Hankel transform of a sequence obtained by series reversion

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Abstract

We study the Hankel transform of a sequence $(u_n)_{n\in\mathbb{N}_0}$ defined by the series reversion of a certain rational function $A(x)$. Using the method based on orthogonal polynomials, we give closed-form evaluations of the Hankel transform of $(u_n)_{n\in\mathbb{N}_0}$ and shifted sequences. It is also shown that the Hankel transforms satisfy certain ratio conditions which recover the sequence $(a_n)_{n\in\mathbb{N}_0}$ whose generating function is $A(x)$. Therefore, we indicate that the term-wise ratio of Hankel transforms of shifted sequences are noteworthy objects of study, giving us more insight into the processes involved in the Hankel transform.

Key words: Hankel transform, Catalan numbers, series reversion.

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1 Introduction

The Hankel transform of an integer sequence has attracted an increasing amount of attention recently. Although Hankel determinants are well-known for a long time, Layman [18] first introduced the term "Hankel transform" in 2001. This is a transformation on the set of integer sequences defined as follows.

Definition 1.1. *For a given sequence $a = (a_n)_{n\in\mathbb{N}_0}$ let us consider the $(n+1)\times(n+1)$ matrix $[a_{i+j}]_{0\le i,j\le n}$. The* **Hankel transform** *$h = (h_n)_{n\in\mathbb{N}_0}$ of the sequence $(a_n)_{n\in\mathbb{N}_0}$ is defined by*

$$h_n = \det\left([a_{i+j}]_{0\le i,j\le n}\right), \quad (n\in\mathbb{N}_0) \tag{1}$$

and is denoted by $h = \mathcal{H}(a)$.

It is well-known that the Hankel transform is invariant under the binomial and k-binomial transformations [18, 29] and hence it is not invertible. The Hankel transform provides the connection between certain well-known integer sequences. This is shown in the following examples.

Example 1.1. The Hankel transform of the sequence of Catalan numbers ($C_n = \frac{1}{n+1}\binom{2n}{n}$) A000108 is the sequence of all 1's [18]. Thus each of the determinants

$$\begin{vmatrix} 1 \end{vmatrix}, \quad \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}, \quad \begin{vmatrix} 1 & 1 & 2 \\ 1 & 2 & 5 \\ 2 & 5 & 14 \end{vmatrix}, \ldots$$

has value 1. A unique feature of the Catalan numbers is that the shifted sequence C_{n+1} also has a Hankel transform of all 1's. An interesting feature of the Catalan numbers is that the sequence $C_n - \delta_{n0}$, or $0, 1, 2, 5, 14, 42, \ldots$ has Hankel transform with general term $-n$. This result will be proven at the end of section 5 (Corollary 5.5). Note that the evaluation of the Hankel-like determinant $\det[C_{\lambda_i+j}]_{0\le i,j\le n}$ is given in the recent paper of Krattenthaler [16].

Example 1.2. The central binomial coefficients A000984, defined by $a_n = \binom{2n}{n}$, have generating function $\frac{1}{\sqrt{1-4x}}$. The Hankel transform of the central binomial coefficients is given by $h_n = 2^n$ [24, 28]. In other words, it holds that

$$\begin{vmatrix} 1 \end{vmatrix} = 1, \quad \begin{vmatrix} 1 & 2 \\ 2 & 6 \end{vmatrix} = 2, \quad \begin{vmatrix} 1 & 2 & 6 \\ 2 & 6 & 20 \\ 20 & 70 & 252 \end{vmatrix} = 4, \ldots$$

Many other Hankel transform evaluations are known in the literature. Papers [5, 26] provide the Hankel transform evaluation of the sum of two consecutive Catalan and generalized Catalan numbers. Brualdi and Kirkland used the Hankel transformation of the large Schröder numbers A006318 for counting a number of tiling of an aztec diamond with dominoes [2]. In recent papers, Eğecioğlu, Redmond and Ryavec [7, 8] introduced a method for Hankel transform evaluation based on differential-convolution equations which is applied to several different sequences. Another method based on exponential generating functions is shown by Junod [13]. One of the earlier contributors to our stock of knowledge about the Hankel transform, Christian Radoux, had published several proofs of this result, along with other interesting examples [22, 23, 25]. Different Hankel transform evaluations, as well as the evaluations of other types of determinants, are given in the papers of Krattenthaler [14, 15].

The Gessel-Viennot-Lindström (G-V-L) method [11, 19, 31] provides the connection between Hankel transform evaluation and lattice paths. A recent example of Hankel transform evaluation using the G-V-L method is shown in [3]. Further connections between Hankel transforms and lattice paths are shown for example in [12, 30, 32]. Links between orthogonal polynomials, lattice paths and continued fractions have been studied by Viennot [31] and Flajolet [9].

In this paper, we use a method based on orthogonal polynomials for Hankel transform evaluation which is used in [5, 26] and is similar to one used in [2].

Let $(a_n)_{n\in\mathbb{N}_0}$ be the moment sequence with respect to some measure $d\lambda(x)$. In other words, let

$$a_n = \int_{\mathbb{R}} x^n d\lambda(x) \quad (n = 0, 1, 2, \ldots)\ . \tag{2}$$

Then the Hankel transform $h = \mathcal{H}(a)$ of the sequence $a = (a_n)_{n\in\mathbb{N}_0}$ can be expressed by the following relation known as the Heilermann formula (for example, see Krattenthaler [15])

$$h_n = a_0^{n+1}\beta_1^n\beta_2^{n-1}\cdots\beta_{n-1}^2\beta_n. \tag{3}$$

The sequence $(\beta_n)_{n\in\mathbb{N}_0}$ appears as a sequence of coefficients in the three-term recurrence relation

$$P_{n+1}(x) = (x-\alpha_n)P_n(x) - \beta_n P_{n-1}(x), \tag{4}$$

satisfied by the sequence $(P_n(x))_{n\in\mathbb{N}_0}$ of monic orthogonal polynomials with respect to the measure $d\lambda(x)$.

