Gravity, Astrophysics, and Strings'05
eds. P. P. Fiziev and M. D. Todorov
St. Kliment Ohridski University Press, Sofia, 2006

Notes on Ultrametric Extra Dimensions and Noncommutative Quantum Cosmology

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Abstract. We consider the (4+D)-dimensional Kaluza-Klein cosmological model with two scaling factors filled with an exotic fluid. One of the scaling factors corresponds to the D-dimensional internal space, and the the second one to the 4-dimensional universe. In standard quantum cosmology, *i.e.*, over the field of real numbers R, it leads to dynamical compactification of additional dimensions and to the accelerating evolution of 4-dimensional universe. We construct a corresponding p-adic quantum model and explore the existence of its p-adic ground state. In addition, we explore the evolution of this model and the possibility for its adelic generalization, which is necessary for further investigation of space-time discreteness at very short distances, *i.e.*, in a very early universe. A special attention is paid to the $(4+1)$- dimensional "empty" model. The corresponding propagators on real (commutative), p-adic and noncommutative (real) spaces are calculated. Forms of these propagators are compared and discussed.

Keywords: Kaluza-Klein quantum cosmology, noncommutativity, p-adic numbers

1. Introduction

Many considerations in quantum gravity (*e.g.*, [1]) suggest that theoretical uncertainty of measuring distances is greater or equal to the Planck distance. It could be concluded that at very short distances the Archimedean axiom is not valid, *i.e.*, the space can posses ultrametric features. Because geometry is always connected with a concrete number field [2], in the case of nonarchimedean geometry it is used to be the field of p-adic numbers Q_p. In high energy physics, these numbers have been used for almost twenty years. The motivation comes from string theory [3]. Generally , the p-adic approach should be useful in describing a very early phase of the universe and processes around the Planck scale.

A significant number of papers motivated by [3] have been published up to now (for a review see [4, 5]). In this short review we are especially interested in applications of p-adic numbers and analysis in quantum cosmology [6]. p-Adic quantum mechanics [7] (QM) has been successfully applied in minisuperspace quantum cosmology [8]. We have treated many cosmological models, mainly constructed in four space-time dimensions [6]. Based on that, one can ask: is it possible to extend this approach to multidimensional quantum cosmological models? The first part of this paper is devoted to the formulation of such a model, with two scaling factors and an exotic fluid. We use these models to consider a possible p-adic structure of extra dimensions. In addition, we explore this possibility to formulate a consistent real, p-adic and adelic (4+D)-dimensional Kaluza-Klein model.

It should be noted that the structure of space-time around the Planck scale is a very interesting problem. One very attractive approach is based on noncommutative (NC) geometry. Besides noncommutative quantum mechanics [9] (mainly used to formulate some interesting toy models) and a lot of papers related to the NC Quantum Field Theory [10] and NC Standard Model [11], there have been a few attempts to formulate NC Quantum Cosmology [12]. In the second part of this paper we formulate and compare the (4+D)-dimensional model, in particular the $(4+1)$-dimensional, Kaluza-Klein "empty" model on real (and p-adic) commutative space with its noncommutative version. First of all, we consider and calculate their corresponding quantum propagators, *i.e.*, examine the evaluation of the same models constructed on different spaces as possible candidates for true geometrical background at a very early phase of the universe.

After a brief mathematical introduction in Section 2, a short review of p-adic and adelic quantum mechanics and cosmology is given in Section 3. Section 4 is devoted to the classical (4+D)-dimensional cosmological models filled with an "exotic" fluid [13]. A corresponding p-adic model is considered in Section 5. In Section 6 we consider a particular (4+1) cosmological model with spacetime metric of Friedman-Robertson-Walker type. We calculate and consider a quantum propagator in the case of real commutative space (Subsection 6.1), p-adic commutative space (Subsection 6.2) and (real) noncommutative space (in principle it can be generalized for p-adic case). Similarities and differences in forms of the propagators are briefly examined. This paper is ended by short conclusion, including adelic generalization and suggestions for advanced research.

