On Warped Product Manifolds Satisfying Ricci-Hessian Class Type Equations

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Outline

Definition of Ricci Solitons

Ricci Solitons

(M, g, X) is a *Ricci soliton*[a], where X is a smooth vector field on M, if there exists a constant λ such that

$$Ric + \frac{1}{2}\mathcal{L}_X g = \lambda g \tag{1}$$

[a] R. S. Hamilton. *Three-manifolds with positive Ricci curvature*, J. Dif. Geo. 17, 255–306, (1982).

- X is called a *potential vector field.*
- Ricci solitons are said to be:
  - *expanding* if $\lambda > 0$,
  - *shrinking* if $\lambda < 0$ and
  - *steady* if $\lambda = 0$.

Motivation for the Study of Ricci Solitons

- If X is identically zero or a Killing vector field, then Ricci soliton reduces to an Einstein manifold (that is, it becomes trivial.) Thus, Ricci solitons generalize the Einstein manifolds.
- The main motivation for the study of Ricci solitons comes from the theory of Ricci flow[1].

Hamilton's Ricci Flow

$$\frac{\partial}{\partial t}g(t) = -2Ric(t) \tag{2}$$

- The fixed points of Ricci flow are Ricci flat manifolds ($Ric = 0$).
- A soliton of the Ricci flow is said to be *self-similar* if we allow the initial metric to change by homotheties and diffeomorphisms:

$$g(t) = \varphi(t)\phi(t)^{*}g(0)$$

where $\varphi : I \to \mathbb{R}$ and $\phi(t) : M \to M$.

- Ricci solitons are the self-similar solutions of the Ricci flow.
- Ricci solitons are fixed points of the Ricci flow as a dynamical system, up to diffeomorphisms and scalings.

[1] R. S. Hamilton. *Three-manifolds with positive Ricci curvature*, J. Dif. Geo. 17, 255–306, (1982).

Gradient Ricci Solitons

Gradient Ricci Soliton

If the potential vector field X is gradient, that is $X = \nabla f$, for some smooth function f then the soliton (M, g, f) satisfying

$$Ric + Hessf = \lambda g \tag{3}$$

is called a *gradient Ricci soliton*[a].

[a] R. S. Hamilton. *Three-manifolds with positive Ricci curvature*, J. Dif. Geo. 17, 255–306, (1982).

Some Examples of Gradient Ricci Solitons

1. Shrinker Ricci Solitons
    - Gaussian Soliton: $(\mathbb{R}^n, dx^2)$ with the potential function $f = \frac{1}{4}|x|^2$.
    - Warped Product: $\mathbb{R}^k \times \mathbb{S}^{n-k}$ round cylinder.
2. Steady Ricci Solitons
    - Hamilton's Cigar Soliton: $(\sum, g) = (\mathbb{R}^2, \frac{dx^2+dy^2}{1+x^2+y^2})$ with potential function $f = -ln(1 + x^2 + y^2)$.
    - Bryant Soliton: Higher dimensioanl analogue of Cigar soliton.
3. Expanding Ricci Solitons
    - Gaussian Soliton: $(\mathbb{R}^n, dx^2)$ with potential function $f = -\frac{1}{4}|x|^2$.
    - Doubly warped product manifold $\mathbb{R}^{n+1} \times N$ where N^n $(n \geqslant 2)$ is an Einstein manifold of positive scalar curvature.

m-Bakry-Emery Ricci Tensor

- In recent years, the Bakry-Emery-Ricci tensor

$$Ric_f = Ric + Hessf \tag{4}$$

  has become an important object in the study of Riemannian geometry, particularly in the study of Ricci flow and Ricci solitons.
- Then, some weakening convergence conditions on a semi-Riemannian manifold $(M^n; g)$ has been considered by introducing the m-Bakry-Emery-Ricci curvature[2] which is an important object related to the study in Riemannian geometry, particularly in the study of Ricci flow and Ricci solitons.

m-Bakry-Emery Ricci Tensor

$$Ric_f^m = Ric + Hessf - \frac{1}{m} df \otimes df; \quad 0 < m \leqslant \infty \tag{5}$$

where f is a smooth function and m is a positive integer.

- When f is constant, the m-Bakry-Emery-Ricci tensor becomes the usual Ricci tensor so it gives an analog of the Ricci tensor for a Riemannian manifold.