The following theorem and corollary provide the way how to explicitly find the measure $d\lambda(x)$ with prescribed moment sequence.

Theorem 1.1. (Stieltjes-Perron inversion formula) [4, 17] *Let $(\mu_n)_{n\in\mathbb{N}_0}$ be a sequence such that all elements of its Hankel transform are non-negative. Denote by $G(z)=\sum_{n=0}^{+\infty}\mu_n z^n$ the generating function of the sequence $(\mu_n)_{n\in\mathbb{N}_0}$ and $F(z)=z^{-1}G(z^{-1})$. Also let the function $\lambda(t)$ be defined by*

$$\lambda(t)-\lambda(0) = -\frac{1}{2\pi i}\lim_{y\to 0^+}\int_0^t \Big[F(x+iy)-F(x-iy)\Big]dx. \tag{5}$$

Then we have $\mu_n = \int_{\mathbb{R}} x^n d\lambda(x)$, i.e. the sequence $(\mu_n)_{n\in\mathbb{N}_0}$ is the moment sequence of the measure $\lambda(t)$.

Corollary 1.2. *Under the assumptions of the previous lemma, let additionally $F(\bar{z})=\overline{F(z)}$. Then*

$$\lambda(t)-\lambda(0) = -\frac{1}{\pi}\lim_{y\to 0^+}\int_0^t \Im F(x+iy)dx. \tag{6}$$

For our further discussion we need the definition of the series reversion of a (generating) function $f(x)$ which satisfies $f(0)=0$ (see [1]).

Definition 1.2. *For a given (generating) function $v=f(u)$ with the property $f(0)=0$, the series reversion is the sequence $(s_n)_{n\in\mathbb{N}_0}$ such that*

$$u = f^{-1}(v) = s_1 v + s_2 v^2 + \cdots + s_n v^n + \cdots ,$$

where $u=f^{-1}(v)$ is the inverse function of $v=f(u)$. Note that since $f(0)=0$, there must hold $s_0=f^{-1}(0)=0$.

In this paper, we consider the Hankel transform evaluation of a series reversion of the function $\frac{x}{1+\alpha x+\beta x^2}$. As will be seen in the next section, that sequence generalizes several well-known integer sequences. Note that the similar evaluation is given by Xin in a recent paper [33]. In our approach (sections 3, 4 and 5), we mainly used a method based on orthogonal polynomials.

In the last section, we show that the Hankel transforms satisfy certain ratio conditions which recover the sequence $(q_n)_{n\in\mathbb{N}_0}$ whose generating function $Q(x)$ was reverted.

2 The series reversion of $\frac{x}{1+\alpha x+\beta x^2}$

Let us consider the sequence $(u_n)_{n\in\mathbb{N}_0}$ given by the series reversion of

$$Q(x) = \frac{x}{1+\alpha x+\beta x^2}.$$

That sequence is already investigated in [1] where several expressions are given for it. Using Definition 1.2, we find that the generating function $U(x)$ of the sequence $(u_n)_{n\in\mathbb{N}_0}$ is the solution of the equation

$$Q(U(x)) = \frac{U(x)}{1+\alpha U(x)+\beta U(x)^2} = x \tag{7}$$

and is given by

$$U(x) = \frac{1-\alpha x-\sqrt{1-2\alpha x+(\alpha^2-4\beta)x^2}}{2\beta x}. \tag{8}$$

According to Proposition 9 in [1], the general term of the sequence $(u_n)_{n\in\mathbb{N}_0}$ is

$$u_n = \sum_{k=0}^{[\frac{n-1}{2}]} \binom{n-1}{2k} C_k \alpha^{n-2k-1}\beta^k. \tag{9}$$

Consider the shifted sequences $(u_n^*)_{n\in\mathbb{N}_0}$ and $(u_n^{**})_{n\in\mathbb{N}_0}$ defined by $u_n^* = u_{n+1}$ and $u_n^{**} = u_{n+2}$. Also denote by h_n, h_n^* and h_n^{**}, the Hankel transforms of the sequences u_n, u_n^* and u_n^{**} respectively. The Hankel transforms h_n^* and h_n^{**} will be used in the evaluation of the Hankel transform h_n.

Putting $\alpha = 2$ and $\beta = 1$ in (8) gives the generating function

$$\frac{1-2x-\sqrt{1-4x}}{2x} = -1+\sum_{n=0}^{+\infty} C_n x^n$$

of the sequence $(C_n-\delta_{n0})_{n\in\mathbb{N}_0}$ which is mentioned in Example 1.1.

More generally, if we put $\alpha = z+1$ and $\beta = z$ we obtain the generating function of the sequence $((N_n(z)-\delta_{n0})/z)_{n\in\mathbb{N}_0}$ where $N_n(z)$ is the n-th Narayana polynomial. This comes directly from the expression for the generating function of Narayana polynomials (see for example [6] or [20]).

Furthermore, note that for specific values of α and β, the sequence $(u_n^*)_{n\in\mathbb{N}_0}$ reduces to the following well-known sequences:

- Motzkin numbers A001006, for $\alpha = \beta = 1$. This follows directly from the fact that $U(x)/x$ reduces to $M(x) = (1-x-\sqrt{1-2x-3x^2})/(2x^2)$ which is the generating function of Motzkin numbers [3, 27].
- Aerated Catalan numbers A126120, for $\alpha = 0$ and $\beta = 1$. Again this follows from the fact that $U(x)/x$ reduces to $(1-\sqrt{1-4x^2})/(2x^2)$ which is the generating function of aerated Catalan numbers [27].