2. *p*-Adic Numbers and Adeles

Perhaps the most easier way to understand p-adic numbers is if one starts with the notion of norm. It is well known that any norm must satisfy three conditions: non-negativity, homogeneity, and triangle inequality. The completion of the field of rational numbers Q with respect to the absolute value, or standard norm $|.|_\infty$, gives the field of real numbers $R \equiv Q_\infty$. Besides this norm, there are others that satisfy the first two conditions and the third one in a stronger way

$$\|x+y\| \leq \max(\|x\|, \|y\|), \tag{1}$$

the so called strong triangle inequality. The most important of them is p-adic norm $|.|_p$ [4]. The feature (1), also called ultrametricity, is one of the most important characteristics of the p-adic norm (p denotes a prime number). The number fields obtained by completion of Q with respect to this norm are called p-adic number fields Q_p. It is known that any nonzero rational number x can be expressed as $x = \pm p^{\gamma} a/b$, where γ is a rational number, and a and b are natural numbers which are not divisible with the prime number p and have no common divisor. Then, p-adic norm of x is, by definition, $|x|_p = p^{-\gamma}$.

Because Q_p is a local compact commutative group, the Haar measure can be introduced, which enables integration. For advanced discussion on this and related topics see, for example, [14]. In particular, the Gauss integral will be employed

$$\int\limits_{Q_p} \chi_p(\alpha x^2 + \beta x)\mathrm{d}x = \lambda_p(\alpha)|2\alpha|_p^{-1/2}\chi_p\left(-\frac{\beta^2}{4\alpha}\right), \quad \alpha \neq 0, \tag{2}$$

where $\chi_p = \exp 2\pi i\{x\}_p$ (a real counterpart $\chi_\infty(x) = \exp(-2\pi i x)$) is an additive character ($\{x\}_p$ is the fractional part of x), and $\lambda_p(\alpha)$ is an arithmetic complex-valued function [4]. It should be noted that there is a real counterpart to λ_p

$$\lambda_\infty(\alpha) = \frac{1}{\sqrt{2}}(1 - \mathrm{i}\,\mathrm{sgn}\alpha), \quad \alpha \in Q_\infty. \tag{3}$$

Commonly, the main properties of λ_v, (v denotes ∞ or any p) are:

$$\lambda_v(0) = 1, \quad \lambda_v(a^2\alpha) = \lambda_v(\alpha), \tag{4}$$

$$\lambda_v(\alpha)\lambda_v(\beta) = \lambda_v(\alpha+\beta)\lambda_v(\alpha^{-1}+\beta^{-1}), \quad |\lambda_v(\alpha)|_\infty = 1. \tag{5}$$

A very simple but rather important function in p-adic analysis and p-adic QM is

$$\Omega(|x|_p) = \begin{cases} 1, & |x|_p \leq 1, \\ 0, & |x|_p > 1, \end{cases} \tag{6}$$

which is the characteristic function of Z_p. Note that $Z_p = \{x \in Q_p : |x|_p \leq 1\}$ is the ring of p-adic integers.

Simultaneous treatment of real and p-adic numbers can be realized by via the concept of adeles. An adele $a \in \mathcal{A}$ is an infinite sequence $a = (a_\infty, a_2, \cdots, a_p, \cdots)$, where $a_\infty \in R$ and $a_p \in Q_p$, with the restriction that $a_p \in Z_p$ for all but a finite set S of primes p. The set of all adeles $\mathcal{A}$ can be written in the form

$$\mathcal{A} = \underset{S}{U}\,\mathcal{A}(S), \quad \mathcal{A}(S) = R \times \prod_{p\in S} Q_p \times \prod_{p \notin S} Z_p\,. \tag{7}$$

Also, $\mathcal{A}$ is a topological space. Algebraically, it is a ring with respect to the component-wise addition and multiplication [15].

