Sobolev orthogonal polynomials in computing of Hankel determinants

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Abstract. In this paper, we study closed form evaluation for some special Hankel determinants arising in combinatorial analysis, especially for the bidirectional number sequences. We show that such problems are directly connected with the theory of quasi-definite discrete Sobolev orthogonal polynomials. It opens a lot of procedural dilemmas which we will try to exceed. A few examples deal with Fibonacci numbers and power sequences will illustrate our considerations. We believe that our usage of Sobolev orthogonal polynomials in Hankel determinant computation is quite new.

2010 *MS Classification*: 33C45, 11B36.
Keywords: Hankel determinants, Orthogonal Polynomials, Sobolev space, Recurrence relations.

1 Introduction

The *Hankel transform* of a given number sequence A is the sequence of Hankel determinants H given by

$$A = \{a_n\}_{n\in\mathbb{N}_0} \quad \to \quad H = \{h_n\}_{n\in\mathbb{N}}: \quad h_n = |a_{i+j-2}|_{i,j=1}^n.$$

The closed-form computation of Hankel determinants is of great combinatorial interest related to partitions and permutations. A lot of methods are known for evaluation of these determinants. A long list of known determinant evaluations can be seen in [11]. A few papers deal with special number sequences and their Hankel determinants, e.g. [6] and [7].

Between the methods for evaluating the Hankel determinants, our attention is occupied by the method based on the theory of distributions and orthogonal polynomials. We have published our considerations about it in the papers [9], [14], [4], and [13].

Namely, the Hankel determinant h_n of sequence $\{a_n\}_{n\in\mathbb{N}_0}$ equals

$$h_n = a_0^n\beta_1^{n-1}\beta_2^{n-2}\cdots\beta_{n-2}^2\beta_{n-1} \qquad (n = 1, 2, \ldots), \tag{1}$$

where sequence $\{\beta_n\}_{n\in\mathbb{N}_0}$ is the sequence of the coefficients in the recurrence relation

$$P_{n+1}(x) = (x-\alpha_n)P_n(x) - \beta_n P_{n-1}(x), \qquad P_{-1}\equiv 0, \quad P_0 \equiv 1. \tag{2}$$

Here, $\{P_n(x)\}_{n\in\mathbb{N}_0}$ is the monic polynomial sequence orthogonal with respect to the inner product

$$(f,\ g) = \mathcal{U}[f(x)g(x)],$$

where $\mathcal{U} : \mathcal{P} \to \mathbb{R}$ ($\mathcal{P}$ is the space of one-variable polynomials with real coefficients) is a linear real quasi-definite functional determined by

$$a_n = \mathcal{U}[x^n] \quad (n = 0, 1, 2, \ldots).$$

In some cases, there exists a weight function $w(x)$ such that the functional $\mathcal{U}$ can be represented by

$$\mathcal{U}[f] = \int_{\mathbb{R}} f(x)\ w(x)\ dx \qquad \big(f \in \mathcal{P};\ w(x) \geq 0\big).$$

So, we can join a two-dimensional sequence of coefficients to every weight $w(x)$, i.e.

$$w(x)\ \mapsto\ \{\alpha_n,\ \beta_n\}_{n\in\mathbb{N}_0}\ ,$$

by

$$\alpha_n = \frac{\mathcal{U}[xP_n^2(x)]}{\mathcal{U}[P_n^2(x)]}, \qquad \beta_n = \frac{\mathcal{U}[P_n^2(x)]}{\mathcal{U}[P_{n-1}^2(x)]} \qquad (n \in \mathbb{N}_0)\ .$$

The statements of the following lemma are very useful (see proofs, for example, in [10]).

Lemma 1. *Let $w(x)$ be a weight function with the support $supp(w) = (a, b)$ and $\{\alpha_n,\ \beta_n\}_{n\in\mathbb{N}_0}$ the corresponding sequences of coefficients in the three-term recurrence relation (2). Also, let $\tilde{w}(x)$ be a modified weight and $\{\tilde{\alpha}_n,\ \tilde{\beta}_n\}_{n\in\mathbb{N}_0}$ its sequences of coefficients. It is valid:*

(i) *if* $\tilde{w}(x) = c_1 w(x) \quad \Rightarrow \quad \tilde{\alpha}_n = \alpha_n, \quad \tilde{\beta}_0 = c_1\beta_0, \quad \tilde{\beta}_n = \beta_n\ (n \in \mathbb{N});$

(ii) *if* $\tilde{w}(x) = w(c_1x + d_1)$ *then*

$$\tilde{\alpha}_n = \frac{\alpha_n - d_1}{c_1}\ , \quad \tilde{\beta}_0 = \frac{\beta_0}{|c_1|}\ , \quad \tilde{\beta}_n = \frac{\beta_n}{c_1^2} \quad (n \in \mathbb{N}), \quad \mathrm{supp}(\tilde{w}) = \Big(\frac{a - d_1}{c_1}, \frac{b - d_1}{c_1}\Big);$$