We show later that the corresponding expressions for h_n and h_n^* also reduce to known ones in the mentioned special cases.

Note that $Q(x)$ is the generating function of the sequence $(q_n)_{n\in\mathbb{N}_0}$ satisfying the linear difference equation

$$q_{n+2}+\alpha q_{n+1}+\beta q_n = 0$$

with initial conditions $q_0 = 0$ and $q_1 = 1$. Therefore, the closed-form expression for q_n is given by

$$\begin{aligned} q_n &= \frac{(-1)^n}{2^n\sqrt{\alpha^2-4\beta}}\left[\left(\alpha-\sqrt{\alpha^2-4\beta}\right)^n-\left(\alpha+\sqrt{\alpha^2-4\beta}\right)^n\right] \\ &= \frac{(-1)^{n-1}}{2^{n-1}}\sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n}{2k+1}(\alpha^2-4\beta)^k\alpha^{n-2k-1}. \end{aligned} \tag{10}$$

Example 2.1. The first few terms of the previously mentioned sequences are:

$$\begin{aligned} u_n &= \left(0,1,\alpha,\alpha^2+\beta,\alpha^3+3\alpha\beta,\alpha^4+6\alpha^2\beta+2\beta^2,\alpha^5+10\alpha^3\beta+10\alpha\beta^2,\alpha^6+15\alpha^4\beta+3\alpha^2\beta^2+5\beta^3\ldots\right) \\ q_n &= \left(0,1,-\alpha,\alpha^2-\beta,2\alpha\beta-\alpha^3,\alpha^4-3\alpha^2\beta+\beta^2,-\alpha^5+4\alpha^3\beta-3\alpha\beta^2,\ldots\right) \\ h_n &= \left(0,-1,-\alpha\beta,-\alpha^2\beta^3+\beta^4,-\alpha^3\beta^6+2\alpha\beta^7,\ldots\right) \\ h_n^* &= \left(1,\beta,\beta^3,\beta^6,\beta^{10},\ldots\right) \\ h_n^{**} &= \left(\alpha,\alpha^2\beta-\beta^2,\alpha^3\beta^3-2\alpha\beta^4,\alpha^4\beta^6-3\alpha^2\beta^7+\beta^8,\ldots\right) \end{aligned}$$

It can be observed that the following ratio conditions are satisfied by the first few terms of the previously mentioned sequences

$$\frac{(-1)^{n+1}h_{n+1}}{h_n^*} = q_{n+1}, \qquad \frac{(-1)^{n+1}h_n^{**}}{h_n^*} = q_{n+2}.$$

These conditions will be proven for general n in the last section of the paper.

3 Moment representations

The following theorem gives an explicit expression for the weight function whose moment sequence is $(u_n)_{n\in\mathbb{N}_0}$. The proof is based on the Stieltjes-Perron inversion formula (Theorem 1.1).

Theorem 3.1. *The sequence $(u_n)_{n\in\mathbb{N}_0}$, is the moment sequence for the weight function:*

$$w(x) = w_{ac}(x) + \frac{\alpha-\sqrt{\alpha^2-4\beta}}{2\beta}\delta(x) \tag{11}$$

where

$$w_{ac}(x) = \begin{cases} \dfrac{\sqrt{4\beta-(x-\alpha)^2}}{2\pi\beta x}, & x\in[\alpha-2\sqrt{\beta},\alpha+2\sqrt{\beta}] \\ 0, & \text{otherwise} \end{cases}.$$

and $\delta(x)$ is the Dirac delta function.

Proof. We start from

$$F(z) = z^{-1}U(z^{-1}) = \frac{z-\alpha-\sqrt{z^2-2\alpha z+\alpha^2-4\beta}}{2\beta z} = \frac{z-\alpha-\sqrt{(z-\alpha)^2-4\beta}}{2\beta z},$$

and let $x_{1,2} = \alpha \pm 2\sqrt{\beta}$ be the branch points of the function

$$\rho(z) = \sqrt{(z-\alpha)^2-4\beta}.$$

We take a regular branch of $\rho(z)$ such that $\arg(z-x_1)=\arg(z-x_2)=0$ for $z\in(x_2,+\infty)$. The selected branch is defined on $\mathbb{C}\setminus(x_1,x_2)$ and we have $\overline{F(z)}=F(\overline{z})$. By direct evaluation we find the following expression for the primitive function of $F(z)$:

$$F_1(z)=\int F(z)dz=\frac{1}{2\beta}\left[z-\rho(z)-\left(\alpha-\sqrt{\alpha^2-4\beta}\right)L_1(z)+\alpha L_2(z)-\sqrt{\alpha^2-4\beta}L_3(z)\right]$$

where

$$\begin{aligned}L_1(z)&=\log z,\\ L_2(z)&=\log\left(z-\alpha+\rho(z)\right),\\ L_3(z)&=\log\left(\alpha^2-4\beta-\alpha z+\rho(z)\sqrt{\alpha^2-4\beta}\right).\end{aligned}$$

In the previous expression, we take a regular branch of the log function on the set $\mathbb{C}\setminus[0,+\infty)$ such that $\lim_{y\to0+}\log(x+iy)=\log x$ when $x>0$. The following then holds

$$G_\rho(x)=\lim_{y\to0}\Im\rho(x+iy)=\begin{cases}\sqrt{4\beta-(x-\alpha)^2}, & x\in(x_1,x_2)\\ 0, & \text{otherwise}\end{cases}. \tag{12}$$