3. Quantum Cosmology

Quantum cosmology [16, 17] is the application of quantum theory to the universe as a whole. However, since gravity is the dominating interaction on cosmic scales, a quantum theory of gravity is needed as a formal prerequisite for quantum cosmology. Most work in quantum cosmology is based on the Wheeler-DeWitt equation of quantum geometrodynamics. The method is used to restrict the configuration space to a finite number of variables (scale factor, matter field, *etc.*) and then to quantize canonically. Since the full configuration space of three-geometries is called "superspace", the ensuing models are called "minisuperspace models".

In quantum mechanics and quantum field theory, path integrals provide a convenient tool for a wide range of applications. In quantum gravity, a path-integral formulation would have to employ a sum over all four-metrics for a given topology,

$$Z[g] = \int \mathcal{D}g_{\mu\nu}(x)\mathrm{e}^{\mathrm{i}S[g_{\mu\nu}(x)]/\hbar}. \tag{8}$$

In addition, one would expect that a sum over all topologies has to be performed. Since four-manifolds are not classifiable, this is an impossible task. Attention is therefore restricted to a given topology (or to a sum over few topologies). Still, the evaluation of an expression such as this meets great mathematical and conceptual difficulties. While full (superspace) quantum cosmological models are usually unsolvable, the minisuperspace ones can be handled with the available mathematical tools.

For minisuperspace models, we use the metric in the standard 3+1 decomposition

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -N^2\mathrm{d}t^2 + h_{ij}\mathrm{d}x^i\mathrm{d}x^j, \tag{9}$$

where N is the lapse function. For these models the functional integral in (8) is reduced to a functional integral over three-metric and configuration of matter fields, and to another usual integral over the lapse function N. For the boundary condition $q_\alpha(t_2) = q''_\alpha$, $q_\alpha(t_1) = q'_\alpha$ in the gauge $\dot{N} = 0$, we have the minisuperspace propagator

$$\langle q''_\alpha; q'_\alpha\rangle = \int \mathrm{d}N \mathcal{K}(q''_\alpha, N; q'_\alpha, 0), \tag{10}$$

where

$$\mathcal{K}(q''_\alpha, N; q'_\alpha, 0) = \int \mathcal{D}q_\alpha \chi(-S[q_\alpha]), \tag{11}$$

is an ordinary quantum-mechanical propagator between fixed minisuperspace coordinates (q'_α, q''_α) in a fixed "time" N. Quantity S is the action of the minisuperspace model, *i.e.*,

$$S[q_\alpha] = \int_0^1 \mathrm{d}tN\left[\frac{1}{2N^2}f_{\alpha\beta}(q)\dot{q}^\alpha\dot{q}^\beta - U(q)\right], \tag{12}$$

where $f_{\alpha\beta}$ is a minisuperspace metric $(\mathrm{d}s_m^2 = f_{\alpha\beta}\mathrm{d}q^\alpha\mathrm{d}q^\beta)$ with an indefinite signature $(-,+,+,\dots)$. This metric includes spatial (gravitational) components, as well as also matter variables for the given model. It should be noted that the necessary condition for the existence of an adelic quantum model is the existence of the p-adic ground state $\Omega(|q_\alpha|_p)$ defined by [6]

$$\int_{|q_\alpha'|_p\le 1} \mathcal{K}_p(q_\alpha'', N; q_\alpha', 0)\mathrm{d}q_\alpha' = \Omega(|q_\alpha''|_p). \tag{13}$$