(iii) *if* $\tilde{w}_{c_1}(x) = (x - c_1)\ w(x)$ *such that* $c_1 < a < b$ *then*

$$\begin{aligned} \tilde{\alpha}_{c_1,0} &= \alpha_0 + r_1 - r_0\ , & \tilde{\alpha}_{c_1,n} &= \alpha_{n+1} + r_{n+1} - r_n, \\ \tilde{\beta}_{c_1,0} &= -r_0\beta_0, & \tilde{\beta}_{c_1,n} &= \beta_n\ \frac{r_n}{r_{n-1}} \qquad (n \in \mathbb{N}), \end{aligned}$$

where

$$r_0 = c_1 - \alpha_0, \quad r_{n+1} = c_1 - \alpha_{n+1} - \frac{\beta_{n+1}}{r_n} \quad (n \in \mathbb{N}_0);$$

(iv) *if* $\tilde{w}_{d_1}(x) = (d_1 - x)\ w(x)$ *such that* $a < b < d_1$ *then*

$$\tilde{\alpha}_{d_1,n} = d_1 + q_{n+1} + e_{n+1}\ (n \in \mathbb{N}_0), \quad \tilde{\beta}_{d_1,0} = \int_{\mathbb{R}} \tilde{w}_{d_1}(x)\ dx, \quad \tilde{\beta}_{d_1,n} = q_{n+1}e_n \quad (n \in \mathbb{N}),$$

where $e_0 = 0, \quad q_n = \alpha_{n-1} - e_{n-1} - d_1, \quad e_n = \beta_n/q_n \quad (n \in \mathbb{N});$

(v) *if*

$$\tilde{w}_{c_1}(x) = \frac{w(x)}{x - c_1} \qquad (c_1 < a < b),$$

then

$$\begin{aligned} \tilde{\alpha}_{c_1,0} &= \alpha_0 + r_0, & \tilde{\alpha}_{c_1,n} &= \alpha_n + r_n - r_{n-1}, & \\ \tilde{\beta}_{c_1,0} &= -r_{-1}, & \tilde{\beta}_{c_1,n} &= \beta_{n-1}\frac{r_{n-1}}{r_{n-2}} & (n \in \mathbb{N}) \ , \end{aligned}$$

where

$$r_{-1} = -\int_{\mathbb{R}} \tilde{w}_{c_1}(x) \ dx, \qquad r_n = c_1 - \alpha_n - \frac{\beta_n}{r_{n-1}} \quad (n \in \mathbb{N}_0);$$

(vi) *if*

$$\tilde{w}_{d_1}(x) = \frac{w(x)}{d_1 - x} \qquad (a < b < d_1),$$

then

$$\begin{aligned} \tilde{\alpha}_{d_1,0} &= \alpha_0 + r_0, & \tilde{\alpha}_{d_1,n} &= \alpha_n + r_n - r_{n-1}, & \\ \tilde{\beta}_{d_1,0} &= r_{-1}, & \tilde{\beta}_{d_1,n} &= \beta_{n-1}\frac{r_{n-1}}{r_{n-2}} & (n \in \mathbb{N}), \end{aligned}$$

where

$$r_{-1} = \int_{\mathbb{R}} \tilde{w}_{d_1}(x) \ dx, \qquad r_n = d_1 - \alpha_n - \frac{\beta_n}{r_{n-1}} \quad (n \in \mathbb{N}_0).$$

Corollary 2. *The coefficients in the three-term recurrence relation for polynomials orthogonal with respect to the inner product defined by*

$$w^{(2)}(x) = \frac{1}{2} \cdot \sqrt{\frac{2-x}{2+x}}, \quad x \in (-2, 2),$$

are given by

$$\alpha_0^{(2)} = 1, \quad \alpha_n^{(2)} = 0; \qquad \beta_0^{(2)} = \pi, \qquad \beta_n^{(2)} = 1 \quad (n \in \mathbb{N}). \tag{3}$$

Proof. Let us start with the monic orthogonal polynomials $\{V_n(x)\}_{n\in\mathbb{N}_0}$ with respect to

$$w^*(x) = p^{(1/2,-1/2)}(x) = \sqrt{\frac{1-x}{1+x}}, \qquad x \in (-1, 1).$$

These polynomials are monic Chebyshev polynomials of the fourth kind and they can be expressed (see Szegö [15]) by

$$V_n(\cos\theta) = \frac{\sin(n + \frac{1}{2})\theta}{2^n \sin\frac{1}{2}\theta}.$$

They satisfy the three-term recurrence relation (see, for example [8]):

$$V_{n+1}(x) = (x - \alpha_n^*) \ V_n(x) - \beta_n^* V_{n-1}(x) \quad (n = 0, 1, \ldots),$$

with initial values $V_{-1}(x) = 0, \qquad V_0(x) = 1$, where

$$\alpha_0^* = -\frac{1}{2}, \quad \alpha_n^* = 0; \qquad \beta_0^* = \pi, \quad \beta_n^* = \frac{1}{4} \quad (n \in \mathbb{N}).$$