Let $G_k(x)=\lim_{y\to0+}L_k(x+iy)$ where $k=1,2,3$. It is well-known that

$$G_1(x)=\lim_{y\to0+}\Im\log(x+iy)=\begin{cases}\pi, & x<0\\ \frac{\pi}{2}, & x=0\\ 0, & x>0\end{cases}. \tag{13}$$

Similarly there holds

$$G_2(x)=\lim_{y\to0}\Im L_2(x+iy)=\begin{cases}\pi, & x<x_1\\ \pi+\arctan\frac{\sqrt{4\beta-(x-\alpha)^2}}{x-\alpha}, & x\in[x_1,\alpha)\\ \frac{\pi}{2}, & x=\alpha\\ \arctan\frac{\sqrt{4\beta-(x-\alpha)^2}}{x-\alpha}, & x\in(\alpha,x_2]\\ 0, & x>x_2\end{cases} \tag{14}$$

and

$$G_3(x)=\lim_{y\to0}\Im L_3(x+iy)=\begin{cases}\pi, & x<x_1\\ \pi+\arctan\frac{\sqrt{4\beta-(x-\alpha)^2}\sqrt{\alpha^2-4\beta}}{\alpha^2-4\beta-\alpha x}, & x\in[x_1,\frac{\alpha^2-4\beta}{\alpha})\\ \frac{\pi}{2}, & x=\frac{\alpha^2-4\beta}{\alpha}\\ \arctan\frac{\sqrt{4\beta-(x-\alpha)^2}\sqrt{\alpha^2-4\beta}}{\alpha^2-4\beta-\alpha x}, & x\in(\frac{\alpha^2-4\beta}{\alpha},x_2]\\ 0, & x>x_2\end{cases}. \tag{15}$$

Now from Corollary 1.2 we obtain $\lambda(t)=-\frac{1}{\pi}(G(t)-G(0))$ where the function $G(x)$ is given by

$$\begin{aligned}G(x)&=\lim_{y\to0}\Im F_1(x+iy)\\ &=\frac{1}{2\beta}\left[-G_\rho(x)-\left(\alpha-\sqrt{\alpha^2-4\beta}\right)G_1(x)+\alpha G_2(x)-\sqrt{\alpha^2-4\beta}G_3(x)\right].\end{aligned}$$

Expression (11) is now obtained by differentiation of $\lambda(t)$ in the sense of distributions. Note that the functions $G_\rho(x)$, $G_2(x)$ and $G_3(x)$ are differentiable for all $x\in(x_1,x_2)$ and therefore they form an absolutely continuous part of the measure $d\lambda(x)$ (given by $w_{ac}(x)$) while $G_1(x)$ forms the delta function term.

This completes the proof of the theorem. □

Let $\bar{u}_n$ be the n-th moment of the weight function $w_{ac}(x)$, i.e. $\bar{u}_n = \int_{\mathbb{R}} x^n w_{ac}(x)dx$, for $n \in \mathbb{N}_0$. From Theorem 3.1 we obtain

$$\begin{aligned} u_n &= \int_{\mathbb{R}} x^n w(x)dx = \int_{\mathbb{R}} x^n w_{ac}(x)dx + \frac{\alpha - \sqrt{\alpha^2 - 4\beta}}{2\beta} \int_{\mathbb{R}} x^n \delta(x)dx \\ &= \bar{u}_n + \frac{\alpha - \sqrt{\alpha^2 - 4\beta}}{2\beta}\delta_{n0}. \end{aligned}$$

where δ_{n0} is Kronecker delta. In other words, the sequences $(u_n)_{n\in\mathbb{N}_0}$ and $(\bar{u}_n)_{n\in\mathbb{N}_0}$ differ only in the elements with index $n = 0$. Hence, it holds that

$$\bar{u}_n = \begin{cases} \dfrac{-\alpha + \sqrt{\alpha^2 - 4\beta}}{2\beta}, & n = 0 \\ u_n \ , & n \geq 1 \end{cases} . \tag{16}$$

As we will see in section 5, the Hankel transform of the sequence $(u_n)_{n\in\mathbb{N}_0}$ will be evaluated using the Hankel transforms of the sequences $(\bar{u}_n)_{n\in\mathbb{N}_0}$ and $(u_n^*)_{n\in\mathbb{N}_0}$.

Moment representations of the shifted sequences $(u_n^*)_{n\in\mathbb{N}_0}$ and $(u_n^{**})_{n\in\mathbb{N}_0}$ holds directly from Theorem 3.1.

Corollary 3.2. *The weight function of the sequence* $(u_n^*)_{n\in\mathbb{N}_0} = (u_{n+1})_{n\in\mathbb{N}_0}$ *is*

$$w^*(x) = \begin{cases} \dfrac{\sqrt{4\beta - (x-\alpha)^2}}{2\pi\beta}, & x \in [\alpha - 2\sqrt{\beta}, \alpha + 2\sqrt{\beta}] \\ 0, & \text{otherwise.} \end{cases} \tag{17}$$

Corollary 3.3. *The weight function of the sequence* $(u_n^{**})_{n\in\mathbb{N}_0} = (u_{n+2})_{n\in\mathbb{N}_0}$ *is*

$$w^{**}(x) = \begin{cases} \dfrac{x\sqrt{4\beta - (x-\alpha)^2}}{2\pi\beta}, & x \in [\alpha - 2\sqrt{\beta}, \alpha + 2\sqrt{\beta}] \\ 0, & \text{otherwise.} \end{cases} \tag{18}$$

4 Hankel transforms of the sequences u_n^* and u_n^{**}

In order to compute the Hankel transforms h_n^* and h_n^{**} using the Heilermann formula (3), we need the coefficients α_n and β_n of the three-term recurrence relation. These coefficients will be obtained by applying weight function transformations. The following lemmas provide relations between the coefficients α_n and β_n of the original and transformed weight function.