[2] D. Bakry, M. Emery , *Diffusions hypercontractives, In Seminaire de probabilities*, XIX, 1983/84, Vol. 1123, Lecture Notes in Math. 177-206, Springer, Berlin (1985).

Generalized Quasi Einstein Manifolds

Generalized Quasi Einstein Manifolds (Catino, 2012)

A smooth manifold (M^n, g) $(n > 2)$ is said to be *generalized quasi Einstein manifold* which was introduced by Catino[a] if there exist three smooth functions f, α and λ such that

$$Ric + Hessf - \alpha df \otimes df = \lambda g \tag{6}$$

[a] G. Catino, *Generalized quasi Einstein manifolds with harmonic Weyl tensor*, Math. Z. **271** (2012), 751–756.

- According to him, a complete n-dimensional generalized quasi Einstein manifold with harmonic Weyl tensor and with zero radial Weyl curvature is locally a warped product whose fiber is an $(n-1)$-dimensional Einstein manifold.
- Also, Jauregui and Wylie investigated the conformal diffeomorphisms of such manifold.

Generalized Quasi Einstein Manifolds

This generalized class reduces to:

- *gradient Ricci soliton*[a] when $\alpha = 0$ and $\lambda \in \mathbb{R}$;
- *m-quasi Einstein manifold*[b] when $\alpha = \frac{1}{m}$, $m \in \mathbb{N}$ and $\lambda \in \mathbb{R}$. Moreover, if $\lambda \in \mathbb{R}$ in (6), then this manifold is called gradient quasi Einstein manifold. Also, it is said to be expanding, steady or shrinking, when $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$, respectively.
- *m-generalized quasi Einstein manifold*[c] when $\alpha = \frac{1}{m}$, $m \in \mathbb{N}$ and $\lambda \in C^\infty(M)$.
- *(m, ρ)-quasi Einstein manifold* [d] when $\alpha = \frac{1}{m}$ and $\lambda \mapsto \rho r + \lambda$, where r denotes the scalar curvature and $\rho, \lambda \in \mathbb{R}$.

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[a] H. D. Cao, *Recent Progress on Ricci soliton*, Adv. Lect. Math. (ALM), **11** (2009), 1–38.

[b] J. S. Case, Y. Shu, G. Wei, *Rigidity of quasi Einstein metrics*, Differ. Geom. Appl., **29** (2011), 93–100.

[c] G. Catino, *Generalized quasi Einstein manifolds with harmonic Weyl tensor*, Math. Z. **271** (2012), 751–756.

[d] G. Huang, Y. Wei, *The classification of (m, ρ)−quasi Einstein manifolds*, Ann. Glob. Anal. Geom., **44** (2013), 269–282.

Mixed Super Quasi Einstein Manifolds

(Nivas et. al., 2011)

A Riemannian manifold (M^n, g) $(n > 2)$ is called a mixed super quasi Einstein manifold[a] if its Ricci tensor of type $(0,2)$ is non-zero and satisfies the following condition

$$S(X,Y) = ag(X,Y) + bA(X)A(Y) + cB(X)B(Y) + d[A(X)B(Y) + A(Y)B(X)] + eD(X,Y) \quad (7)$$

where a, b, c, d and e are real valued, non-zero scalar functions on (M^n, g), A, B are two non-zero 1-forms and D is symmetric $(0,2)-$tensor field such that

$$A(X) = g(X,U),\ B(X) = g(X,V);\ g(U,U) = g(V,V) = 1,\ g(U,V) = 0,\ D(X,U) = 0,\ tr(D) = 0 \quad (8)$$

[a] R. Nivas, A. Bajpai, *Certain Properties of Mixed Super Quasi Einstein Manifolds*, Gen. Math. Notes, Vol. 5, No. 1, 2011, 15–26 (2011).

Mixed Super Quasi Einstein Manifolds

Special classes:

- If $e = c = 0$, then the manifold reduces to a generalized quasi Einstein manifold[3] denoted by $G(QE)_n$..
- If $e = d = c = 0$, then the manifold reduces to a quasi Einstein manifold [4] denoted by $(QE)_n$.
- If $e = d = b = c = 0$, then the manifold becomes an Einstein manifold whose Ricci tensor is proportional to the metric tensor.

[3] M. C. Chaki, On Generalized quasi-Einstein manifold, *Publ. Math. Debrecen* **58** (2001) 638–691.