Let

$$w^{(1)}(x) = w^*\Big(-\frac{1}{2}\,x\Big) = \sqrt{\frac{2+x}{2-x}}, \quad x \in (-2, 2).$$

By Lemma 1, case (ii), we have

$$\alpha_n^{(1)} = 2\alpha_n^* = 0 \quad (n \in \mathbb{N}_0), \qquad \beta_0^{(1)} = 2\beta_0^* = \pi, \qquad \beta_n^{(1)} = 4\beta_n^* = 1 \quad (n \in \mathbb{N}).$$

At last, considering the weight

$$w^{(2)}(x) = \frac{1}{2} \cdot w^{(1)}(x), \quad x \in (-2, 2),$$

by Lemma 1, case (i), we find coefficients (3). □

2 The quasi-definite case of discrete Sobolev inner product

To date, the orthogonality on Sobolev spaces has been considered because of the spectral theory of ordinary and partial differential equations and the numerical methods for their solving, and expansions of the functions in the special Fourier series (see [1], [2] and [12]).

This paper considers a polynomial sequence orthogonal with respect to a discrete Sobolev inner product, that is, an ordinary inner product on the real line plus an atomic inner product. According to our knowledge, their application in the computing of any Hankel determinant has not been noted.

Let $(\cdot,\cdot)$ be a positive definite inner product and $\{P_n(x)\}_{n\in\mathbb{N}_0}$ the corresponding monic orthogonal polynomial sequence which satisfies the three-term recurrence relation (2).

The sequence of monic polynomials $\{Q_n(x)\}_{n\in\mathbb{N}_0}$ orthogonal with respect to the discrete Sobolev inner product

$$\langle f,\ g\rangle = A(f,\ g)\ +\ Bf(c)g(c) \qquad (A, B, c \in \mathbb{R}), \tag{4}$$

is quite determined by $\{P_n(x)\}_{n\in\mathbb{N}_0}$ and constants A, B and c. If we take $B = 0$ we get the standard continuous case, while by taking $A = 0$ we get the discrete case. F. Marcellan and A. Ronveaux in [12] researched the case $A = 1$ and $B = 1/\lambda > 0$, where λ is a parameter. In the present paper, we let A and B be any real numbers which assure the existence of the sequence of Sobolev orthogonal polynomials $\{Q_n(x)\}_{n\in\mathbb{N}_0}$.

Since $\{P_k(x)\}_{0\le k\le n}$ is a basis in the subspace $\mathcal{P}_n$ of all polynomials whose degree do not exceed n, we can write

$$Q_n(x) = P_n(x)\ +\ \sum_{j=0}^{n-1} \frac{(Q_n, P_j)}{(P_j, P_j)} P_j(x).$$

Based on the orthogonality, we can write

$$0 = \langle Q_n,\ P_j \rangle = A(Q_n, P_j) + BQ_n(c)P_j(c),$$

wherefrom

$$Q_n(x) = P_n(x)\ -\frac{B}{A}\ Q_n(c)\sum_{j=0}^{n-1}\frac{P_j(c)P_j(x)}{(P_j, P_j)}\ .$$

By introducing the kernel

$$K_n(x, y) = \sum_{j=0}^{n}\frac{P_j(x)P_j(y)}{(P_j, P_j)}\ , \tag{5}$$

we yield

$$Q_n(x) = P_n(x)\ -\frac{B}{A}\ Q_n(c)\ K_{n-1}(x, c)\ . \tag{6}$$

Putting $x = c$ in the previous relation, we get

$$Q_n(c) = P_n(c)\ -\frac{B}{A}\ Q_n(c)K_{n-1}(c, c) \quad \Rightarrow \quad Q_n(c) = \frac{AP_n(c)}{A + BK_{n-1}(c, c)}\ .$$

By applying (6), for n and $n + 1$, after some computation, we have

$$\begin{vmatrix} Q_n(x) & Q_{n+1}(x) \\ Q_n(c) & Q_{n+1}(c) \end{vmatrix} = \frac{A}{A + BK_{n-1}(c, c)}\ \begin{vmatrix} P_n(x) & P_{n+1}(x) \\ P_n(c) & P_{n+1}(c) \end{vmatrix}.$$

Also, multiplying (6) with $(x-c)$ and using the three-term recurrence relation for $\{P_n(x)\}_{n\in\mathbb{N}_0}$, we yield

$$(x - c)Q_n(x) = P_{n+1}(x) + \big(\alpha_n - c - \rho_n P_{n-1}(c)\big)P_n(x) + \rho_n P_n(c)P_{n-1}(x),$$

where

$$\rho_n = \frac{B}{(P_{n-1}, P_{n-1})}\cdot\frac{P_n(c)}{A + BK_{n-1}(c, c)} \qquad (n \in \mathbb{N}).$$

Directly from the orthogonality in the discrete Sobolev space, it follows

$$(x - c)Q_n(x) = Q_{n+1}(x) + e_{n,n}Q_n(x) + e_{n,n-1}Q_{n-1}(x), \tag{7}$$

where $e_{n,n}$ and $e_{n,n-1}$ are some real constants.