Lemma 4.1. *Let $w(x)$ and $\tilde{w}(x)$ be weight functions and denote by $(\pi_n(x))_{n\in\mathbb{N}_0}$ and $(\tilde{\pi}_n(x))_{n\in\mathbb{N}_0}$ the corresponding orthogonal polynomials. Also denote by $(\alpha_n)_{n\in\mathbb{N}_0}, (\beta_n)_{n\in\mathbb{N}_0}$ and $(\tilde{\alpha}_n)_{n\in\mathbb{N}_0}, \left(\tilde{\beta}_n\right)_{n\in\mathbb{N}_0}$ the three-term relation coefficients corresponding to $w(x)$ and $\tilde{w}(x)$ respectively. The following transformation formulas are valid:*

(1) *If $\tilde{w}(x) = Cw(x)$ where $C > 0$ then we have $\tilde{\alpha}_n = \alpha_n$ for $n \in \mathbb{N}_0$ and $\tilde{\beta}_0 = C\beta_0$, $\tilde{\beta}_n = \beta_n$ for $n \in \mathbb{N}$. Additionally holds $\tilde{\pi}_n(x) = \pi_n(x)$ for all $n \in \mathbb{N}_0$.*

(2) *If $\tilde{w}(x) = w(ax + b)$ where $a, b \in \mathbb{R}$ and $a \neq 0$ there holds $\tilde{\alpha}_n = \frac{\alpha_n - b}{a}$ for $n \in \mathbb{N}_0$ and $\tilde{\beta}_0 = \frac{\beta_0}{|a|}$ and $\tilde{\beta}_n = \frac{\beta_n}{a^2}$ for $n \in \mathbb{N}$. Additionally holds $\tilde{\pi}_n(x) = \frac{1}{a^n}\pi_n(ax + b)$.*

Proof. In both cases, we directly check the orthogonality of $\bar{\pi}_n(x)$ and obtain the coefficients $\bar{\alpha}_n$ and $\bar{\beta}_n$ by putting $\bar{\pi}_n(x)$ in the three-term recurrence relation for $\pi_n(x)$. □

Lemma 4.2. (Linear multiplier transformation) [10] *Consider the same notation as in Lemma 4.1. Let the sequence $(r_n)_{n\in\mathbb{N}_0}$ be defined by*

$$r_0 = c - \alpha_0, \qquad r_n = c - \alpha_n - \frac{\beta_n}{r_{n-1}} \qquad (n \in \mathbb{N}_0). \tag{19}$$

If $\tilde{w}(x) = (x-c)w(x)$ where $c < \inf \operatorname{supp}(w)$, there holds

$$\begin{aligned} \tilde{\beta}_0 &= \int_{\mathbb{R}} \tilde{w}(x)\, dx, \quad \tilde{\beta}_n = \beta_n \frac{r_n}{r_{n-1}}, \qquad (n \in \mathbb{N}), \\ \tilde{\alpha}_n &= \alpha_{n+1} + r_{n+1} - r_n, \quad (n \in \mathbb{N}_0). \end{aligned} \tag{20}$$

Now we can apply the transformation method to the weight functions $w^*(x)$ and $w^{**}(x)$.

Theorem 4.3. *The Hankel transform of the sequence $(u_n^*)_{n\in\mathbb{N}_0}$ is given by*

$$h_n^* = \beta^{\binom{n+1}{2}}. \tag{21}$$

Proof. We use the Heilermann formula (3) and the weight function transformation given by Lemma 4.1. Recall that

$$w^*(x) = \begin{cases} \dfrac{\sqrt{4\beta - (x-\alpha)^2}}{2\pi\beta}, & x \in [\alpha - 2\sqrt{\beta}, \alpha + 2\sqrt{\beta}] \\ 0, & \text{otherwise.} \end{cases}.$$

Hence, we start from the monic Chebyshev polynomials of the second kind

$$Q_n^{(0)}(x) = S_n(x) = \frac{\sin\big((n+1)\arccos x\big)}{2^n \cdot \sqrt{1-x^2}}$$

which are orthogonal with respect to the weight function

$$w^{(0)}(x) = \begin{cases} \sqrt{1-x^2}, & x \in [-1,1] \\ 0, & \text{otherwise} \end{cases}.$$

The corresponding coefficients α_n and β_n of the three-term recurrence relation are

$$\beta_0^{(0)} = \frac{\pi}{2}, \quad \beta_n^{(0)} = \frac{1}{4} \quad (n \geq 1), \qquad \alpha_n^{(0)} = 0 \quad (n \geq 0)\ .$$

Now, we introduce a new weight function $w^{(1)}(x)$ by

$$w^{(1)}(x) = w^{(0)}\left(\frac{x-\alpha}{2\sqrt{\beta}}\right) = \begin{cases} \dfrac{\sqrt{4\beta - (x-\alpha)^2}}{2\sqrt{\beta}}, & x \in [\alpha - 2\sqrt{\beta}, \alpha + 2\sqrt{\beta}] \\ 0, & \text{otherwise} \end{cases}.$$

and use part **(2)** of Lemma 4.1 with $a = 1/(2\sqrt{\beta})$ and $b = -\alpha/(2\sqrt{\beta})$. Hence we obtain

$$\alpha_n^{(1)} = \alpha \quad (n \in \mathbb{N}_0), \qquad \beta_0^{(1)} = \pi\sqrt{\beta}, \qquad \beta_n^{(1)} = \beta \quad (n \in \mathbb{N}). \tag{22}$$

