Quasitoric Manifolds and Small Covers over Neighborly Polytopes

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Convex Sets

A point set $K \subseteq \mathbb{R}^n$ is *convex* if for any two points $\mathbf{x}$, $\mathbf{y} \in K$ we have that the straight line segment $[x, y] = \{\lambda\mathbf{x} + (1-\lambda)\mathbf{y} \mid 0 \leq \lambda \leq 1\}$ entirely lies in K (see Figure 1).

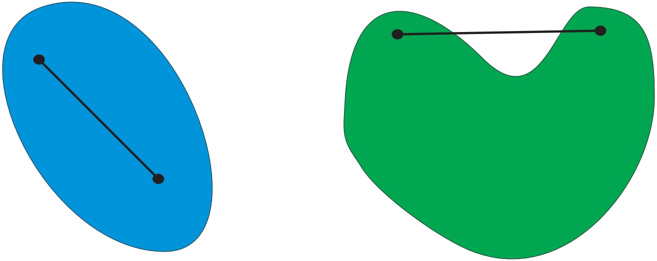


Figure: A convex set and a non-convex set

Convex Polytopes

The 'smallest' convex set containing a given set K is called *the convex hull* of K and it is equal to the intersection of all convex set that contain K:

$$\mathrm{conv}(K) := \bigcap \{L \subseteq \mathbb{R}^n \mid K \subseteq L, L \text{ is convex}\}.$$

Definition

A convex polytope is the convex hull of a finite set of points in some $\mathbb{R}^n$.

Examples of Convex Polytopes

Example

Polygons are convex polygons in $\mathbb{R}^2$ (see Figure 2).

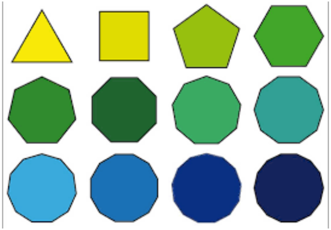


Figure: Polygons

Example

The standard n-simplex Δ^n is convex hull of $n+1$ points $\{\mathbf{O}, \mathbf{e}_1, \ldots, \mathbf{e}_d\}$ where $\mathbf{O}$ is origin and $\mathbf{e}_1, \ldots, \mathbf{e}_n$ the standard base in $\mathbb{R}^n$.

Faces of Polytopes

The *dimension* of a polytope is the dimension of its affine hull.

Every linear form $l = l_{\mathbf{a}} : \mathbb{R}^n \to \mathbb{R}$ has the form $\mathbf{x} \mapsto \mathbf{ax}$, where $\mathbf{a} \in (\mathbb{R}^n)^*$ and $\mathbf{ax}$ is the scalar obtained as the matrix product assuming that the point x is represented with a column vector in $\mathbb{R}^n$ and $\mathbf{a}$ is represented by a row vector in $(\mathbb{R}^n)^*$. We say that a linear inequality $\mathbf{mx} \leq r$ is *valid* for a convex polytope $P \subseteq \mathbb{R}^n$ if it is satisfied for all points $\mathbf{x} \in P$.

A *face* of P is any set of the form

$$F = P \cap \{\mathbf{x} \in \mathbb{R}^n \,|\, \mathbf{mx} = r\}.$$

The dimension of a face is the dimension of its affine hull.

Proper Faces

From the obvious inequalities $\mathbf{0x} \leq 0$ and $\mathbf{0x} \leq 1$, we deduce that the polytope P itself and $\emptyset$ are faces of P.

All other faces of P are called *proper* faces. The faces of dimension 0, 1, $\dim(P) - 2$, and $\dim(P) - 1$ are called *vertices*, *edges*, *ridges* and *facets*, respectively.

The faces of a polytope P are polytopes of smaller dimension and every intersection of finite number of faces is a face of P.

Definition

A polytope P is called *simplicial* if all its proper faces are simplices.

Face Lattice

The faces of a convex polytope P form a partially ordered structure with respect to inclusion.

Definition

The *face lattice* of a convex polytope P is the poset $L := L(P)$ of all faces of P, partially ordered by inclusion.

We say that two polytopes P_1 and P_2 are *combinatorially equivalent* if their face lattices $L(P_1)$ and $L(P_2)$ are isomorphic. A *combinatorial polytope* is a class of combinatorially equivalent polytopes.

Polar Set

For any convex polytope $P \subset \mathbb{R}^n$ we define *its polar set* $P^* \subset (\mathbb{R}^n)^*$ by

$$P^* := \{\mathbf{c} \in (\mathbb{R}^n)^* \mid \mathbf{cx} \leq 1 \text{ for all } \mathbf{x} \in P\}.$$

It is well known fact from convex geometry that the polar set P^* is a convex set in $(\mathbb{R}^n)^*$ that contains $\mathbf{0}$ in its interior. Morover, if $O \in P$ then P^* is convex polytope and $(P^*)^* = P$. The face lattice $L(P^*)$ is the opposite of the face lattice $L(P)$ of P.