[4] M. C. Chaki and R. K. Maity, On-quasi Einstein Manifolds, *Publ. Math. Debrecen* . **57** (2000) 297–306.

Warped Product Manifolds

- In 1969, Bishop and O'Neill[5], [6] introduced the notion of warped product manifolds and the many authors have been considered such manifolds with regards to geometric and physical applications of these manifolds.
- For example, every surface of revolution (not crossing the axis of revolution) is isometric to a warped product $B \times_f F$, with B the generating curve, F the circle of unit radius, and $f(b)$ the distance to the axis of revolution.
- Morover, sphere and $\mathbb{R}^n - \{0\}$ are locally isometric to the warped product manifolds.
- Also, it is known that many well-known spacetimes, for instance Robertson-Walker, Schwarzchild, Reissner- Nordström-de Sitter spacetimes are warped products.
- Thus warped product manifolds play an important role in geometry as well as in general theory of relativity.

[5] R. L. Bishop and B. O'Neill, Manifolds of negative curvature, *Transactions of the American Mathematical Society*. **145** (1969) 1-49.

[6] B. O'Neill, Semi-Riemannian Geometry with Applications to Relativity, New York, Academic Press, (1983).

Warped Product Manifolds

- Let (M, g) and $(N, \bar{g})$ with $dimM = q, dimN = n - q; (1 \leq q < n)$ be Riemannian manifolds covered by systems of charts $\{\phi, x^{\alpha}\}$ and $\{\psi, x^{\alpha}\}$,respectively.
- Let f be a positive C^{∞}-function on M.
- The *warped product* $M \times_f N$ of (M, g) and $(N, \bar{g})$ [7] ,[8] is the manifold $M \times N$ with the metric $\tilde{g} = g \times_f \bar{g}$.
- o More precisely,

$$g \times_f \bar{g} = \pi_1^* g + (f \circ \pi_1)^2 \pi_2^* \bar{g} \tag{9}$$

where π_i $(1 \leq i \leq 2)$ are natural projections from $M \times N \to M$ and $M \times N \to N$, respectively.

[7] R. L. Bishop and B. O'Neill, Manifolds of negative curvature, *Transactions of the American Mathematical Society*. **145** (1969) 1-49.

[8] B. O'Neill, Semi-Riemannian Geometry with Applications to Relativity, New York, Academic Press, (1983).

Warped Product Manifolds

- The manifold (M, g) is called the base manifold.
- $(N, \bar{g})$ the fiber manifold.
- The function f is called the warping function.
- The warped product manifold with constant warping function is simply called Riemannian product[9] ,[10].

[9] R. L. Bishop and B. O'Neill, Manifolds of negative curvature, *Transactions of the American Mathematical Society*. **145** (1969) 1-49.

[10] B. O'Neill, Semi-Riemannian Geometry with Applications to Relativity, New York, Academic Press, (1983).

Warped Product Manifolds

- Let $\{\phi \times \psi : x^1, \cdots, x^q, x^{q+1} = y^1, \cdots, x^n = y^{n-q}\}$ be a product chart for $M \times N$.
- The local components of the metric $\tilde{g} = g \times_f \bar{g}$ with respect to this chart are

$$\tilde{g}_{ij} = \begin{cases} g_{ab} & \text{, if } i = a, j = b \\ f\bar{g}_{\alpha\beta} & \text{, if } i = \alpha, j = \beta \\ 0 & \text{, otherwise} \end{cases} \tag{10}$$

  where $a, b, c \cdots \in \{1, \cdots, q\}$, $\alpha, \beta, \gamma \cdots \in \{q+1, \cdots, n\}$ and $i, j, k \cdots \in \{1, \cdots, n\}$.
- Here throughout this paper each object denoted by "tilde" is assumed to be from the warped product $M \times_f N$, each unmarked object is assumed to be from M and each object denoted by "bar" is assumed to be from N.

Warped Product Manifolds

Lemma

On the warped product manifold $M \times_f N$, if $X, Y \in \chi(M)$ and $V, W \in \chi(N)$, then

(1) $\tilde{\nabla}_X Y = \nabla_X Y$

(2) $\tilde{\nabla}_X V = \tilde{\nabla}_V X = (\frac{Xf}{f})V$

(3) $\tilde{\nabla}_V W = -f\bar{g}(V, W)\nabla f + \bar{\nabla}_V W$

where ∇f denotes the gradient of f.

