{aligned}$$

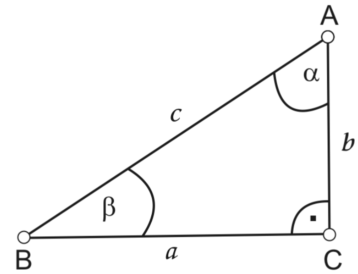
$$\alpha \in [0, \frac{\pi}{2}],\ \sin\alpha \nearrow_0^1,\ \cos\alpha \searrow_0^1,\ \tan\alpha \nearrow_0^{+\infty},\ \cot\alpha \searrow_0^{+\infty},$$

$$\alpha \in [\frac{\pi}{2}, \pi],\ \sin\alpha \searrow_0^1,\ \cos\alpha \searrow_{-1}^0,\ \tan\alpha \nearrow_{-\infty}^0,\ \cot\alpha \searrow_{-\infty}^0,$$

$$\alpha \in [\pi, \frac{3\pi}{2}],\ \sin\alpha \searrow_{-1}^0,\ \cos\alpha \nearrow_{-1}^0,\ \tan\alpha \nearrow_0^{+\infty},\ \cot\alpha \searrow_0^{+\infty},$$

$$\alpha \in [\frac{3\pi}{2}, 2\pi],\ \sin\alpha \nearrow_{-1}^0,\ \cos\alpha \nearrow_0^1,\ \tan\alpha \nearrow_{-\infty}^0,\ \cot\alpha \searrow_{-\infty}^0 .$$

Ako su a i b katete, c hipotenuza pravouglog trougla $\triangle ABC$, $\alpha, \beta, \gamma = 90^o$ uglovi redom u temenima A, B i C, onda važi:



$$\sin\alpha = \frac{a}{c}, \quad \cos\alpha = \frac{b}{c},$$

$$\tan\alpha = \frac{a}{b}, \quad \cot\alpha = \frac{b}{a},$$

$$\sin(90^o - \alpha) = \cos\alpha, \quad \cos(90^o - \alpha) = \sin\alpha,$$

$$\tan(90^o - \alpha) = \cot\alpha, \quad \cot(90^o - \alpha) = \tan\alpha.$$

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta,$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta,$$

$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}, \quad \cos\alpha\cos\beta \neq 0,$$

$$\cot(\alpha \pm \beta) = \frac{\cot\alpha\cot\beta \mp 1}{\cot\beta \pm \cot\alpha}, \quad \sin\alpha\sin\beta \neq 0.$$

$$\sin 2\alpha = 2\sin\alpha\cos\alpha, \quad \cos 2\alpha = \cos^2\alpha - \sin^2\alpha,$$

$$\tan 2\alpha = \frac{2\tan\alpha}{1 - \tan^2\alpha}, \quad \cot 2\alpha = \frac{\cot^2\alpha - 1}{2\cot\alpha},$$

$$\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1 + \cos\alpha}{2}}, \quad \sin\frac{\alpha}{2} = \pm\sqrt{\frac{1 - \cos\alpha}{2}},$$

$$\tan\frac{\alpha}{2} = \pm\sqrt{\frac{1 - \cos\alpha}{1 + \cos\alpha}}, \quad \cot\frac{\alpha}{2} = \pm\sqrt{\frac{1 + \cos\alpha}{1 - \cos\alpha}},$$

$$\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2},$$
$$\sin\alpha - \sin\beta = 2\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2},$$
$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2},$$
$$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2},$$
$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)],$$
$$\cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha+\beta) - \sin(\alpha-\beta)],$$
$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha+\beta) + \cos(\alpha-\beta)],$$
$$\sin\alpha\sin\beta = -\frac{1}{2}[\cos(\alpha+\beta) - \cos(\alpha-\beta)],$$
$$\tan\alpha \pm \tan\beta = \frac{\sin(\alpha\pm\beta)}{\cos\alpha\cos\beta}, \quad \cot\alpha \pm \cot\beta = \frac{\sin(\beta\pm\alpha)}{\sin\alpha\sin\beta}.$$

Sinusna teorema. Odnos stranice i sinusa ugla naspram te stranice jednak je prečniku opisanog kruga oko trougla, tj.

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} = 2R,$$

gde je $2R$ prečnik opisanog kruga.