Simple Polytopes

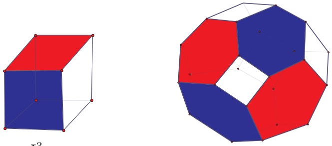
Definition

A polytope P is called *simple* if any combinatorially polar polytope is simplicial.

Now we observe that polytope P^n is simple if there are exactly n facets meeting at each vertex of P^n and each face of simple polytope is again a simple polytope. Any combinatorially polar polytope of a simple polytope is simplicial.

Example

The cube I^n and the permutahedron Π^n are simple polytopes (see Figure 3



I^3 Π_3

f *and* h *vectors*

Definition

Let P be a simplicial n-polytope. The f-vector is the integer vector

$$\mathbf{f}(P) = (f_{-1}, f_0, f_1, \ldots, f_{n-1}),$$

where $f_{-1} = 1$ and $f_i = f_i(P)$ denotes the number of i-faces of P, for all $i = 1, \ldots, n-1$.

The **f**-polynomial of a simplicial polytope P is

$$\mathbf{f}(t) = t^n + f_0 t^{n-1} + \cdots + f_{n-1}.$$

The **h**-polynomial is the polynomial

$$\mathbf{h}(t) = \mathbf{f}(t-1), \tag{1}$$

and the coefficients $h_0, \ldots, h_d$ of the **h**-polynomial $\mathbf{h}(t) = h_0 t^d + \cdots + h_{n-1} t + h_n$ define the **h**-vector by

$$\mathbf{h}(P) = (h_0, h_1, \ldots, h_n).$$

$\mathbf{g}$-*theorem*

The **f**-vector and the **f**-vector are a combinatorial invariant of P and mutually determined each other by means of linear relations coming from the equation (1)

$$h_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{n-i}{d-k} f_{i-1},\; f_{n-k-1} = \sum_{j=k}^{n} \binom{j}{k} h_{n-j},\; k = 0, \ldots, d. \tag{2}$$

Theorem (Stanley, Billera & Lee)

An integer vector $(f_{-1}, f_0, f_1, \ldots, f_{n-1})$ *is the* **f**-*vector of a simple n-polytope if and only if the corresponding sequence* $(h_0, h_1, \ldots, h_n)$ *determined by* (1) *satisfies the following conditions:*

1. $h_i = h_{n-i}$, $i = 0, \ldots, n$ *(the Dehn-Sommervillle equations);*
2. $h_0 \leq h_1 \leq \cdots \leq h_{[\frac{n}{2}]}$, $i = 0$;
3. $h_0 = 1$, $h_{i+1} - h_i \leq (h_i - h_{i-1})^{\langle i \rangle}$, $i = 0, \ldots, [\frac{n}{2}] - 1$.

$\mathbf{g}$-*theorem*

Recall that for any two integers a and i $a^{\langle i \rangle}$ is defined as

$$a^{\langle i \rangle} = \binom{a_i + 1}{i + 1} + \binom{a_{i-1}}{i} + \cdots + \binom{a_j + 1}{j + 1},$$

where $a_i > a_{i-1} > \cdots > a_j \geq j \geq 1$ are the unique integers such that

$$a = \binom{a_i}{i} + \binom{a_{i-1}}{i-1} + \cdots + \binom{a_j}{j}.$$

The latter representation of a is known as *the binomial i-expansion of a*.

Neighborly Polytopes

A n-polytope P is said to be *k-neighborly* if any subset of k or less vertices is the vertex set of a face of P. It is straightforward to check that for $k > \left[\frac{n}{2}\right]$, the simplex Δ^n is the only k-neighborly polytope. Thus, polytopes that are $\left[\frac{n}{2}\right]$-neighborly are of particular interests and are called *neighborly* polytopes.

Note that for a neighborly n-polytope $P^n(m)$ with m vertices it holds

$$f_i(P^n(m)) = \binom{m}{i+1} \text{ for } i = 0, \ldots, \left[\frac{d}{2}\right] - 1. \quad (3)$$

Neighborly Polytopes

By the Dehn-Sommerville equations we straightforwardly deduce the following claim.