Observe that $w^*(x) = w^{(1)}(x)/(\pi\sqrt{\beta})$. From part **(1)** of Lemma 4.1 we see that $\alpha_n^* = \alpha_n^{(1)} = \alpha$ for every $n \in \mathbb{N}_0$ and $\beta_n^* = \beta_n^{(1)} = \beta$, while $\beta_0^* = \beta_0^{(1)}/(\pi\sqrt{\beta}) = 1$. Hence

$$\alpha_n^* = \alpha \quad (n \in \mathbb{N}_0), \qquad \beta_0^* = 1, \quad \beta_n^* = \beta \quad (n \in \mathbb{N}). \tag{23}$$

The statement of the theorem now holds directly from (23) and the Heilermann formula (equation (3)):

$$h_n^* = (u_0^*)^{n+1}(\beta_1^*)^n(\beta_2^*)^{n-1}\cdots\beta_n^* = \beta^{\binom{n+1}{2}}. \tag{24}$$

□

Now observe that the expression for h_n^* does not depend on α, which means that all sequences obtained for a fixed value of β have the same Hankel transform. Moreover, for $\beta = 1$ we have $h_n^* = 1$, which is the Hankel transform of both the sequence of aerated Catalan numbers and the sequence of Motzkin numbers.

Theorem 4.4. *The Hankel transform of the sequence* $(u_n^{**})_{n\in\mathbb{N}_0}$ *is given by*

$$h_n^{**} = \frac{\beta^{\binom{n+1}{2}}}{2^{n+1}\sqrt{\alpha^2-4\beta}}\left[(\alpha+\sqrt{\alpha^2-4\beta})^{n+2} - (\alpha-\sqrt{\alpha^2-4\beta})^{n+2}\right]. \tag{25}$$

Proof. Recall that $w^{**}(x) = xw^*(x)$. Hence, we need to apply one linear multiplier transformation to the weight function $w^*(x)$. According to Lemma 5.1, we introduce the sequence $(r_n)_{n\in\mathbb{N}_0}$ by

$$r_0 = -\alpha_0^* = -\alpha, \qquad r_n = -\alpha_n^* - \frac{\beta_n^*}{r_{n-1}} \quad (n \in \mathbb{N})$$

and then obtain the coefficients β_n^{**} by

$$\beta_0^{**} = \int_{-\infty}^{+\infty} w^{**}(x)\, dx = \alpha,$$
$$\beta_n^{**} = \beta_n^* \cdot \frac{r_n}{r_{n-1}} \quad (n \geq 1).$$

Recall that coefficients α_n^* and β_n^* are given by (23). Since we are not able to guess a nice solution of this recursive equation with initial value $\beta_0^{**} = \alpha$, we have to use another approach. According to the Heilermann formula (3) there holds

$$\begin{aligned}\frac{h_{n+1}^{**}}{h_n^{**}} &= \frac{(\beta_0^{**})^{n+1}(\beta_1^{**})^n\cdots(\beta_{n+1}^{**})}{(\beta_0^{**})^n(\beta_1^{**})^{n-1}\cdots(\beta_n^{**})} = \beta_0^{**}\beta_1^{**}\beta_2^{**}\cdots\beta_{n+1}^{**}\\ &= \alpha\cdot\beta_1^*\frac{r_1}{r_0}\cdot\beta_2^*\frac{r_2}{r_1}\cdots\beta_{n+1}^*\frac{r_{n+1}}{r_n} = \alpha\cdot\beta^{n+1}\frac{r_{n+1}}{r_0}\\ &= -\beta^{n+1}r_{n+1},\end{aligned}$$

which implies

$$r_n = -\frac{h_n^{**}}{\beta^n h_{n-1}^{**}}, \qquad (n \geq 1).$$

By replacing the previous expression in

$$r_n = -\alpha - \frac{\beta}{r_{n-1}},$$

we obtain the following difference equation

$$h_n^{**} = \alpha\beta^n h_{n-1}^{**} - \beta^{2n} h_{n-2}^{**}, \quad (n \geq 2). \tag{26}$$

where the initial values are given by $h_0^{**} = \alpha$ and $h_1^{**} = \alpha^2\beta - \beta^2$. In order to solve the equation (26), we introduce a new sequence $(y_n)_{n\in\mathbb{N}_0}$ defined by

$$y_n = h_n^{**}\beta^{-\frac{n^2}{2}}.$$

Substituting into the previous equation yields

$$y_n = \alpha\sqrt{\beta} y_{n-1} - \beta^2 y_{n-2}. \tag{27}$$

By solving the linear difference equation (27) with the initial values $y_0 = \alpha$ and $y_1 = \alpha^2\sqrt{\beta} - \beta\sqrt{\beta}$, we obtain

$$y_n = \frac{\beta^{\frac{n}{2}}}{2^{n+1}\sqrt{\alpha^2 - 4\beta}}\left[(\alpha + \sqrt{\alpha^2 - 4\beta})^{n+2} - (\alpha - \sqrt{\alpha^2 - 4\beta})^{n+2}\right].$$

Finally, by replacing $h_n^{**} = y_n\beta^{\frac{n^2}{2}}$ we finish the proof of the theorem. □

5 The Hankel transforms of the sequences $\bar{u}_n$ and u_n

Recall that the sequence

$$\bar{u}_n = \begin{cases} \dfrac{-\alpha + \sqrt{\alpha^2 - 4\beta}}{2\beta}, & n = 0 \\ u_n\ , & n \geq 1 \end{cases}$$

is the moment sequence of the absolutely continuous part $w_{ac}(x)$ of the weight $w(x)$, which is given by

$$w_{ac}(x) = \begin{cases} \dfrac{\sqrt{4\beta - (x-\alpha)^2}}{2\pi\beta x}, & x \in [\alpha - 2\sqrt{\beta}, \alpha + 2\sqrt{\beta}] \\ 0, & \text{otherwise} \end{cases}.$$

Observe that $w_{ac}(x) = w^*(x)/x$. We can derive the coefficients $\bar{\alpha}_n$ and $\bar{\beta}_n$, corresponding to the weight $w_{ac}(x)$, by applying a linear divisor transformation given by the following lemma.