Theorem 3. *The polynomial sequence $\{Q_n(x)\}_{n\in\mathbb{N}_0}$ orthogonal with respect to the discrete Sobolev inner product (4) satisfies the three-term recurrence relation of the form:*

$$Q_{n+1}(x) = (x - \sigma_n)Q_n(x) - \tau_n Q_{n-1}(x) \quad (n \in \mathbb{N}), \qquad Q_{-1} \equiv 0,\ Q_0 \equiv 1, \tag{8}$$

where

$$\sigma_n = \alpha_n + \rho_{n+1}P_n(c) - \rho_n P_{n-1}(c),$$
$$\tau_n = \frac{\rho_n P_n(c) + \beta_n}{\rho_{n-1}P_{n-1}(c) + \beta_{n-1}} \qquad (n \in \mathbb{N}).$$

Here $\tau_0 = \langle 1, 1\rangle$.

Proof. The recurrence relation (7) can be rearranged in the form (8). By multiplying it with $(x - c)$ and applying (6), from linear independence of the sequence $\{P_n(x)\}_{n\in\mathbb{N}_0}$, we find σ_n and τ_n. □

3 The Hankel determinants of a special integer sequence

We are interested in investigating sequences that are indirectly defined by the expression of bidirectional sequences. Such sequences have been studied, for instance, by Basor and Ehrhardt [5]. They arise naturally in the Fourier analysis of signals and systems. The authors in [5] were interested in deriving the asymptotic behavior of the Hankel determinants which are important in statistical mechanics, random matrix theory, and the theory of orthogonal polynomials.

3.1 The Fibonacci case

Let

$$b_n = \sum_{k=0}^{n} \binom{n}{k} (a_{n-2k+1} + a_{n-2k}), \quad \text{where} \quad a_n = F_{|n|}. \tag{9}$$

Here $F_{|n|}$ is the $|n|$-th Fibonacci number:

$$F_0 = 0, \quad F_1 = 1, \quad F_{n+1} = F_n + F_{n-1} \qquad (n \in \mathbb{N}).$$

The first members of $\{b_n\}_{n\in\mathbb{N}_0}$ are $\{1, 3, 7, 17, 39, \ldots\}$. P. Barry and A. Hennessy in [3] proved that the generating function of the sequence $\{b_n\}_{n\in\mathbb{N}_0}$ is

$$\sum_{n=0}^{\infty} b_n x^n = \frac{1+2x}{1-5x^2}\left(1 + \frac{x}{\sqrt{1-4x^2}}\right). \tag{10}$$

Lemma 4. *The sequence $\{b_n\}_{n\in\mathbb{N}_0}$ satisfies the following recurrence relation*

$$b_n - 5b_{n-2} = \begin{cases} 2^{(n-1)/2}\frac{(n-2)!!}{((n-1)/2)!}\ , & n - \text{odd}, \\ \\ 2^{n/2}\frac{(n-3)!!}{(n/2-1)!}, & n - \text{even}. \end{cases} \tag{11}$$

Proof. Directly from the expression (10), we have

$$\begin{aligned}\sum_{n=2}^{\infty}(b_n - 5b_{n-2})x^n &= \frac{x(1+2x)}{\sqrt{1-4x^2}} - x = -x + (x+2x^2)\sum_{n=0}^{\infty}\binom{-1/2}{n}(-4)^n x^{2n} \\ &= \sum_{n=2}^{\infty}\binom{-1/2}{n-1}(-4)^{n-1}x^{2n-1} + 2\sum_{n=1}^{\infty}\binom{-1/2}{n-1}(-4)^{n-1}x^{2n} \\ &= \sum_{n=2}^{\infty} 2^{n-1}\frac{(2n-3)!!}{(n-1)!}x^{2n-1} + \sum_{n=1}^{\infty} 2^n \frac{(2n-3)!!}{(n-1)!}x^{2n}.\end{aligned}$$

□

Lemma 5. *The following moment representation is valid*

$$b_n = -\frac{1}{\pi}\int_{-2}^{2} \frac{x^n}{5-x^2}\sqrt{\frac{2+x}{2-x}}\,dx + \left(1+\frac{2}{\sqrt{5}}\right)c^n \qquad \left(c=\sqrt{5}\right)\,(n\in\mathbb{N}). \tag{12}$$