Lemma

Let $M \times_f N$ be a warped product manifold. For all $X, Y, Z \in \chi(M)$ and $U, V, W \in \chi(N)$,

(1) $\tilde{R}(X, Y)Z = R(X, Y)Z$

(2) $\tilde{R}(V, X)Y = -(\frac{H^f(X,Y)}{f})V$

(3) $\tilde{R}(X, Y)V = \tilde{R}(V, W)X = 0$

(4) $\tilde{R}(X, W)V = -(\frac{\tilde{g}(V,W)}{f})\tilde{\nabla}_X gradf$

(5) $\tilde{R}(V, W)U = \bar{R}(V, W)U - \frac{\|gradf\|^2}{f^2}[g(W, U)V - g(V, U)W]$

where H^f denotes the Hessian of f.

Warped Product Manifolds

Lemma

Let $M \times_f N$ be a warped product manifold with $dimM = q, dimN = n - q$. For all $X, Y, Z \in \chi(M)$ and $U, V, W \in \chi(N)$,

(1) $\tilde{S}(X, Y) = S(X, Y) - \frac{n-q}{f} H^f(X, Y)$

(2) $\tilde{S}(X, V) = 0$

(3) $\tilde{S}(V, W) = \bar{S}(V, W) - \left[\frac{\Delta f}{f} + (n - q - 1)\frac{\|gradf\|^2}{f^2}\right] g(V, W)$

where H^f and Δf denote the Hessian and the Laplacian of f, respectively [a].

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[a] B. O'Neill, Semi-Riemannian Geometry with Applications to Relativity, New York, Academic Press, (1983).

Warped Product Manifolds Satisfying Ricci-Hessian Class Type Equations

- In this section, we express the Ricci tensor of warped product manifold with respect to the base and the fiber when the warped product manifold is also a generalized quasi Einstein manifold in the sense of Chaki.

Theorem 1

Let $M \times_f N$ be a warped product manifold which is also a $G(QE)_n$ with $dimM = m$, $dimN = d > 1$ and the generator vector fields ξ_1 and ξ_2. Then for all $X, Y \in \chi(M)$ and $U, V \in \chi(N)$, the followings hold:

(1) If $\xi_1, \xi_2 \in \chi(M)$, then

$$S(X,Y) = ag(X,Y) + \frac{d}{f}H^f(X,Y) + bg(X,\xi_1)g(Y,\xi_1) + c[g(X,\xi_1)g(Y,\xi_2) + g(X,\xi_2)g(Y,\xi_1)] \quad (11)$$

$$\bar{S}(U,V) = f^2\Big(a + \frac{\Delta f}{f} + \frac{d-1}{f^2}||gradf||^2\Big)\bar{g}(U,V) \quad (12)$$

Warped Product Manifolds Satisfying Ricci-Hessian Class Type Equations

Theorem Continued..

(2) If $\xi_1, \xi_2 \in \chi(N)$, then

$$S(X,Y) = ag(X,Y) + \frac{d}{f}H^f(X,Y) \tag{13}$$

$$\begin{aligned}\bar{S}(U,V) =& f^2\Big(a + \frac{\Delta f}{f} + \frac{d-1}{f^2}\|gradf\|^2\Big)\bar{g}(U,V) + bf^4\bar{g}(U,\xi_1)\bar{g}(V,\xi_1) \\ &+ cf^4[\bar{g}(U,\xi_1)\bar{g}(V,\xi_2) + \bar{g}(U,\xi_2)\bar{g}(V,\xi_1)]\end{aligned} \tag{14}$$

(3) If $\xi_1 \in \chi(M)$ and $\xi_2 \in \chi(N)$, then

$$S(X,Y) = ag(X,Y) + \frac{d}{f}H^f(X,Y) + bg(X,\xi_1)g(Y,\xi_1) \tag{15}$$

$$\bar{S}(U,V) = f^2\Big(a + \frac{\Delta f}{f} + \frac{d-1}{f^2}\|gradf\|^2\Big)\bar{g}(U,V) \tag{16}$$

Warped Product Manifolds Satisfying Ricci-Hessian Class Type Equations

Theorem Continued...

