FOURIER TRANSFORMATION AND PSEUDODIFFERENTIAL OPERATOR WITH RATIONAL PART

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ABSTRACT

Fourier transformation for a wide class of complex-valued p-adic functions is calculated. Corresponding integral formulas are presented. Eigenproblem and application of the pseudodifferential operator with rational part are discussed.

Keywords: p-adic analysis, p-adic quantum mechanics, pseudodifferential operator

1. Introduction

During last 15 years there has been much interest around p-adic numbers and ultrametric spaces in theoretical and mathematical physics (see [1]-[4]). The most attractive investigations have been on Planck scale physics and quantum cosmology. However, p-adic analysis applies in study of deformation quantization, spin glasses, disordered systems, quasi-crystals and some other dynamical systems.

The advent of the "New Quantum Mechanics" of Heisenberg, Schrodinger and Dirac in the late 1920's started a revolution in the mathematical outlook of quantum physics. Here, accent was on noncommutativity of the quantum world instead of the Planck discreteness in the "Old Quantum Mechanics". Its manifest was the famous H. Weyl book [5]. In particular it presents the Weyl rule for quantization of classical observables – the functions $f(p,q)$ of $2n$ canonical variables $p=(p_1,....,p_n)$, $q=(q_1,...,q_n)$ on the phase space. The corresponding quantum observables are operators

$$f(P,Q)=\frac{1}{(2p)^{2n}}\int_{R^{2n}} d^m x\, d^n x\, \tilde{f}(x,x)\exp(i(xP+xQ)) \tag{1}$$

where P and Q are unbounded canonical operators of partial differentation. The Weyl operators $f(P,Q)$ are now known as pseudodifferential (PD) operators. There are very reach literatures concerning PD operators (e.g. see [6]).

The formalism of p-adic quantum mechanics proposed by Vladimirov and Volovich [7] is based on the direct construction of the unitary representation of an evolution operator *U(t)* and its kernel K_p on the Hilbert space $L_2(Q_p)$

$$U_p(t'')y_p(x'') = \int_{Q_p} K_p(x'',t'';x',t')y_p(x',t')dx'. \tag{2}$$

? lso, the Weyl quantization is understood.

The standard Hamiltonian approach fails in this situation, since Q_p is totally disconnected. It excludes possibility to define infinitesimal generators of space and time translation. Nevertheless, there is a natural desire and a need to find some (pseudo)differential equation for wave function.

2. Mathematical preliminaries

Usually, a p-adic number is expressed as follows

$$x = p^{g(x)}\sum_{j=0}^{\infty} x_j p^j, \quad g(x) \in Z, \quad x_j \in \{0,1,2,\ldots,p-1\}, \quad x_0 \neq 0. \tag{3}$$

Corresponding p-adic norm of a number presented above is: $|x|_p = p^{-g(x)}$. In calculation related to a new pseudodifferential operator we will use rational part of a p-adic number $\{x\}_p$:

$$\{x\}_p = \begin{cases} 0, \quad g \geq 0 \quad or \quad x = 0 \\ p^{g(x)}\left(x_0 + x_1 p + x_2 p^2 + \ldots + x_{|g|-1} p^{|g|-1}\right), \quad g < 0. \end{cases} \tag{4}$$

It is worth noting that for $x, y \in Q_p$: (1) $\{x+y\}_p = \{x\}_p + \{y\}_p - N$, where $N = 0,1$; (2) $\{-x\}_p = 1 - \{x\}_p$, $|x|_p > 1$. Also, let us state the notation for the ring of p-adic integers, p-adic disc and circle, respectively:

$$\begin{aligned} &Z_p = \{x \in Q_p : |x|_p \leq 1\}, \\ &S_g(a) = \{x \in Q_p : |x-a|_p = p^g\}, \quad g \in Z \\ &B_g(a) = \{x \in Q_p : |x-a|_p \leq p^g\}, \quad g \in Z, \quad \bigcup_{n=-\infty}^{g} S_g(a) = B_g(a) \end{aligned} \tag{5}$$

An additive character $c_p(x) = \exp(2p\,i\{x\}_p)$ on Q_p is a complex-valued continuous function. Recall that in the real case one has $c_\infty(x) = \exp(-2p\,ix)$.

There is well-defined Haar measure dx on Q_p. Fourier transform $\tilde{f}(e)$ of $f(x)$ is defined by

$$\tilde{f}(e) = F[f](x) = \int_{Q_p} f(x)c_p(ex)dx \tag{6}$$

3. Quantization and p-Adic Quantum Mechanics

As we already said, in p-adic case we are forced to use Weyl operators of finite transformations, which, in an analogy with real case, should be symbolically written as

$$\hat{Q}(a) = c_p\left(\frac{a\,\hat{q}}{h}\right) = \exp\left(2p\,i\left\{\frac{a\,\hat{q}}{h}\right\}_p\right), \quad \hat{P}(b) = c_p\left(\frac{b\,\hat{k}}{h}\right) = \exp\left\{b\,\frac{d}{dx}\right\}_p \tag{7}$$

where a and b are p-adic parameters. In this way commutation relation obtains form

$$\hat{Q}(a)\hat{P}(b) = \exp\left(2p\,i\left\{-\frac{a\,b}{h}\right\}_p\right)\hat{P}(b)\hat{Q}(a). \tag{8}$$

It is possible to introduce the family of unitary operators

$$\hat{W}_p(z) = c_p\left(-\frac{1}{2}qk\right)\hat{K}_p(b)\hat{Q}_p(a), \quad z \in Q_p \times Q_p \tag{9}$$

that is a unitary representation of the Heisenberg-Weyl group. Dynamics of a *p*-adic quantum model is described by a unitary operator of evolution *U(t)* formulated in terms of its kernel. This way *p*-adic quantum mechanics is given by a triple $\left(L_2(Q_p), W_p(z_p), U_p(t_p)\right)$ [1,6].