Proof. Let us denote by B_n the right side of (12). Then

$$B_n - 5B_{n-2} = -\frac{1}{\pi}\int_{-2}^{2} \frac{x^n - 5x^{n-2}}{5-x^2}\sqrt{\frac{2+x}{2-x}}\,dx + \left(1+\frac{2}{\sqrt{5}}\right)(c^n - 5c^{n-2}),$$

wherefrom

$$B_{2n} - 5B_{2n-2} = \frac{1}{\pi}\int_{-2}^{2} x^{2n-2}\sqrt{\frac{2+x}{2-x}}\,dx = \frac{2^{2n-1}\Gamma(n-1/2)}{\sqrt{\pi}\Gamma(n)}\ ,$$

and

$$B_{2n-1} - 5B_{2n-3} = \frac{1}{\pi}\int_{-2}^{2} x^{2n-3}\sqrt{\frac{2+x}{2-x}}\,dx = \frac{2^{2n}\Gamma(n-1/2)}{\sqrt{\pi}\Gamma(n)}\ .$$

It proves that both sequences $\{B_n\}_{n\in\mathbb{N}_0}$ and $\{b_n\}_{n\in\mathbb{N}_0}$ satisfy the same recurrence relation (11). To prove that $B_n = b_n$ for every $n \in \mathbb{N}_0$, we just need to show that initial terms B_0 and b_0 are equal. That can be done by direct evaluation of both terms:

$$B_0 = -\frac{1}{\pi}\int_{-2}^{2} \frac{1}{5-x^2}\sqrt{\frac{2+x}{2-x}}\,dx + \left(1+\frac{2}{\sqrt{5}}\right) = 1 = b_0.$$

That completes the proof of the lemma. □

Note that (12) is the moment sequence corresponding to the special case of discrete Sobolev inner product (4) for $A=-1$ and $B = 1+\frac{2}{\sqrt{5}}$:

$$\langle f,\ g\rangle = -(f,\ g)\ +\ \left(1+\frac{2}{\sqrt{5}}\right)f(c)g(c) \qquad (A,B,c\in\mathbb{R}) \tag{13}$$

with

$$(f,\ g) = \int_{-2}^{2} f(x)g(x)w(x)\,dx, \tag{14}$$

where

$$w(x) = \frac{1}{\pi(5-x^2)}\sqrt{\frac{2+x}{2-x}}, \quad x\in(-2,2). \tag{15}$$

Our goal is to evaluate the corresponding coefficients τ_n which will lead to the expression for the Hankel transform of the sequence b_n. Evaluation will be performed in two steps. The first step is to find expressions for the coefficients α_n and β_n corresponding to the weight function $w(x)$ (i.e. the absolute continuous part $(\cdot,\cdot)$ of (13)). It is done in the following lemma.

Lemma 6. *The coefficients α_n and β_n of the three-term recurrence relation, corresponding to the weight function* (15)*, are given by:*

$$\begin{aligned} \alpha_0 &= \frac{5-\sqrt{5}}{2}, \quad \alpha_1 = \frac{\sqrt{5}-3}{2}, \quad \alpha_n = 0; \\ \beta_0 &= \frac{2}{\sqrt{5}}, \quad \beta_1 = \frac{3\sqrt{5}-5}{2}, \quad \beta_n = 1 \quad (n \geq 2). \end{aligned} \tag{16}$$

Proof. We will continue the considerations started in Corollary 2. Let

$$w^{(3)}(x) = \frac{w^{(2)}(x)}{\sqrt{5}-x}, \qquad x \in (-2, 2).$$

Applying Lemma 1 (vi), for $d_1 = \sqrt{5}$, we have $r_{-1} = \pi(1+\sqrt{5})/2,\ r_k = (\sqrt{5}-1)/2\ (k \in \mathbb{N}_0)$. Hence

$$\alpha_0^{(3)} = \frac{1+\sqrt{5}}{2},\ \alpha_n^{(3)} = 0\ (n \in \mathbb{N}); \quad \beta_0^{(3)} = \frac{\pi}{2}(1+\sqrt{5}),\ \beta_1^{(3)} = \frac{3-\sqrt{5}}{2},\ \beta_n^{(3)} = 1\ (n \geq 2).$$

Similarly, let

$$w^{(4)}(x) = \frac{w^{(3)}(x)}{x-(-\sqrt{5})}, \quad x \in (-2, 2).$$

According to Lemma 1 (v), for $c_1 = -\sqrt{5}$, we have $r_{-1} = -\pi/\sqrt{5},\ r_0 = 2-\sqrt{5},\ r_k = (1-\sqrt{5})/2\ (k \in \mathbb{N})$. Hence

$$\begin{aligned} \alpha_0^{(4)} &= \frac{5-\sqrt{5}}{2}, \quad \alpha_1^{(4)} = \frac{\sqrt{5}-3}{2}, \quad \alpha_n^{(4)} = 0; \\ \beta_0^{(4)} &= \frac{\pi}{\sqrt{5}}, \quad \beta_1^{(4)} = \frac{3\sqrt{5}-5}{2}, \quad \beta_n^{(4)} = 1 \qquad (n = 2, 3, \ldots). \end{aligned}$$