Lemma 5.1. (Linear divisor transformation) [10] *Consider the same notation as in the Lemma 4.1. Let the sequence $(r_n)_{n\in\mathbb{N}_0}$ be defined by*

$$r_{-1} = -\int_{\mathbb{R}} \tilde{w}(x)\, dx, \qquad r_n = c - \alpha_n - \frac{\beta_n}{r_{n-1}} \qquad (n \in \mathbb{N}_0). \tag{28}$$

If $\tilde{w}(x) = \frac{w(x)}{x-c}$, where $c < \inf \operatorname{supp}(w)$ there holds

$$\begin{aligned} \tilde{\alpha}_0 &= \alpha_0 + r_0, \quad \tilde{\alpha}_n = \alpha_n + r_n - r_{n-1}, \\ \tilde{\beta}_0 &= -r_{-1}, \quad \tilde{\beta}_n = \beta_{n-1}\frac{r_{n-1}}{r_{n-2}} \quad (n \in \mathbb{N}). \end{aligned} \tag{29}$$

Theorem 5.2. *The Hankel transform of the sequence* $(\bar{u}_n)_{n\in\mathbb{N}_0}$ *is given by*

$$\bar{h}_n = \beta^{\binom{n+1}{2}} \left(\frac{\alpha - \sqrt{\alpha^2 - 4\beta}}{2\beta} \right)^{n+1}. \tag{30}$$

Proof. According to Lemma 5.1 we need to introduce the sequence

$$r_n = -\alpha_n^* - \frac{\beta_n^*}{r_{n-1}} \quad (n \in \mathbb{N}_0), \qquad r_{-1} = -\int_{\mathbb{R}} w_{ac}(x)dx = -\frac{\alpha - \sqrt{\alpha^2 - 4\beta}}{2\beta} \tag{31}$$

and compute coefficients $\bar{\beta}_n$ by

$$\bar{\beta}_0 = -r_{-1} = \frac{\alpha - \sqrt{\alpha^2 - 4\beta}}{2\beta}, \quad \bar{\beta}_n = \beta_{n-1}\frac{r_{n-1}}{r_{n-2}} \quad (n \in \mathbb{N}).$$

Recall that

$$\alpha_n^* = \alpha \quad (n \in \mathbb{N}_0), \qquad \beta_0^* = 1, \quad \beta_n^* = \beta \quad (n \in \mathbb{N}).$$

We proceed similarly as in the case of the sequence u_n^{**} (Theorem 4.4). According to the Heilermann formula (equation (3)), it holds that:

$$\begin{aligned}\frac{\bar{h}_{n+1}}{\bar{h}_n} &= \bar{\beta}_0\bar{\beta}_1\bar{\beta}_2\cdots\bar{\beta}_{n+1}\\ &=(-r_{-1})\cdot\beta_0^*\frac{r_0}{r_{-1}}\cdot\beta_1^*\frac{r_1}{r_0}\cdots\beta_n^*\frac{r_n}{r_{n-1}}\\ &=-\beta^n r_n,\end{aligned}$$

which implies

$$r_n = -\frac{\bar{h}_{n+1}}{\beta^n\bar{h}_n}.$$

Recurrence relation (31) now becomes

$$\bar{h}_{n+1} = \alpha\beta^n\bar{h}_n - \beta^{2n}\bar{h}_{n-1}. \tag{32}$$

We introduce a new sequence $(v_n)_{n\in\mathbb{N}_0}$ defined by $v_n = \bar{h}_n\beta^{-\frac{n^2}{2}}$. Substituting into the previous equation yields:

$$v_{n+1} - \frac{\alpha}{\sqrt{\beta}}v_n + v_{n-1} = 0, \qquad (n \geq 1). \tag{33}$$

The first two values of the sequence $(v_n)_{n\in\mathbb{N}_0}$ are given by

$$v_0 = \frac{\alpha - \sqrt{\alpha^2 - 4\beta}}{2\beta}, \quad v_1 = \frac{\alpha^2 - 2\beta - \alpha\sqrt{\alpha^2 - 4\beta}}{2\beta\sqrt{\beta}}.$$

By solving the linear difference equation (33) we obtain

$$v_n = \frac{1}{2^{n+1}\beta^{\frac{n+2}{2}}}\left(\alpha - \sqrt{\alpha^2 - 4\beta}\right)^{n+1}$$

and hence

$$\bar{h}_n = \beta^{\binom{n+1}{2}} \left(\frac{\alpha - \sqrt{\alpha^2 - 4\beta}}{2\beta} \right)^{n+1}. \tag{34}$$

This completes the proof of the theorem. □

The following lemma shows the connection between Hankel transforms of sequences which differ only in the term with index 0.