Kosinusna teorema. U svakom trouglu je kvadrat bilo koje stranice jednak zbiru kvadrata ostalih dveju stranica umanjenom za dvostruki proizvod dužina tih stranica i kosinusa ugla izmedju njih, tj.

$$a^2 = b^2 + c^2 - 2bc\cos\alpha,$$
$$b^2 = a^2 + c^2 - 2ac\cos\beta,$$
$$c^2 = a^2 + b^2 - 2ab\cos\gamma.$$

Tangesna teorema. Odnos zbira i razlike dveju stranica trougla jednak je odnosu tangensa poluzbira i tangensa polurazlike naspramnih uglova tj.

$$(a+b) : (a-b) = \tan\frac{\alpha+\beta}{2} : \tan\frac{\alpha-\beta}{2},$$
$$(b+c) : (b-c) = \tan\frac{\beta+\gamma}{2} : \tan\frac{\beta-\gamma}{2},$$
$$(a+c) : (a-c) = \tan\frac{\alpha+\gamma}{2} : \tan\frac{\alpha-\gamma}{2}.$$

Zadatak 1.1 *Odrediti vrednosti trigonometrijskih funkcija* sin *i* cos *ugla* $\alpha = 18^o$.

Zadatak 1.2 *Rešiti trougao, ako je:*
a) $b = 4$, $\alpha = 105^o$, $\gamma = 60^o$,
b) $b = 2\sqrt{3}$, $c = 3\sqrt{2}$, $\gamma = 60^o$.

Zadatak 1.3 *Ako su* α, β *i* γ *uglovi trougla, dokazati da je*

$$\tan\frac{\alpha}{2}\tan\frac{\beta}{2} + \tan\frac{\gamma}{2}\tan\frac{\beta}{2} + \tan\frac{\alpha}{2}\tan\frac{\gamma}{2} = 1.$$

Zadatak 1.4 *Za uglove* $\triangle ABC$ *važi*

$$\tan\alpha : \tan\beta : \tan\gamma = 1 : 2 : 3.$$

Izračunati obim ovog trougla, ako je stranica naspram ugla γ *jednaka* $\overline{AB} = 3$.

Zadatak 1.5 *Ako su* α, β, γ *uglovi nekog trougla, onda je*

$$\sin\alpha + \sin\beta - \sin\gamma = 4\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\cos\frac{\gamma}{2}.$$

Zadatak 1.6 *Dokazati da će trougao čiji su uglovi* α *i* β *oštri biti tupougli akko je* $\tan\alpha \cdot \tan\beta < 1$.

Zadatak 1.7 *Svako teme jednakostraničnog trougla* $\triangle A_1B_1C_1$ *je na po jednoj stranici pravouglog trougla* $\triangle ABC$*. Jedna stranica dužine* $\frac{c}{3}$ *trougla* $\triangle A_1B_1C_1$ *je paralelna hipotenuzi* c *trougla* $\triangle ABC$*. Odrediti uglove trougla* $\triangle ABC$*.*

Zadatak 1.8 *Dokazati da za stranice i uglove trougla važi:*

a) $a(b\cos\gamma - c\cos\beta) = b^2 - c^2$,
b) $a + b + c = (c + b)\cos\alpha + (c + a)\cos\beta + (a + b)\cos\gamma$,
c) $(b + c)\sin\frac{\alpha}{2} = a\cos\frac{\beta-\gamma}{2}$.

Zadatak 1.9 *Dokazati da u trouglu važi:*

a) $r \cdot (\sin\alpha + \sin\beta + \sin\gamma) = 2R\sin\alpha\sin\beta\sin\gamma$,
b) $a\cos\alpha + b\cos\beta + c\cos\gamma = 4R\sin\alpha\sin\beta\sin\gamma$,

gde su r *i* R *redom poluprečnik upisanog, opisanog kruga oko trougla i* P *površina trougla.*

Zadatak 1.10 *Dokazati da u oštrouglom trouglu* $\triangle ABC$ *važi:*
a) $h_c = c\frac{\tan\alpha\cdot\tan\beta}{\tan\alpha+\tan\beta}$;
b) ako u tom trouglu važi $h_c = c\frac{\tan\alpha\cdot\tan\beta}{\cot\alpha+\cot\beta}$*, dokazati da je* $\triangle ABC$ *pravougli trougao.*