4. (4+D)-Dimensional Cosmological Models Over the Field of Real Numbers

The old idea that the four dimensional universe in which we exist is just our observation of physical multidimensional space-time is receiving much attention nowadays. In such models compactification of extra dimensions plays the key role and, in the some of them, leads to the period of accelerated expansion of the universe [13, 18, 19]. This approach is supported and encouraged by recent results of astronomical observations. We briefly recapitulate some facts of the real multidimensional cosmological models, necessary for p-adic and adelic generalization. The metric of such a Kaluza-Klein model with D-dimensional internal space can be presented in the form [13, 20]

$$rmds^2 = -\tilde{N}^2(t)\mathrm{d}t^2 + R^2(t)\frac{\mathrm{d}r^i\mathrm{d}r^i}{(1+\frac{kr^2}{4})^2} + a^2(t)\frac{\mathrm{d}\rho^a\mathrm{d}\rho^a}{(1+k'\rho^2)^2}, \tag{14}$$

where $\tilde{N}(t)$ is a lapse function, $R(t)$ and $a(t)$ are the scaling factors of 4-dimensional universe and internal space, respectively; $r^2 \equiv r^i r^i (i = 1, 2, 3)$, $\rho^2 \equiv \rho^a\rho^a(a = 1, ...D)$, and $k, k' = 0, \pm 1$. The form of the energy-momentum tensor is

$$T_{AB} = \mathrm{diag}(-\rho, p, p, p, p_D, p_D, ..., p_D), \tag{15}$$

where indices A and B run over both spacetime coordinates and the internal space dimensions. If we want the matter to be confined to the four-dimensional universe, we set all $p_D = 0$.

Now, we examine the case for which the pressure along all extra dimensions vanishes $p_D = 0$ (in the braneworld scenarios the matter is confined to the four-dimensional universe), so that all components of T_{AB} are set to zero except the spacetime components [13]. We assume the energy-momentum tensor of spacetime to be an exotic fluid χ with the equation of state

$$p_\chi = \left(\frac{m}{3} - 1\right)\rho_\chi, \tag{16}$$

(p_χ and ρ_χ are pressure and energy density of the fluid, parameter m has value between 0 and 2).

4.1. Classical model

Dimensionally extended Einstein-Hilbert action (without a cosmological term) is

$$S = \int \sqrt{-g}\tilde{R}\mathrm{d}t\mathrm{d}^3R\mathrm{d}^D\rho + S_m = \kappa\int \mathrm{d}tL, \tag{17}$$

where κ is an irrelevant constant and $\tilde{R}$ is the scalar curvature of the metric. So we can read off the Lagrangian of the model (for flat internal space)

$$L = \frac{1}{2\tilde{N}}Ra^D\dot{R}^2 + \frac{D(D-1)}{12\tilde{N}}R^3a^{D-2}\dot{a}^2 + \frac{D}{2\tilde{N}}R^2a^{D-1}\dot{R}\dot{a} - \frac{1}{2}k\tilde{N}Ra^D + \frac{1}{6}\tilde{N}\rho_\chi R^3a^D. \tag{18}$$

For closed universe ($k = 1$), substitution of the equation of state in the continuity equation

$$\dot{\rho}_\chi R + 3(p_\chi + \rho_\chi)\dot{R} = 0 \tag{19}$$

leads to energy density of the form

$$\rho_\chi(R) = \rho_\chi(R_0)\left(\frac{R_0}{R}\right)^m, \tag{20}$$

where R_0 is the value of the scaling factor in arbitrary reference time t_0. If we define the cosmological constant as $\Lambda = \rho_\chi(R)$, the Lagrangian becomes

$$L = \frac{1}{2\tilde{N}} R a^D \dot{R}^2 + \frac{D(D-1)}{12\tilde{N}} R^3 a^{D-2}\dot{a}^2 + \frac{D}{2\tilde{N}} R^2 a^{D-1}\dot{R}\dot{a} - \frac{1}{2}\tilde{N} R a^D + \frac{1}{6}\tilde{N}\Lambda R^3 a^D. \tag{21}$$