Lemma

The **h***-vector of a neighborly n-polytope* $P^n(m)$ *with m vertices is given by*

$$h_i(P^n(m)) = h_{n-i}(P^n(m)) = \binom{m-n+i-1}{i} \; i = 0, \ldots, \left[\frac{n}{2}\right].$$

Corollary

The **f***-vector of a neighborly n-polytope* $P^n(m)$ *with m vertices is given by*

$f_i(P^n(m)) = \sum_{j=0}^{\left[\frac{n}{2}\right]} \binom{j}{n-1-i}\binom{m-n+j-1}{j} + \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{n-k}{i+1-k}\binom{m-n+k-1}{k}$,

for $i = -1, \ldots, n-1$, *where we assume* $\binom{k}{j} = 0$ *for* $k < j$.

The Upper Bound Theorem

Theorem (The Upper Bound Theorem)

From all simplicial d-polytopes Q with m vertices, any simplicial neighborly d-polytope $P^n(m)$ with m vertices has the maximal number of i-faces, $2 \leq i \leq n-1$. That is

$$f_i(Q) \leq f_i(P^d(m)) \text{ for } i = 0, \ldots, n-1.$$

The equality in the above formula holds if and only if Q is simplicial neighborly d-polytope with m vertices.

The Upper Bound Theorem implies that for a simplicial n-polytope Q with m vertices the following inequalities for h-vector are true

$$h_i(Q) \leq \binom{m-n+i-1}{i}, \; i = 0, \ldots, \left[\frac{n}{2}\right] - 1.$$

Cyclic Polytopes

A classical example of a neighborly n-polytope with m vertices is *the cyclic polytope* $C^n(m)$. Recall that *the moment curve* γ in $\mathbb{R}^n$ is defined by $\gamma : \mathbb{R} \to \mathbb{R}^n$, $t \mapsto \gamma(t) = (t, t^2, \ldots, t^n) \in \mathbb{R}^n$. The cyclic polytope $C^n(m)$ is the convex hull

$$C^n(m) := \operatorname{conv}\{\gamma(t_1), \gamma(t_2), \ldots, \gamma(t_m)\},$$

for m distinct points $\gamma(t_i)$ with $t_1 < t_2 < \cdots < t_m$ on the moment curve (see Figure 4).

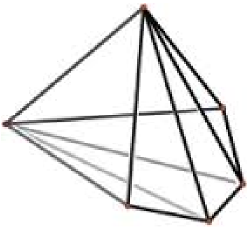


Figure: The cyclic polytope $C^3(6)$

Gale's evenness condition

The combinatorial class of $C^n(m)$ does not depend on the specific choices of the parameters t_i due to Gale's evenness condition.

Theorem (Gale's evenness condition)

Let $m > d \geq 2$ and $C^n(m)$ the cyclic polytope with vertices $\gamma(t_i)$ with $t_1 < t_2 < \cdots < t_m$ on the moment curve. A n-subset $S \subseteq \{1, 2, \ldots, m\}$ forms a facet of $C^n(m)$ if and only if the following 'evenness condition' is satisfied:
If $i < j$ are not in S then the number of $k \in S$ such that $i < k < j$ is even.

Cyclic polytopes are simplicial polytopes and it can be proved even-dimensional neighborly polytopes are necessarily simplicial, but this is not true in general. For example, any 3-dimensional polytope is neighborly by definition.

Non-cyclic Neighborly Polytopes

If the number of vertices m of a neighborly n-polytope is not grater than $n+3$ then combinatorially the polytope is isomorphic to a cyclic polytope. However, there are many neighborly polytopes which are not cyclic. Barnette in 1981 constructed an infinite family of duals of neighborly n-polytopes by using facet splitting and Shemer in 1982 introduced a sewing construction that allows to add a vertex to a neighborly polytope in such a way as to obtain a new neighborly polytope. Both constructions show that for a fixed n the number of combinatorially different neighborly polytopes grows superexponentially with the number of vertices m. Duals (polar set) of simplicial neighborly n-polytopes are simple polytopes with property that each $\left[\frac{n}{2}\right]$ facets have nonempty intersections. We shall call such polytopes also *neighborly* and in the rest of the talk under therm neighborly we assume simple neighborly polytope.

G_d^n-*manifolds*

Let

$$G_d = \begin{cases} S^0, & \text{if } d = 1 \\ S^1, & \text{if } d = 2 \end{cases}, \mathbb{R}_d = \begin{cases} \mathbb{Z}_2, & \text{if } d = 1 \\ \mathbb{Z}, & \text{if } d = 2 \end{cases} \text{ and } \mathbb{K}_d = \begin{cases} \mathbb{R}, & \text{if } d = 1 \\ \mathbb{C}, & \text{if } d = 2 \end{cases}$$

where $S^0 = \{-1, 1\}$ and $S^1 = \{z || z| = 1\}$ are multiplicative subgroups of real and complex numbers, respectively. *The standard action* of group G_d^n on $\mathbb{K}_d^n$ is given as

$$G_d^n \times \mathbb{K}_d^n \to \mathbb{K}_d^n : (t_1, \dots, t_n) \cdot (x_1, \dots, x_n) \mapsto (t_1 x_1, \dots, t_n x_n).$$

A G_d^n*-manifold* is a differentiable manifold with a smooth action of G_d^n.