Establishing a differnet approach to *p*-adic quantum mechanics Vladimirov has introduced a pseudodifferential operator [1]

$$D^a y\,(x) = \int_{Q_p} |\,e\,|_p^a\, \tilde{y}(e)\, c_p(-ex)\,de \tag{10}$$

where $|\,e\,|_p^a$ is the symbol of the operator D^a, and $F[D^a y\,] = |\,x\,|_p^a\, \tilde{y}$. It leads to nonstationary Schreodinger-type equation [8] (in fact, diffusion equation) with potential $V(x)$

$$D_t\, y\,(x,t) = \frac{1}{|\,4\,|_p} D_x^2\, y\,(x,t) + V(x)\,y\,(x,t). \tag{11}$$

For $V(x) = 0$, one has free-particle equation with the solution which can be interpreted as a decomposition into plane waves

$$y\,(x,t) = \int_{Q_p} r(k)\, c_p\!\left(\frac{k^2}{4}t - kx\right) dk. \tag{12}$$

While in the real case there is solution of the Cauchy problem such a solution does not exist in *p*-adic case (for more details see [9]), since

$$D_t\, e(x,t) - D_x^2\, e\,(x,t) \neq d\,(x,t). \tag{13}$$

It should be added that a new, very interesting class of *p*-adic PD has been introduced recently [10]. The constructed operators, unlike the Vladimirov operator are not diagonalizable by the Fourier transform.

4. Pseudodifferential Operator with Rationa Part

As a consequence of relation (13), it is fully justified to explore other possibilities for PD in p-adic quantum mechanics [11].

Analysing (8) we find a possibility to introduce an operator which acts on the character as

$$D\, c_p\!\left(\frac{a\,x}{h}\right) = \{b\,\frac{d}{dx}\}_p^a\, c_p\!\left(\frac{a\,x}{h}\right) = 2p\,i\,\{\frac{a\,b}{h}\}_p\, c_p\!\left(\frac{a\,x}{h}\right). \tag{14}$$

In accordance with (14) and to provide the form of Vladimorov operator we propose the action of new operator

$$\{b\,\frac{d}{dx}\}_p^a\, c_p\!\left(\frac{a\,x}{h}\right) = \{b\,\frac{d}{dx}\}_p^a \int_{Q_p} \tilde{y}(k)\, c_p\!\left(-\frac{kx}{h}\right) dk = (2p\,i)^a \int_{Q_p} \tilde{y}(k)\{-\frac{bk}{h}\}_p^a\, c_p\!\left(-\frac{kx}{h}\right) dk. \tag{15}$$

Any application of this operator in the investigation of some particular *p*-adic models requires calculation of many new integrals. We list here a few of these results.

It should be noted that integral (15) converges if $y\,(x)$ is locally constant function. Recall any locally constant function on B_N (and its Fourier transformation) could be written as follows [1]

$$f(x) = \sum_{n=1}^{p^{N-g}} f(a^n)\,\Omega(p^{-g}\,|\,x - a^n\,|_p), \quad x, a^n \in B_N. \tag{16}$$

$$\tilde{f}(k) = \sum_{n=1}^{p^{N-g}} f(a^n)\Omega(p^g \, |k - a^n|_p). \tag{17}$$

According to (15) we have $(b, a, h = 1)$

$$D_x f(x) = \sum_{n=1}^{p^{N-g}} f(a^n) \sum_{h=-\infty}^{-g} \int_{S_h} p^h \{-k\}_p \, c_p(-kx)\, dk. \tag{18}$$

For $g \geq 0 \Rightarrow D_x f(x) = 0$. For $g < 0$ we have

$$D_x f(x) = \sum_{n=1}^{p^{N-g}} f(a^n) \sum_{h=1}^{|g|} p^h \left(\int_{S_h} c_p(kx)dk - \int_{S_h} \{k\}_p \, c_p(-kx)\, dk \right) \tag{19}$$

where the last integral is ($|x|_p = p^M$):

$$\int_{S_h} \{k\}_p \, c_p(-kx)dk = \begin{cases} 0, & for \quad h \leq 0 \; or \; M > 0 \\ p^h \dfrac{p-1}{2p}, & for \quad h > 0, M \leq 0, h \leq |M| \\ \dfrac{1}{c_p(-xp^{-h}) - 1} - \dfrac{p^M - 1}{2}, & for \quad h > 0, M \leq 0, h = |M| + 1 \\ \dfrac{1}{c_p(-xp^{-h}) - 1} - \dfrac{1}{c_p(-xp^{-h+1}) - 1}, & for \quad h > 0, M \leq 0, h \geq |M| + 2. \end{cases} \tag{20}$$

Equation (18) is a good basis for further investigation of the eigenvalue problem of a wide range of function important in p-adic quantum mechanics.

All these results express very complex mathematical nature of PD with rational part. It serves a good starting point for investigation of its spectral properties, related to standard quantum mechanical models as well as to other parts of physics treated by means of p-adic analysis.

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6. References

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