(4) If $\xi_1 \in \chi(N)$ and $\xi_2 \in \chi(M)$ ise,

$$S(X,Y) = ag(X,Y) + \frac{d}{f} H^f(X,Y) \tag{17}$$

$$\bar{S}(U,V) = f^2\Big(a + \frac{\Delta f}{f} + \frac{d-1}{f^2}||gradf||^2\Big)\bar{g}(U,V) + bf^4\bar{g}(U,\xi_1)\bar{g}(V,\xi_1) \tag{18}$$

Warped Product Manifolds Satisfying Ricci-Hessian Class Type Equations

Now, by using Theorem 1, we give some characterizations about this manifold with related to the certain Ricci-Hessian type equations such as $Ric_f^m = \lambda g$, for some smooth function λ:

Theorem 2

Let $M \times_f N$ be a warped product manifold which is also a $G(QE)_n$ with $dimM = m$, $dimN = d > 1$ and the generator vector fields ξ_1 and ξ_2. Then, the followings hold:

(1) If $\xi_1, \xi_2 \in \chi(M)$ and $\Delta f = \frac{1}{f}||\nabla f||^2$, then M is a mixed super quasi Einstein and N is an Einstein manifold.

(2) If $\xi_1, \xi_2 \in \chi(N)$, then M is a $d-$quasi Einstein manifold in the sense of Case and N is a generalized quasi Einstein manifold in the sense of Chaki.

(3) If $\xi_1 \in \chi(M)$ and $\xi_2 \in \chi(N)$, then M is a generalized quasi Einstein manifold in the sense of Catino and N is an Einstein manifold.

(4) If $\xi_1 \in \chi(N)$ and $\xi_2 \in \chi(M)$, then M is an Einstein and N is a quasi Einstein manifold in the sense of Chaki.

Example of $G(QE)_4$ Warped Product Manifold

We consider a Riemannian metric g on the 4-dimensional real number space $\mathbb{R}^4$ by

$$ds^2 = g_{ij}dx^i dx^j = (dt)^2 + t^4[(dx)^2 + (dy)^2] + e^{2t}(dz)^2 \tag{19}$$

where $-1 - \sqrt{7} < t < 1 - \sqrt{3}$ or $-1 + \sqrt{7} < t < 1 + \sqrt{3}$ and $\{t, x, y, z\}$ are the standard coordinates of $\mathbb{R}^4$. Then the only non vanishing components of the Christoffel symbols, the curvature tensor and the Ricci tensor are

$$\Gamma^t_{xx} = \Gamma^t_{yy} = -2t^3, \ \ \Gamma^t_{zz} = -e^{2t}, \ \ \Gamma^x_{tx} = \Gamma^y_{ty} = \frac{2}{t}, \ \ \Gamma^z_{zt} = 1 \tag{20}$$

$$R_{xyyx} = 4t^6, \ \ R_{txxt} = R_{tyyt} = 2t^2, \tag{21}$$

$$R_{tzzt} = e^{2t}, \ \ R_{xzzx} = R_{yzzy} = 2e^{2t}t^3, \tag{22}$$

$$R_{tt} = 1 + \frac{4}{t^2}, \ \ R_{xx} = R_{yy} = 2t^2[t + 3], \ \ R_{zz} = \frac{4e^{2t}}{t} + e^{2t} \tag{23}$$

and the components which can be obtained from these by symmetry properties. Also it can be shown that the scalar curvature is

$$r = \frac{16}{t^2} + \frac{8}{t} + 2 \tag{24}$$

which is non zero and non constant.

Example of $G(QE)_4$ Warped Product Manifold

Let us now define associated scalar functions as

$$a = \frac{6}{t^2} + \frac{2}{t}, \quad b = 2 - \frac{8}{t^2}, \quad c = (\frac{4}{t^2} - 1)tan(2\lambda) \tag{25}$$

and the 1-forms

$$A = \left(e^t sin(\lambda), 0, 0, cos(\lambda)\right) \tag{26}$$

and

$$B = \left(e^t cos(\lambda), 0, 0, -sin(\lambda)\right) \tag{27}$$

where λ is some non-zero function of (x^4) satisfying the conditions

$$sin^2(\lambda) = \frac{t^2 + 2t - 6}{4(t-1)} \ \ and \ \ cos^2(\lambda) = \frac{t^2 - 2t - 2}{-4(t-1)}$$

.