Locally Standard Action

Definition

A map $f : X \to Y$ between to G-spaces X and Y is called weekly equivariant if for any $x \in X$ and $g \in G$ holds

$$f(g \cdot x) = \psi(g) \cdot f(x),$$

where $\psi : G \to G$ is some automorphism of group G.

Let M^{dn} be a dn-dimensional G_d^n-manifold. A *standard chart* on M^{dn} is a couple (U, f), where U is a G_d^n-stable open subset of M^{dn} and f is a weekly equivariant diffeomorphism from U onto some G_d^n-stable open subset of $\mathbb{K}_d^n$. A *standard atlas* is an atlas which consists of standard charts. A G_d^n action on a G_d^n-manifold M^{dn} is called locally standard if manifold M^{dn} has a standard atlas. The orbit space for a locally standard action is naturally regarded as a manifold with corners.

Small Covers and Quasitoric Manifolds

Definition

A G_d^n -manifold $\pi_d : M^{dn} \to P^n$ $(d = 1, 2)$ is a smooth closed (dn)-dimensional G_d^n - manifold admitting a locally standard G_d^n -action such that its orbit space is a simple convex n-polytope P^n regarded as a manifold with corners. If $d = 1$ such a G_d^n-manifold is called a *small cover* and if $d = 2$ a *quasitoric manifold*.

Proposition

Let M_1^{dn} and M_2^{dn} be G_d^n-manifolds over simple polytopes P and P' such that there is a weekly equivariant homeomorphism $f : M_1^{dn} \to M_2^{dn}$. Then f descends to a homeomorphism from P to P' as manifolds with corners.

Characteristic Map

Let P^n be a simple polytope with m facets $F_1, \ldots, F_m$. By Definition it follows that every point in $\pi^{-1}(\mathrm{rel.int}(F_i))$ has the same isotropy group which is one-dimensional subgroup of G_d^n. We denote it $G_d(F_i)$.

Each G_d^n -manifold $\pi_d : M^{dn} \to P^n$ determines a *characteristic map* l_d on P^n

$$l_d : \{F_1, \ldots, F_m\} \to R_d^n$$

defined by mapping each facet of P^n to nonzero elements of $\mathbb{R}_d^n$ such that $l_d(F_i) = \boldsymbol{\lambda_i} = (\lambda_{1,i}, \ldots, \lambda_{n,i})^t \in \mathbb{R}_d^n$, where λ_i is primitive vector such that

$$G_d(F_i) = \left\{(t^{\lambda_{1,i}}, \ldots, t^{\lambda_{n,i}}) | t \in \mathbb{K}_d, |t| = 1\right\}.$$

Characteristic Matrix

From the characteristic map we obtain an integer $(n \times m)$-matrix $\Lambda_{\mathbb{R}_d}(M^{dn}) := (\lambda_{i,j})$ which is called *the characteristic matrix of* M^{dn}. For $d = 2$ each $\boldsymbol{\lambda}_i$ is determined up to sign. Since the G_d^n-action on M^{dn} is locally standard, the characteristic matrix $\Lambda_{\mathbb{R}_d}(M^{dn})$ satisfies the non-singular condition for P^n, i. e. if n facets $F_{i_1}, \ldots, F_{i_n}$ of P^n meet at vertex, then $\left|\det \Lambda_{\mathbb{R}_d}^{(i_1,\ldots,i_n)}(M^{dn})\right| = 1$, where $\Lambda_{\mathbb{R}_d}^{(i_1,\ldots,i_n)}(M^{dn}) := (\boldsymbol{\lambda}_{i_1}, \ldots, \boldsymbol{\lambda}_{i_n})$. Any integer $(n \times m)$-matrix satisfying the non-singular condition for P^n is also called a *characteristic matrix on* P^n.

Characteristic Pair

The construction of a small cover and a quasitoric manifold from the *characteristic pair* $(P^n, \Lambda_{\mathbb{R}_d})$ where $\Lambda_{\mathbb{R}_d}$ is a characteristic matrix.