Growth of the scaling factor R, according to (20), leads to decrease in the cosmological constant by the relation

$$\Lambda(R) = \Lambda(R_0)\left(\frac{R_0}{R}\right)^m. \tag{22}$$

This decaying Λ term may also explain the smallness of the present value of the cosmological constant since, as the universe evolves from very small to very large in size, the large initial value of Λ decays to small values. If we take $m = 2$, initial condition for cosmological constant and scaling factor $\Lambda(R_0)R_0^2 = 3$, for laps function $\tilde{N}(t) = R^3(t)a^D(t)N(t)$, the Lagrangian (21) becomes

$$L = \frac{1}{2N}\frac{\dot{R}^2}{R^2} + \frac{D(D-1)}{12N}\frac{\dot{a}^2}{a^2} + \frac{D}{2N}\frac{\dot{R}\dot{a}}{Ra}. \tag{23}$$

It should be noted that there are no parameters k and Λ in the Lagrangian. This means that although they are not zero in this model, it is equivalent to a flat universe with zero cosmological term. In other words, there is no difference between a four dimensional universe which looks flat and is not filled with fluid and a closed universe filled with an exotic fluid.

The solutions to corresponding equations of motion are

$$R(t) = C_1 \mathrm{e}^{\alpha t}, \tag{24}$$

$$a(t) = C_2 \mathrm{e}^{\beta t}, \tag{25}$$

where the constants C_1, C_2, α and β depend on initial conditions. A reasonable assumption is that the size of all spatial dimensions is the same at $t = 0$. It may be assumed that this size would be the Planck size, *i.e.*, $R(0) = a(0) = l_P$. The above solutions can be read in terms of the Huble parameter $H = \dot{R}/R$ [13]

$$R(t) = l_P \mathrm{e}^{Ht}, \tag{26}$$

$$a(t) = l_P \mathrm{e}^{-Ht}. \tag{27}$$

Depending on the dimensionality of the internal space, we have

$$R(t) = l_P \mathrm{e}^{Ht}, \tag{28}$$

$$a_\pm(t) = l_P \mathrm{e}^{\frac{2Ht}{D}[-1\pm\sqrt{1-\frac{2}{3}(1-\frac{1}{D})}]^{-1}}, \tag{29}$$

for $D = 1$ and

$$R_\pm(t) = l_P \mathrm{e}^{\frac{D\beta t}{2}[-1\pm\sqrt{1-\frac{2}{3}(1-\frac{1}{D})}]}, \tag{30}$$

$$a(t) = l_P \mathrm{e}^{\beta t}, \tag{31}$$

for $D > 1$. The solution corresponding to $D = 1$ predicts an accelerating (de Sitter) universe and a contracting internal space with exactly the same rates. If $D > 1$, analysis is complicated but the results are similar.

4.2. Quantum Model

Quantum solutions are obtained from the Wheeler-DeWitt equation

$$H\Psi(R, a) = 0, \tag{32}$$

where H is the Hamiltonian and Ψ is the wave function of the universe. For this model, the above equation is read in new variables ($X = \ln R$ and $Y = \ln a$)

$$\left[(D-1)\frac{\partial^2}{\partial X^2} + \frac{6}{D}\frac{\partial^2}{\partial Y^2} - 6\frac{\partial}{\partial X}\frac{\partial}{\partial Y}\right]\Psi(X,Y) = 0. \tag{33}$$

By introducing a new change

$$x = X\frac{3}{D+3} + Y\frac{D}{D+3}, \qquad y = \frac{X-Y}{D+3},$$

the Wheeler-DeWitt equation takes a simple form

$$\left(-3\frac{\partial^2}{\partial x^2} + \frac{D+2}{D}\frac{\partial^2}{\partial y^2}\right)\Psi(x,y) = 0. \tag{34}$$

Eq.(34) has four possible solutions [13]

$$\Psi_D^\pm(x,y) = A^\pm \mathrm{e}^{\pm\sqrt{\frac{\gamma}{3}}x \pm \sqrt{\frac{\gamma D}{D+2}}y}, \tag{35}$$

$$\Psi_D^\pm(x,y) = B^\pm \mathrm{e}^{\pm\sqrt{\frac{\gamma}{3}}x \mp \sqrt{\frac{\gamma D}{D+2}}y}, \tag{36}$$

where $A^\pm$ and $B^\pm$ are normalization constants. It is possible to impose boundary conditions to get $\Psi_D(R, a) = 0$. For further details see [13].