Finally the last transformation

$$w(x) = \frac{2}{\pi}\, w^{(4)}(x),$$

directly implies (16). □

Lemma 7. *The coefficients $\{\tau_n\}$ of the three-term recurrence relation, satisfied by Sobolev orthogonal polynomials $\{Q_n(x)\}_{n \in \mathbb{N}_0}$ orthogonal with respect to* (13)*, are given by:*

$$\tau_0 = 1, \quad \tau_1 = -2, \quad \tau_n = 1 \quad (n \geq 2).$$

Proof. Denote by $\{P_n(x)\}_{n \in \mathbb{N}_0}$ the sequence of monic orthogonal polynomials with respect to the weight function $w(x)$ (i.e. the absolute continuous scalar product (14)). Their squared norms are

$$(P_0,\ P_0) = \frac{2}{\sqrt{5}}, \qquad (P_n,\ P_n) = \beta_n\beta_{n-1}\ldots\beta_0 = 3-\sqrt{5} \qquad (n \in \mathbb{N}).$$

By mathematical induction, we can prove that polynomials $\{P_n(x)\}_{n\in\mathbb{N}_0}$, at the point $c=\sqrt{5}$, have the following values:

$$P_0(c)=1,\quad P_1(c)=\frac{3\sqrt{5}-5}{2}\ ,\quad P_n(c)=(5-2\sqrt{5})\left(\frac{1+\sqrt{5}}{2}\right)^n\quad (n=2,3,\ldots).$$

According to (5), $K_n=K_n(c,c)$ $(n\in\mathbb{N}_0)$ is given by

$$\begin{aligned}K_0&=\frac{\sqrt{5}}{2},\quad K_1=\frac{3}{4}(5-\sqrt{5}),\\ K_n&=\frac{\sqrt{5}+1}{8}\left(14\sqrt{5}-30+5(3\sqrt{5}-7)\left(\frac{1+\sqrt{5}}{2}\right)^{2n}\right)\quad (n=2,3,\ldots),\end{aligned}$$

and

$$\rho_1=\frac{1+\sqrt{5}}{2},\qquad \rho_n=\frac{15+7\sqrt{5}}{10}\left(\frac{\sqrt{5}-1}{2}\right)^n,\quad \rho_n\ P_n(c)=\frac{1+\sqrt{5}}{2}\quad (n=2,3,\ldots).$$

Hence, according to Theorem 3, we have $\tau_n=1\quad(n\in\mathbb{N};\ n\geq 2)$. It is known that $\tau_0=b_0=1$. The first members of Sobolev orthogonal polynomial sequence are

$$Q_0(x)=1,\quad Q_1(x)=x-3,\quad Q_2(x)=x^2-2x-1=(x+1)Q_1(x)-(-2)Q_0(x),$$

wherefrom $\tau_1=-2$. □

Now, we have all elements for formula (1) and we can compute h_n by

$$h_n=\tau_0^n\tau_1^{n-1}\tau_2^{n-2}\cdots\tau_{n-2}^2\tau_{n-1}\ .$$

That completes the proof of the following theorem:

Theorem 8. *The Hankel transform of* $\{b_n\}_{n\in\mathbb{N}_0}$ *defined by (9) is given by* $h_n=(-2)^{n-1}$ $(n\in\mathbb{N})$.

3.2 The power case

Let

$$b_n=\sum_{k=0}^{n}\binom{n}{k}(a_{n-2k+1}+a_{n-2k}),\quad \text{where}\quad a_n=r^{|n|}\quad (r>1).\tag{17}$$

The sequence $\{b_n\}_{n\in\mathbb{N}_0}$ has the following generating function (see [3])

$$\sum_{n=0}^{\infty}b_nx^n=\frac{r+1}{r-(r^2+1)x}\left(\frac{r+1}{2}+\frac{(r-1)(x+1/2)}{\sqrt{1-4x^2}}\right).$$

Lemma 9. *This sequence also satisfies the following recurrence relation*

$$\frac{r}{r^2-1}\left(b_n-\frac{r^2+1}{r}b_{n-1}\right)=\begin{cases}2^{(n-1)/2}\frac{(n-2)!!}{((n-1)/2)!}\ , & n-\text{odd},\\ \\ 2^{n/2-1}\frac{(n-1)!!}{(n/2)!}, & n-\text{even};\end{cases}\qquad (n\in\mathbb{N}). \tag{18}$$

Lemma 10. *The following moment representation is valid*

$$b_n=\frac{-(r^2-1)}{2\pi r}\int_{-2}^{2}\frac{x^n}{c-x}\sqrt{\frac{2+x}{2-x}}\,dx+\frac{(r+1)^2}{r}c^n\qquad\left(c=\frac{r^2+1}{r}\right)(n\in\mathbb{N}). \tag{19}$$