For each point $x \in P^n$, we denote the minimal face containing x in its relative interior by $F(q)$. The characteristic map l_d corresponding to $\Lambda_{\mathbb{R}_d}$ is a map from the set of the faces of P^n to the set of subtori of G_d^n defined by

$$l_d(F_{i_1} \cap \ldots F_{i_k}) := \mathrm{im}\left(l_d^{(i_1,\ldots,i_k)} : G_d^k \to G_d^n \right),$$

where $l_d^{(i_1,\ldots,i_k)}$ is the map induced from the linear map determined by $\Lambda_{\mathbb{R}_d}^{(i_1,\ldots,i_k)}$. A G_d^n-manifold $M^{dn}(\Lambda_{\mathbb{R}_d})$ over simple polytope P^n is obtained by setting

$$M^{dn}(\Lambda_{\mathbb{R}_d}) := (G_d^n \times P^n)/\sim_{l_d},$$

where $\sim_{l_d}$ is an equivalence relation defined by $(t_1, p) \sim_{l_d} (t_2, q)$ if and only if $p = q$ and $t_1 t_2^{-1} \in l_d(F(q))$.

Characteristic Pair

The free action of G_d^n on $G_d^n \times P^n$ obviously descends to an action on $(G_d^n \times P^n)/\sim_{I_d}$ with quotient P^n. Simple polytope P^n is covered by the open sets U_v obtained by deleting all faces not containing vertex v of P^n. Clearly, U_v is diffeomorphic to $\mathbb{R}_+^n$, so the space $(G_d^n \times P^n)/\sim_{I_d}$ is covered by open sets $(G_d^n \times U_v)/\sim_{I_d}$ homeomorphic to $\mathbb{K}_d$. We easily see that the transition maps are diffeomorphic, so G_d^n-action on $(G_d^n \times P^n)/\sim_{I_d}$ is locally standard and $M^{dn}(\Lambda_{\mathbb{R}_d})$ is a G_d^n-manifold $\pi_d : M^{dn} \to P^n$ over simple convex n-polytope P^n.

Quasitoric manifolds

By the Davis-Januszkiewicz construction of a quasitoric manifolds from the characteristic pair, we can think about a quasitoric manifold as a simple polytope P^n decorated with m lattice vectors, one for every facet $F_1, \ldots, F_m$ of P_n, arranged so that the vectors prescribed to the facets sharing a common vertex span basis of $\mathbb{Z}^n$.

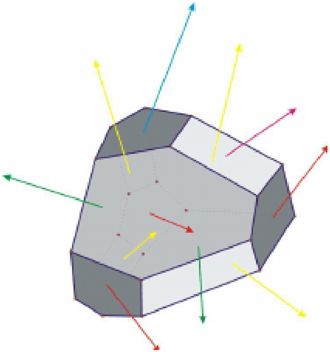


Figure: The Characteristic Pair $(P^n, \Lambda_{\mathbb{R}_d})$

Lifting Problem

Every quasitoric manifold M^{2n} has admits an involution called *conjugation* such that its fixed point set is homeomorphic to a small cover M^n over the same polytope P^n. In this case, the real characteristic matrix $\Lambda_{\mathbb{Z}_2}(M^n)$ is exactly the modulo 2 reduction of the characteristic matrix $\Lambda_{\mathbb{Z}}(M^{2n})$. The following problem posted by Zhi Lü, known as the *lifting problem* asks if the converse is true

Problem

Let P be a simple polytope and M^n a small cover over P. Is it true that there is a quasitoric manifold M^{2n} such that M^n is the fixed point set of the conjugation on M^{2n}?

Lifting Problem

The problem can be reformulated in the following way: For any real characteristic matrix $\Lambda_{\mathbb{Z}_2}(M^n) = \Lambda$ where M^n is a small cover over simple polytope P^n, is it true that there is a characteristic matrix $\widetilde{\Lambda}$ such that $(P^n, \widetilde{\Lambda})$ is the characteristic pair of a quasitoric manifold? In other terms, does the diagram

$$\begin{array}{ccc} & & \mathbb{Z}^n \\ & \overset{\widetilde{\Lambda}}{\nearrow} & \downarrow \ \text{(mod 2)} \\ F_i & \xrightarrow{\Lambda} & \mathbb{Z}_2^n \end{array}$$

commute for each facet F_i of P^n?

Lifting Problem

Since the determinant of any three times three matrix with 0 and 1 is between -2 and 2, the non-singular condition for $\Lambda_{\mathbb{Z}_2}(M^n)$ is satisfied with the same matrix but viewed as the characteristic manifold of a quasitoric manifold. Therefore, the lifting conjecture is true for all simple polytopes in dimension 2 and 3.
The answer to the lifting problem is also affirmative for all small covers over dual cyclic polytopes by Hasui. The hypothesis is also true for the products of simplices by Choi and Masuda.