Example of $G(QE)_4$ Warped Product Manifold

Then, we can show that

1. $R_{tt} = ag_{tt} + bA_tA_t + 2cA_tB_t$
2. $R_{xx} = ag_{xx} + bA_xA_x + 2cA_xB_x$
3. $R_{yy} = ag_{yy} + bA_yA_y + 2cA_yB_y$
4. $R_{zz} = ag_{zz} + bA_zA_z + 2cA_zB_z$

and all the cases other than (1)-(4) are trivial. Thus, the Ricci tensor of this manifold satisfies $G(QE)_4$ metric condition. Moreover, we find

$$\|A\| = \|B\| = 1 \ , \ and \ A \perp B \tag{28}$$

and so

$$r = 4a + b = \frac{16}{(x^4)^2} + \frac{8}{(x^4)} + 2 \tag{29}$$

Therefore, this proves that the manifold under consideration is a generalized quasi Einstein manifold with non-zero and non-constant scalar curvature.

Example of $G(QE)_4$ Warped Product Manifold

Furthermore, to define a warped product metric on $G(QE)_n$, we consider the warping function $f : \mathbb{R}^3 \to \mathbb{R}^+$ defined by $f(t) = e^t$ which is positive definite, smooth function. Thus, we can define the line element on $\mathbb{R}^3 \times_{e^t} \mathbb{R}$, where $\mathbb{R}^3$ and $\mathbb{R}$ denote the base and fiber respectively. That is,

$$ds^2 = g_{ij}dx^i dx^j = [(dt)^2 + t^4(dx)^2 + t^4(dy)^2] + e^{2t}(dz)^2 \tag{30}$$

Then the local components of non vanishing components of the Christoffel symbols, the curvature tensor and the Ricci tensor for the warped product manifold $\tilde{M} = B \times_f I$ endowed with the metric $\tilde{g} = g + f^2\bar{g}$, are obtained as follows:

$$\tilde{\Gamma}^a_{bc} = \Gamma^a_{bc}, \quad \tilde{\Gamma}^a_{11} = -ff^a, \quad \tilde{\Gamma}^1_{1a} = \frac{f_a}{f}, \quad \tilde{\Gamma}^1_{11} = \bar{\Gamma}^1_{11} \tag{31}$$

$$\tilde{R}^a_{bcd} = R^a_{bcd}, \qquad \tilde{R}^1_{d1b} = -\frac{1}{f}\nabla_d f_b \tag{32}$$

$$\tilde{S}_{bc} = S_{bc} + \frac{1}{f}(\nabla_c f_b) \qquad \tilde{S}_{c1} = 0, \qquad \tilde{S}_{11} = f(\Delta f)$$

$$\tilde{r} = r + \frac{2\Delta f}{f}$$

Example of $G(QE)_4$ Warped Product Manifold

Hence we can state that:

Theorem 3

Let $M^4 = \{t, x, y, z \in \mathbb{R}^4 : \ t \in (-1 - \sqrt{7}, 1 - \sqrt{3}) \bigcup (-1 + \sqrt{7}, 1 + \sqrt{3})\}$ be an open subset of $\mathbb{R}^4$ endowed with the Riemannian metric given by

$$ds^2 = g_{ij} dx^i dx^j = [(dt)^2 + t^4 (dx)^2 + t^4 (dy)^2] + e^{2t} (dz)^2$$

where t, x, y, z are the standard coordinates of $\mathbb{R}^4$. Then $M^4 = \mathbb{R}^3 \times_{e^t} \mathbb{R}$ is a generalized quasi Einstein warped product manifold with non zero and non constant scalar curvature.

m-Generalized Quasi Einstein Manifolds

- Recall that a smooth vector field ϕ on a Riemannian manifold (M^n, g) is called *conformal* if it satisfies

$$\mathcal{L}_\phi g = 2\Omega g \tag{33}$$

where $\mathcal{L}_\phi$ denotes the Lie derivative in the direction of ϕ and Ω is some smooth function on M, which is called a conformal factor.

  - If Ω is constant, then the vector field ϕ is said to be *homothetic*.
  - If Ω is identically zero, then the vector field ϕ is said to be *Killing*.