Proof. We proceed similarly as in Lemma 5. Let us denote by B_n the right-hand side of (19). Hence,

$$\begin{aligned}B_n-cB_{n-1}&=\frac{-(r^2-1)}{2\pi r}\int_{-2}^{2}\frac{x^n-cx^{n-1}}{c-x}\sqrt{\frac{2+x}{2-x}}\,dx+\frac{(r+1)^2}{r}(c^n-c\cdot c^{n-1})\\&=\frac{(r^2-1)}{2\pi r}\int_{-2}^{2}x^{n-1}\sqrt{\frac{2+x}{2-x}}\,dx=2^{n-1}\frac{(r^2-1)}{\pi r}\int_{-1}^{1}t^{n-1}\frac{\sqrt{1-t^2}}{1-t}\,dt.\end{aligned}$$

which implies

$$B_{2m+1}-cB_{2m}=2^{2m}\frac{(r^2-1)}{\sqrt{\pi}r}\frac{\Gamma(m+1/2)}{m!},\qquad B_{2m}-cB_{2m-1}=2^{2m-1}\frac{(r^2-1)}{\sqrt{\pi}r}\frac{\Gamma(m+1/2)}{m!}.$$

That proves that sequences $\{B_n\}_{n\in\mathbb{N}_0}$ and $\{b_n\}_{n\in\mathbb{N}_0}$ satisfy the same recurrent relation (18). To prove that $B_n=b_n$ for every $n\in\mathbb{N}_0$, we just need to show that initial terms B_0 and b_0 are equal, which again can be done by direct evaluation:

$$B_0=\frac{-(r^2-1)}{2\pi r}\int_{-2}^{2}\frac{1}{c-x}\sqrt{\frac{2+x}{2-x}}\,dx+\frac{(r+1)^2}{r}=1+r=b_0.$$

That completes the proof of the lemma. □

Therefore, for the further consideration, the important weight function is given by

$$w(x)=\frac{r^2-1}{\pi r}\ w^{(3)}(x)=\frac{r^2-1}{\pi r}\frac{1}{2(c-x)}\sqrt{\frac{2+x}{2-x}},\quad x\in(-2,2), \tag{20}$$

Lemma 11. *The coefficients α_n and β_n of the three-term recurrence relation, corresponding to the weight function (20), are given by:*

$$\alpha_0=\frac{r+1}{r},\quad \alpha_n=0\ (n\in\mathbb{N});\qquad \beta_0=\frac{r+1}{r},\quad \beta_1=\frac{r-1}{r},\quad \beta_n=1\quad(n\geq 2). \tag{21}$$

Proof. Let

$$w^{(3)}(x)=\frac{w^{(2)}(x)}{c-x},\quad x\in(-2,2).$$

Applying Lemma 1 (vi), for $d_1 = c = (r^2+1)/r$, we have $r_{-1} = \pi/(r-1),\ r_k = 1/r \quad (k \in \mathbb{N}_0)$. Hence

$$\alpha_0^{(3)} = \frac{r+1}{r},\ \alpha_n^{(3)} = 0\ (n \in \mathbb{N}); \qquad \beta_0^{(3)} = \frac{\pi}{r-1},\ \beta_1^{(3)} = \frac{r-1}{r}, \quad \beta_n^{(3)} = 1 \quad (n = 2, 3, \ldots).$$

Finally, for

$$w(x) = \frac{r^2-1}{\pi r}\ w^{(3)}(x)$$

we get (21). □

Theorem 12. *Let $a_n = r^{|n|}$, and $\{b_n\}_{n\in\mathbb{N}_0}$ sequence determined by (17). Then the Hankel transform of $\{b_n\}_{n\in\mathbb{N}_0}$ is given by $h_n = (r+1)(1-r^2)^{n-1}$ $(n \in \mathbb{N})$.*

Proof. The squared norms of the polynomials orthogonal with respect to (20) are

$$(P_0,\ P_0) = \frac{r+1}{r}, \qquad (P_n,\ P_n) = \beta_n\beta_{n-1}\cdots\beta_0 = \frac{r^2-1}{r^2} \qquad (n \in \mathbb{N})\ .$$

By mathematical induction, we prove that these polynomials have the following values in the point c:

$$P_0(c) = 1, \qquad P_n(c) = (r-1)r^{n-1} \qquad (n \in \mathbb{N})\ .$$

Now, according to formula (5) and Theorem 3 it is valid

$$K_m(c,c) = \frac{r(r^{2m+1}+1)}{(r+1)^2} \quad (m \in \mathbb{N}_0), \qquad \rho_n = (r+1)r^{1-n} \quad (n \geq 1).$$

Notice that $\rho_n\ P_n(c) = r^2 - 1$ $(n \geq 2)$ implying $\tau_n = 1$ $(n \geq 2)$.