Classification Problem

Problem

Find a combinatorial description of the class of polytopes P^n admitting a characteristic map.

We know that the class admitting a characteristic map contains some important combinatorial simple polytopes such as the simplex, the cube, the permutahedron, polygons, 3-dimensional polyhedrons etc. They all belong to the class of simple polytopes with 'small'chromatic number.

Chromatic Number of Polytope

Example

The coloring into k colors of a simple polytope P^n with m facets $F_1, \ldots, F_m$ is a map

$$c : \{F_1, \ldots, F_m\} \to [k]$$

such that for every i and j, $i \neq j$ and $F_i \cap F_j$ is a codimension-two face of P_n holds $c(F_i) \neq c(F_j)$. The least k for which there exist a coloring of the simple polytope P^n is called *the chromatic number* $\chi(P^n)$. Obviously, $\chi(P^n) \geq n$ for any simple polytope P^n. The class of simple polytopes P^n whose chromatic number is equal to n or $n+1$, allows the characteristic map. The coloring with n colors gives rise to a canonical characteristic function λ where $\lambda(F_i) = e_{c(i)}$, while in the case of colorings with $n+1$ colors for all the facets F_i such that $c(F_i) = n+1$ we assign $\lambda(F_i) = -e_1 - \ldots - e_n$, where $e_1, \ldots, e_n$ are the standard base vectors of $\mathbb{R}^n_d$

Chromatic Number

The existence of a characteristic map impose an immediate obstruction at the chromatic number of P. Namely, there are at most $2^n - 1$ different possibilities for $\boldsymbol{\lambda}_i$ modulo 2. Thus a characteristic map produces a coloring with no more than $2^n - 1$ colors, so for a simple polytope P^n which is the orbit space of a G_d^n manifold it holds

$$\chi(P^n) \leq 2^n - 1. \tag{4}$$

However, also there are examples of polytopes which do not admit a characteristic maps as are neighborly simple polytopes with large number of facets.

Example

Let P^n be a 2-neighborly simple polytope with $m \geq 2^n$. The chromatic number of P^n is equal to the number of its facets m. But than the existence of a G_d^n manifold over P^n would contradict the inequality (4).

Classification Problem

There are two main classification problems of G_d^n-manifolds over a given simple polytope: up to a weekly equivariant diffeomorphism (the equivariant classification) and up to a diffeomorphism (the topological classification).

Let M^{dn} ba a G_d^n-manifold over P^n with characteristic map l_d.

Lemma

There exist a weekly equivariant diffeomorphism $f : M^{dn} \to M^{dn}$ induced by an automorphism ψ of G_d^n such that the characteristic matrix induced by f has the form $(I_{n\times n}|*)$ where $*$ denotes some $n \times (m-n)$ matrix.

Classification Problem

For a given polytope P^n, let us denote the set of all weakly equivariant homeomorphism classes of G_d^n-manifolds over P by ${}_{\mathbb{R}_d}\mathcal{M}_P$ and by ${}_{\mathbb{R}_d}\mathcal{M}_P^{\mathrm{homeo}}$ the set of all homeomorphism classes of G_d^n-manifolds over P. Define ${}_{\mathbb{R}_d}\mathfrak{M}_P$ the set of all $\mathbb{R}_d$ characteristic matrices over P. The map $\Lambda_{\mathbb{R}_d} \mapsto M^{dn}(\Lambda_{\mathbb{R}_d})$ is a surjection of ${}_{\mathbb{R}_d}\mathfrak{M}_P$ onto ${}_{\mathbb{R}_d}\mathcal{M}_P$.

Let us denote by $\mathrm{Aut}(P)$ the group of all automorphisms of the face poset of P, that are the bijection from the set of the facets of P to itself which preserve the structure of all faces of P. Group $GL(n, \mathbb{R}_d)$ acts on ${}_{\mathbb{R}_d}\mathfrak{M}_P$ by left multiplication. In the case $d = 2$, the group ${\mathbb{Z}_2}^m$ acts on ${}_{\mathbb{R}_d}\mathfrak{M}_P$ by multiplication with -1 on each columns. Also, the group $\mathrm{Aut}(P)$ acts on ${}_{\mathbb{R}_d}\mathfrak{M}_P$ by permuting columns. Let

$${}_{\mathbb{R}_d}\mathcal{X}_P = \begin{cases} GL(n, \mathbb{Z}_2)\backslash {}_{\mathbb{Z}_2}\mathfrak{M}_P, & \text{if } d = 1 \\ GL(n, \mathbb{Z})\backslash {}_{\mathbb{Z}}\mathfrak{M}_P / {\mathbb{Z}_2}^m, & \text{if } d = 2 \end{cases}$$

Classification Problem

The action of $\mathrm{Aut}(P)$ on ${}_{\mathbb{R}_d}\mathfrak{M}_P$ descends to the action of $\mathrm{Aut}(P)$ on ${}_{\mathbb{R}_d}\mathcal{X}_P$. Let us denote with $[\Lambda_{\mathbb{R}_d}]$ the orbit of $\Lambda_{\mathbb{R}_d}$ in ${}_{\mathbb{R}_d}\mathcal{X}_P\backslash\mathrm{Aut}(P)$.