- Particularly, if the vector field ϕ satisfies the condition

$$\nabla_X \phi = \Omega X \tag{34}$$

for all X and a smooth function Ω, then it is called *closed conformal vector field*.

m-Generalized Quasi Einstein Manifolds

In the next lemma we prove some formulas which are needed for the proof of our main results:

Lemma 4

Let (M^n, g, f, λ) be an m-generalized quasi Einstein manifold admitting a closed conformal vector field $\phi \in \chi(M)$ with conformal factor Ω. Then the followings hold:

1. $\nabla\phi(f) = \frac{1}{m}\phi(f)\nabla f + (n-1)\nabla\Omega + \lambda\phi$.
2. $Hess(\phi(f)) = \frac{1}{m}[d(\phi(f)) \otimes df + \phi(f)Hessf] + (n-1)Hess\Omega + d\lambda \otimes \phi^\flat$.
3. $\frac{1}{m}d(\phi(f)) \otimes df + d\lambda \otimes \phi^\flat$ is symmetric.

m-Generalized Quasi Einstein Manifolds

Now, we give some characterizations of an m-generalized quasi Einstein manifold endowed with closed conformal vector fields. Then, we obtain some rigidity conditions for this class of manifolds:

Theorem 5

Let (M^n, g, f, λ) be an m-generalized quasi Einstein manifold admitting a closed conformal vector field ϕ with conformal factor Ω. Then, its Ricci tensor is of the form

$$Ric = \lambda g + \Big(\frac{1-n}{||\phi||} d\Omega(U) - \lambda \Big) U^\flat \otimes U^\flat \tag{35}$$

where $U^\flat$ is a 1-form corresponding to the unit vector field U in the direction of ϕ.

*** Theorem 5 says that an m-generalized quasi Einstein manifold admitting closed conformal vector field ϕ with conformal factor Ω is a quasi Einstein manifold in the sense that Chaki.

m-Generalized Quasi Einstein Manifolds

- Also, by using the equation (34), we see that the unit vector field U in the direction of ϕ satisfies

$$\nabla_X U = \frac{\Omega}{||\phi||}[X - U^\flat(X)U] \tag{36}$$

which means that the U is a unit concircular vector field.

- The term *concircular* comes from the concircular transformation introduced first by K. Yano [21].

m-Generalized Quasi Einstein Manifolds

Remark

K. Yano [22] proved that in order that a Riemannian manifold admits a concircular vector field, it is necessary and sufficient that there exists a coordinate system with respect to which the fundamental quadratic differential form may be written in the form

$$ds^2 = (dx^1)^2 + e^q g^*_{\alpha\beta} dx^\alpha dx^\beta \tag{37}$$

where $g^*_{\alpha\beta} = g^*_{\alpha\beta}(x^\gamma)$ are the functions of x^γ only $(\alpha, \beta, \gamma, \delta = 2, 3, \ldots, n)$ and $q = q(x^1) \neq$ *constant* is a function of x^1 only.

- Thus, in view of this remark the first fundamental form of (M^n, g, f, λ) admitting a closed conformal vector field ϕ has the form (37).
- Since $e^{q/2}$ is always a positive function, this also implies that such manifold is a warped product $I \times_{e^{q/2}} M^*$, where (M^*, g^*) is an $(n-1)$-dimensional Riemannian manifold.

m-Generalized Quasi Einstein Manifolds

- Moreover, A. Gebarowski [23] proved that warped product $I \times_{e^{q/2}} M^*$ is conformally conservative, (i.e. $divC = 0$, where C denotes the conformal curvature tensor) if and only if M^* is an Einstein manifold.
- Since every conformally flat manifold is conformally conservative, this result can be summarized as follows:

Theorem 6

Let (M^n, g, f, λ) be an m-generalized quasi Einstein manifold admitting a non-trivial closed conformal vector field ϕ. Then

- (M^n, g) is a warped product manifold $I \times_{e^{q/2}} M^*$, where I is a real interval, (M^*, g^*) is an $(n-1)$-dimensional Riemannian manifold and q is a smooth function on I.
- In addition, if (M^n, g) is conformally flat, then it has the same warped product structure with an $(n-1)$-dimensional Einsteinian fiber.