The first members of the sequence $\{Q_n(x)\}_{n\in\mathbb{N}_0}$ are:

$$Q_0(x) = 1, \quad Q_1(x) = x - r - 1, \quad Q_2(x) = x^2 - (r+1)x + r - 1 = xQ_1(x) - (1-r)Q_0(x).$$

Hence $\tau_0 = b_0 = r+1, \quad \tau_1 = 1 - r$. The Hankel transform of the sequence $\{b_n\}_{n\in\mathbb{N}_0}$ defined by (17), is given by

$$h_n = \tau_0^n \tau_1^{n-1}\tau_2^{n-2}\cdots\tau_{n-2}^2 = (r+1)^n(1-r)^{n-1}\ .$$

□

Example 13. Let $r = 2$. Then $a_n = 2^{|n|}$ and

$$b_n = \sum_{k=0}^{n} \binom{n}{k} \left(2^{|n-2k+1|} + 2^{|n-2k|}\right)$$

which begins with

$$3,\ 9,\ 24,\ 63,\ 162,\ 414,\ 1050,\ 2655, \ldots\ .$$

and has the generating function

$$\sum_{n=0}^{\infty} b_n x^n = \frac{3(2-3x-x^2C(x^2))}{(1-2x)(2-5x)} \ .$$

Here,

$$C(x) = \frac{1-\sqrt{1-4x}}{2x}$$

is the generating function of the sequence of Catalan numbers. We have the following moment representation

$$b_n = \frac{1}{2\pi}\int_{-2}^{2} x^n \frac{3\sqrt{4-x^2}}{(2-x)(2x-5)}\, dx + \frac{9}{2}\left(\frac{5}{2}\right)^n .$$

Hence $\tau_0 = 3,\ \tau_1 = -1,\ \tau_n = 1 \quad (n \in \mathbb{N};\ n \geq 2)$. The Hankel transform of the sequence $\{b_n\}_{n\in\mathbb{N}_0}$ is

$$h_n = 3^n(-1)^{n-1} \qquad (n = 1, 2, \ldots).$$

Acknowledgements. We are very thankful to the anonymous referee for careful proof-reading and useful suggestions.

This research was supported by the Ministry of Science and Education, the projects No. 174011 and 174013.

References

[1] M. Alfaro, J.J. Moreno-Balcázar, A. Peña, M.L. Rezola, A new approach to the asymptotics of Sobolev type orthogonal polynomials, Journal of Approximation Theory **163** (2011) 460-480.

[2] R. Barrio, B. Melendo, S. Serrano, *Generation and evaluation of orthogonal polynomials in discrete Sobolevspaces I: algorithms*, Journal of Computational and Applied Mathematics **181** (2005) 280298.

[3] P. Barry, A. Hennessy, *Riordan arrays and the LDU decomposition of symmetric Toeplitz plus Hankel matrices*, ArXiv: 1101.2605v1[math.CO] 13. Jan 2011. (http://arxiv.org/abs/1101.2605v1).

[4] P. Barry, P.M. Rajković, M.D. Petković, *An Application of Sobolev Orthogonal Polynomials to the Computation of a Special Hankel Determinant*, Springer Optimization and Its Applications **42** (2010) 1–8.

[5] E.L.Basor, T. Ehrhardt, Some identities for determinants of structured matrices, Linear Algebra and its Applications, **343-344** (2002), 5-19.

[6] R.A. Brualdi, S. Kirkland, *Aztec diamonds and digraphs, and Hankel determinants of Schrder numbers*, Journal of Combinatorial Theory, Series B **94** (2005) 334-351.

[7] N.T. Cameron, A.C.M. Yip, Hankel determinants of sums of consecutive Motzkin numbers, Linear Algebra and its Applications **434** (2011) 712-722.

[8] T. S. Chihara, *An Introduction to Orthogonal Polynomials*, Gordon and Breach, New York, 1978.

[9] A. Cvetković, P. Rajković and M. Ivković, *Catalan Numbers, the Hankel Transform and Fibonacci Numbers*, Journal of Integer Sequences **5** (2002) Article 02.1.3.

[10] W. Gautschi, *Orthogonal Polynomials: Computation and Approximation*, Clarendon Press - Oxford, 2003.

[11] C. Krattenthaler, *Advanced determinant calculus: A complement,* Linear Algebra and its Applications **411** (2005) 68-166.

[12] F. Marcellan, A. Ronveaux, *On a class of polynomials orthogonal with respect to a discrete Sobolev inner product*, Indag. Mathem., N.S. **1** (1990) 451–464.

[13] M.D. Petković, P. Barry, P.M. Rajković, *Closed-form expression for Hankel determinants of the Narayana polynomials*, Czechoslovak Mathematical Journal, 62 No. 137 (2012), 39–57.

[14] P.M. Rajković, M.D. Petković, P. Barry *The Hankel Transform of the Sum of Consecutive Generalized Catalan Numbers*, Integral Transforms and Special Functions, **18** No. 4 (2007) 285–296.

[15] G. Szegö, *Orthogonal Polynomials*, AMS, 4th. ed., Vol. 23 (Colloquium publications), 1975.