Theorem

For any simple polytope P^n, the map $[\Lambda_{\mathbb{R}_d}] \mapsto M^{dn}(\Lambda_{\mathbb{R}_d})$ is a bijection between ${}_{\mathbb{R}_d}\mathcal{X}_P\backslash\mathrm{Aut}(P)$ and ${}_{\mathbb{R}_d}\mathcal{M}_P$

Classification Problem

Proposition

- *Any small cover over Δ^n is weekly equivariant (and topologically) diffeomorphic to $\mathbb{R}P^n$.*
- *Any quasitoric manifold over Δ^n is weekly equivariant (and topologically) diffeomorphic to $\mathbb{C}P^n$.*

Theorem (Orlik & Raymond)

- *A small cover over a convex polygon is homeomorphic to the connected sums of $S^1 \times S^1$ and $\mathbb{R}P^2$.*
- *A quasitoric manifold over a convex polygon is homeomorphic to the connected sums of $S^2 \times S^2$, $\mathbb{C}P^2$ and $\overline{\mathbb{C}P^2}$.*

Classification Problem

Garrison and Scott found a 25 small covers up to homeomorphism over dodecahedron using computer search.

Theorem (Hasui)

- *If* $n \geq 4$ *and* $m \geq n+4$*, or* $n \geq 6$ *and* $m \geq n+3$*, there exists no* G_d^n *manifolds over* $C^n(m)^*$*.*
- *There exists* 1 *small cover and* 4 *different quasitoric manifolds over* $C^4(7)$*.*
- *There exists* 1 *small cover and* 46 *different quasitoric manifolds over* $C^5(8)$*.*

Small Covers over Neighborly Polytopes

There are 3 combinatorially distinct neighborly 4-polytopes with 8 vertices. $C^4(8)$ is not the orbit space of a small cover. But, the other 2 polytopes $P_0^4(8)$ and $P_1^4(8)$ allow the characteristic maps. The vertices of $P_0^4(8)$ are

$$F_0\cap F_1\cap F_2\cap F_3,\ F_0\cap F_1\cap F_2\cap F_7,\quad F_0\cap F_1\cap F_3\cap F_4,\ F_0\cap F_1\cap F_4\cap F_5,$$
$$F_0\cap F_1\cap F_5\cap F_6,\ F_0\cap F_1\cap F_6\cap F_7,\quad F_0\cap F_2\cap F_3\cap F_4,\ F_0\cap F_2\cap F_4\cap F_5,$$
$$F_0\cap F_2\cap F_5\cap F_6,\ F_0\cap F_2\cap F_6\cap F_7,\quad F_1\cap F_3\cap F_4\cap F_6,\ F_1\cap F_3\cap F_6\cap F_7,$$
$$F_1\cap F_4\cap F_5\cap F_6,\ F_2\cap F_3\cap F_4\cap F_7,\quad F_0\cap F_1\cap F_5\cap F_6,\ F_2\cap F_4\cap F_5\cap F_7,$$
$$F_2\cap F_5\cap F_6\cap F_7,\ F_3\cap F_4\cap F_5\cap F_6,\quad F_3\cap F_4\cap F_5\cap F_7,\ F_3\cap F_5\cap F_6\cap F_7$$

and the vertices of $P_1^4(8)$ are

$$F_0\cap F_1\cap F_2\cap F_3,\ F_0\cap F_1\cap F_2\cap F_4,\quad F_0\cap F_1\cap F_3\cap F_7,\ F_0\cap F_1\cap F_4\cap F_5,$$
$$F_0\cap F_1\cap F_5\cap F_6,\ F_0\cap F_1\cap F_6\cap F_7,\quad F_0\cap F_2\cap F_3\cap F_4,\ F_0\cap F_3\cap F_4\cap F_5,$$
$$F_0\cap F_3\cap F_5\cap F_6,\ F_0\cap F_3\cap F_6\cap F_7,\quad F_1\cap F_2\cap F_3\cap F_7,\ F_1\cap F_2\cap F_4\cap F_5,$$
$$F_1\cap F_2\cap F_5\cap F_7,\ F_1\cap F_5\cap F_6\cap F_7,\quad F_2\cap F_3\cap F_4\cap F_6,\ F_2\cap F_3\cap F_6\cap F_7,$$