- R. S. Hamilton. *Three-manifolds with positive Ricci curvature*, J. Dif. Geo. 17, 255–306, (1982).
- J. M. Lee, *Riemannian Manifolds: An Introduction to Curvature*, Springer-Verlag, (1997).
- R. S. Hamilton, *The Ricci Flow on Surfaces, Mathematics and general relativity*, 237–262, Contemp. Math., 71, Amer. Math. Soc., Providence, RI, (1988)
- T. Ivey, *Ricci solitons on compact three-manifolds*, Differential Geom. Appl., 3, 301–307, (1993).
- N. Koiso, *On rotationally symmetric Hamilton's equation for Kahler-Einstein metrics*, Recent topics in differential and analytic geometry, 327–337, Adv. Stud. Pure Math., 18-I, Academic Press, Boston, MA, (1990).
- P.Baird,J. Eells, *A conservation law for harmonic maps*, Geometry Symposium, Utrecht, (1980), (Utrecht, 1980), pp. 1–25, Lecture Notes in Math., 894, Springer, Berlin,New York, (1981).
- G. Perelman, *The entropy formula for the Ricci fow and its geometric applications*, Preprint, http://arXiv.org/sbs/Math.DG/0211159.
- D. Bakry, M. Emery , *Diffusions hypercontractives, In Seminaire de probabilities*, XIX, 1983/84, Vol. 1123, Lecture Notes in Math. 177-206, Springer, Berlin (1985).
- J. Case, Y. Shu, G. Wei, *Rigity of quasi-Einstein metrics*, Differ. Geom. Appl., 29, 93–100 (2011).

- D. S. Kim, Y. H. Kim, *Compact Einstein warped product spaces with nonpositive scalar curvature*, Proc. Amer. Math. Soc. 131, 2573–2576, (2003).
- M. Eminenti, G. Nave, C. Mantegazza, *Ricci solitons-the equation point of view*, Manus. Math., 127(3), 345–367, (2008).
- G. Catino, *Generalized quasi Einstein manifolds with harmonic Weyl tensor*, Math. Z. **271** (2012), 751–756.
- P. Petersen, W. Wylie, *Rigidity of gradient Ricci solitons*, Pac. J. Math **241** (2009), 329–345.
- M. Fernandez-Lopez, E. Garcio-Rio, *Rigidity of shrinking Ricci solitons*, Math. Z., **269** (2011), 461–466.
- G. Huang, Y. Wei, *The classification of $(m,\rho)-$quasi Einstein manifolds*, Ann. Glob. Anal. Geom., **44** (2013), 269–282.
- B. Y. Chen, *Some Results On Concircular Vector Fields and Their Applications To Ricci Solitons*, Bull. Korean Math. Soc. **52** (2015), no. 5, 1535–1547.
- J. F. S. Filho, *Quasi Einstein manifolds endowed with a parallel vector field*, Monatsh. Math., **179** (2016), no. 2, 305–320.
- A. L. Besse, *Einstein Manifolds*, Springer Verlag, 1987.
- S. Tanno, W. Weber, *Closed conformal vector fields*, J. Dif. Geom. **3** (1969), 361–366.
- A. Ros, F. Urbano, *Lagrangian submanifolds of $\mathbb{C}^n$ with conformal Maslov form and the Whitney sphere*, J. Math. Soc. Japan, **50** (1988), 203–226.

- K. Yano, *Concircular geometry*. I, II, III, IV, V, Proc. Imp. Acad. Tokyo **16** (1940), 195-200, 354–360,442–148, 505–511, **18** (1942), 446–451.
- K. Yano and T. Adati, *On certain spaces admitting concircular transformation*, Proc. Jap. Acad., **25** (1949), 188–195.
- A. Gebarowski, *Nearly Conformally Symmetric Warped Product Manifolds*, Bulletin of the Institute of Mathematics Academia Sinica, **20** (1992), 359–371.
- Y. Tashiro, *Complete Riemannian manifolds and some vector fields*, Trans. Amer. Math. Soc. **117** (1965), 251–275.
- R. Nivas, A. Bajpai, *Certain Properties of Mixed Super Quasi Einstein Manifolds*, Gen. Math. Notes, Vol. 5, No. 1, 2011, 15–26 (2011).
- R. Bryant, *Local existence of gradient Ricci solitons*, unpublished.
- T. Ivey, *New examples of complete Ricci solitons*, Proc. Amer. Math. Soc. 122 (1994), 241–245.
- R. L. Bishop and B. O'Neill, *Manifolds of negative curvature*, Transactions of the American Mathematical Society, 145 (1969) 11-49.
- M.C. Chaki, and R.K. Maity, *On Quasi Einstein Manifolds*, Publ. Math. Debrecen, 57 (2000) 297–306.
- M.C. Chaki, *On Generalized quasi Einstein manifolds*, Publ. Math. Debrecen, 58 (2001) 638–691.

THANK YOU..!

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