Lemma 5.3. *Let $(u_n)_{n\in\mathbb{N}_0}$ and $(\bar{u}_n)_{n\in\mathbb{N}_0}$ be sequences which differ only in the term with index 0, i.e. $u_n = \bar{u}_n$ for all $n \geq 1$. The Hankel transforms $(h_n)_{n\in\mathbb{N}_0}$ and $\left(\bar{h}_n\right)_{n\in\mathbb{N}_0}$ of these sequences are related by*

$$h_n = \bar{h}_n + (u_0 - \bar{u}_0)h^{**}_{n-1} \qquad (n \in \mathbb{N}_0),$$

where $(h^{**}_n)_{n\in\mathbb{N}_0}$ is the Hankel transform of the sequence $(u^{**}_n)_{n\in\mathbb{N}_0}$, given by $u^{**}_n = u_{n+2}$ for all $n \geq 0$ and $h^{**}_{-1} = 1$.

Proof. Notice at the outset that the determinant $h_n = \det[u_{i+j-2}]_{1\leq i,j\leq n}$ can be written in the form of

$$h_n = \sum_{k=0}^{n-1} u_k M_{1,k+1} \tag{35}$$

where $M_{1,k}$ is the minor corresponding to the matrix element $(1, k)$. Also we can write $\bar{h}_n$ in the same form

$$\bar{h}_n = \sum_{k=0}^{n-1} \bar{u}_k \bar{M}_{1,k+1}. \tag{36}$$

Note that the minors $M_{1,k+1}$ and $\bar{M}_{1,k+1}$ are equal for every $k \geq 0$. Hence, we have

$$h_n - \bar{h}_n = (u_0 - \bar{u}_0)M_{1,1}.$$

Finally, we obtain the statement of the lemma by noticing that

$$M_{1,1} = \det[u_{i+j}]_{1\leq i,j\leq n-1} = \det[u_{i+j+2}]_{0\leq i,j\leq n-2} = \det[u^{**}_{i+j}]_{0\leq i,j\leq n-2} = h^{**}_{n-1}.$$

□

From Lemma 5.3 and expressions (30) and (25) (Theorem 5.2 and Theorem 4.4) we obtain the closed-form expression for $(h_n)_{n\in\mathbb{N}_0}$.

Corollary 5.4. *The Hankel transform of the sequence $(u_n)_{n\in\mathbb{N}_0}$ is given by*

$$h_n = \frac{\beta^{\binom{n}{2}}}{2^n\sqrt{\alpha^2 - 4\beta}} \left[\left(\alpha - \sqrt{\alpha^2 - 4\beta}\right)^n - \left(\alpha + \sqrt{\alpha^2 - 4\beta}\right)^n \right]. \tag{37}$$

By taking $\alpha = z + 1$ and $\beta = z$, we obtain the Hankel transform of $((N_n(z) - \delta_{n0})/z)_{n\in\mathbb{N}_0}$, where $N_n(z)$ is the n-th Narayana polynomial. The case $\beta = \alpha^2/4$ is notweworthy. Expression (37) cannot be used in that case, since the denominator $\sqrt{\alpha^2 - 4\beta}$ is zero. However, due to the polynomiality (and hence continuity) of the expression $h_n = h_n(\alpha, \beta)$ we can find $h_n(\alpha, \alpha^2/4)$ by

$$h_n(\alpha, \alpha^2/4) = \lim_{\beta\to\alpha^2/4} h_n(\alpha, \beta).$$

By setting $t = \sqrt{\alpha^2 - 4\beta}$ we have

$$\begin{aligned} \lim_{\beta\to\alpha^2/4} h_n(\alpha, \beta) &= \lim_{\beta\to\alpha^2/4} \frac{\beta^{\binom{n}{2}}}{2^{n-1}} \cdot \frac{\left(\alpha - \sqrt{\alpha^2 - 4\beta}\right)^n - \left(\alpha + \sqrt{\alpha^2 - 4\beta}\right)^n}{2\sqrt{\alpha^2 - 4\beta}} \\ &= \frac{\alpha^{n(n-1)}}{2^{n^2-1}} \lim_{t\to 0} \frac{(\alpha - t)^n - (\alpha + t)^n}{2t} = -\frac{n\alpha^{n^2-1}}{2^{n^2-1}}. \end{aligned}$$

Therefore, the final expression for $h_n(\alpha, \alpha^2/4)$ is

$$h_n(\alpha, \alpha^2/4) = -n \left(\frac{\alpha}{2}\right)^{n^2-1}.$$

If we put $\alpha = 2$, we obtain the result mentioned in Example 1.1.

Corollary 5.5. *The Hankel transform of the sequence $(C_n - \delta_{n0})_{n\in\mathbb{N}_0}$ is the sequence with general term $h_n = -n$.*

6 Ratio relations and further research

We can easily check (using expressions (10), (21), (25) and (37)) that the following ratio relations

$$\frac{(-1)^{n+1}h_{n+1}}{h_n^*} = q_{n+1}, \qquad \frac{(-1)^{n+1}h_n^{**}}{h_n^*} = q_{n+2} \quad (n \in \mathbb{N}_0) \tag{38}$$

are satisfied. Those relations are general since the coefficients α and β do not appear explicitly. This result suggests the formulation of the following problem which we leave for further research.

Problem 6.1. Characterize the sequences $q = (q_n)_{n\in\mathbb{N}_0}$ (i.e. generating functions $Q(x)$) such that the Hankel transforms $h = \mathcal{H}(u)$, $h^* = \mathcal{H}(u^*)$ and $h^{**} = \mathcal{H}(u^{**})$, where $u = (u_n)_{n\in\mathbb{N}_0}$ is the series reversion of $Q(x)$ while $u^* = (u_{n+1})_{n\in\mathbb{N}_0}$ and $u^{**} = (u_{n+2})_{n\in\mathbb{N}_0}$, satisfy ratio relations (38).

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