Small Covers over Neighborly Polytopes

Proposition

$\mathbb{Z}_2\mathcal{X}_{P_0^4(8)}$ has exactly 7 elements and they are represented by the matrices

$$m_1 = \begin{vmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{vmatrix}, m_2 = \begin{vmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{vmatrix},$$

$$m_3 = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{vmatrix}, m_4 = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{vmatrix},$$

Small Covers over Neighborly Polytopes

$$m_5 = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{vmatrix}, m_6 = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{vmatrix},$$

$$m_7 = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \end{vmatrix}.$$

Small Covers over Neighborly Polytopes

Proposition

$\mathbb{Z}_2\mathcal{X}_{P_1^4(8)}$ has exactly 3 elements and they are represented by the matrices

$$n_1 = \begin{vmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{vmatrix}, n_2 = \begin{vmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{vmatrix},$$

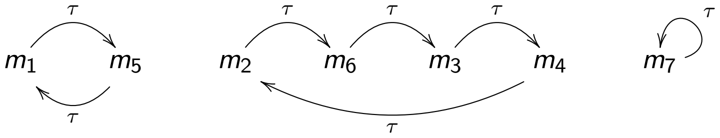
$$n_3 = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \end{vmatrix}.$$

Theorem

There are exactly 3 different small covers $M^4(m_1)$, $M^4(m_2)$ and $M^4(m_3)$ over $P_0^4(8)$ and exactly 3 different small covers $M^4(n_1)$, $M^4(n_2)$ and $M^4(n_3)$ over $P_1^4(8)$.

Proof: $F_0 \cap F_1$, $F_0 \cap F_2$, $F_2 \cap F_5$, $F_2 \cap F_7$, $F_3 \cap F_4$ and $F_5 \cap F_6$ are hexagons, $F_1 \cap F_3$, $F_1 \cap F_6$, $F_3 \cap F_7$ and $F_6 \cap F_7$ are pentagons, $F_0 \cap F_4$, $F_0 \cap F_6$, $F_1 \cap F_4$, $F_1 \cap F_7$, $F_2 \cap F_3$, $F_2 \cap F_4$, $F_2 \cap F_5$, $F_3 \cap F_6$ and $F_5 \cap F_7$ are quadrilaterals and $F_0 \cap F_3$, $F_0 \cap F_7$, $F_1 \cap F_2$, $F_1 \cap F_5$, $F_2 \cap F_6$, $F_3 \cap F_5$, $F_4 \cap F_5$, $F_4 \cap F_6$ and $F_4 \cap F_7$ are triangles. Thus, $\mathrm{Aut}(P_0^4(8))$ is generated by $\tau = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 6 & 4 & 1 & 0 & 2 & 7 & 3 \end{pmatrix}$.

By direct calculation we examine the action of $\mathrm{Aut}(P_0^4(8))$ on $\mathbb{Z}_2\mathcal{X}_{P_0^4(8)}$,
$\tau(m_1) = m_5, \tau(m_2) = m_6, \tau(m_3) = m_4, \tau(m_4) = m_2, \tau(m_5) = m_1, \tau(m_6) = m_3$ and $\tau(m_7) = m_7$. The action is depicted on the following diagram



and the claim for $P_0^4(8)$ follows from Theorem 21. Similarly, we obtain that $\mathrm{Aut}(P_1^4(8))$ is trivial and the claim is therefore proved. □

Small Covers over Neighborly Polytopes

Corollary

The lifting conjecture holds for all duals of neighborly 4-polytopes with 8 vertices.

Theorem

There are 23 different neighborly 4-polytopes on 9 vertices. 4 of them allow characteristic map and the lifting conjecture holds for all small covers over those polytopes.

Theorem

There are 431 different neighborly 4-polytopes on 10 vertices. 41 of them allow characteristic map and the lifting conjecture holds for all small covers over those polytopes.

Small Covers over Neighborly Polytopes

Theorem

There are 13935 different neighborly 4-polytopes on 11 vertices. At least 10 of them allow characteristic map and the lifting conjecture holds for all small covers over those polytopes.

Theorem

There are 126 different neighborly 5-polytopes on 9 vertices. 82 of them allow characteristic map and the lifting conjecture holds for all small covers over those polytopes.

Theorem

There are 37 different neighborly 6-polytopes on 10 vertices. Only 1 of them allows characteristic map and the lifting conjecture holds for all small covers over those polytopes.

Thank you for attention!