5. (4+D)-Dimensional Model Over the Field of p-Adic Numbers

Our consideration of the (4+D)-dimensional Kaluza-Klein model over the field Q_p will start from the Lagrangian of the form (23). All quantities in this Lagrangian will be treated as p-adic ones. Taking again the replacement $X = \ln R$ and $Y = \ln a$, it becomes

$$L = \frac{1}{2N}\dot{X}^2 + \frac{D(D-1)}{12N}\dot{Y}^2 + \frac{D}{2N}\dot{X}\dot{Y}. \tag{37}$$

The corresponding p-adic equations of motion are

$$\ddot{X} + \frac{D}{2}\ddot{Y} = 0, \quad \ddot{X} + \frac{D-1}{3}\ddot{Y} = 0. \tag{38}$$

It is not difficult to see that the above system can be rewritten in the following form

$$\ddot{X} = 0, \quad \ddot{Y} = 0. \tag{39}$$

If we omit from consideration pseudoconstant solutions and concentrate on the analytical ones, we get $X(t) = C_1 t + C_2$, $Y(t) = C_3 t + C_4$. The calculation of the p-adic classical action gives

$$\begin{aligned} &\bar{S}_p(X'', Y'', N; X', Y', 0) \\ &\quad = \frac{1}{2N}(X''-X')^2 + \frac{D(D-1)}{12N}(Y''-Y')^2 + \frac{D}{2N}(X''-X')(Y''-Y'). \end{aligned} \tag{40}$$

Because this action is quadratic with respect to both variables X and Y, we can write down the kernel of p-adic operator of evolution [21, 22]

$$\begin{aligned} &\mathcal{K}_p(X'', Y'', N; X', Y', 0) \\ &\quad = \lambda_p\left[\frac{D(D+2)}{48N^2}\right]\left|\frac{D(D+2)}{12N^2}\right|_p \chi_p[-\bar{S}_p(X'', Y'', N; X', Y', 0)]. \end{aligned} \tag{41}$$

Let us use again the change

$$x = X\frac{3}{D+3} + Y\frac{D}{D+3}, \qquad y = \frac{X-Y}{D+3}$$

to separate variables and make further analysis of this model rather simple. In these variables the classical action and the kernel of evolution operator read

$$\begin{aligned} &\bar{S}_p(x'', y'', N; x', y', 0) \\ &\quad = \frac{1}{2N}\left[1 + \frac{D(D+5)}{6}\right](x''-x')^2 - \frac{1}{2N}D(D+3)(y''-y')^2, \end{aligned} \tag{42}$$

$$\begin{aligned} \mathcal{K}_p(x'', y'', N; x', y', 0) &= \lambda_p\left[\frac{6 + D(D+5)}{6}\right]\lambda_p\left[-\frac{D(D+3)}{12N}\right] \\ &\times \left|\frac{D(D+3)}{2N^2}\left[1 + \frac{D(D+5)}{6}\right]\right|_p^{1/2} \chi_p[-\bar{S}_p(x'', y'', N; x', y', 0)]. \end{aligned} \tag{43}$$

Now, we can examine the case in which the p-adic wave function has the form corresponding to the simplest ground state [23] (here in two dimensions (6))

$$\Psi_p(x,y) = \Omega(|x|_p)\Omega(|y|_p). \tag{44}$$

Putting the kernel of the operator of evolution (43) in Eq.(13) we get that the required state exists if both conditions

$$|N|_p \leq \left|1 + \frac{D(D+5)}{6}\right|_p, \quad |N|_p \leq |D(D+3)|_p,\ p \neq 2, \tag{45}$$

are fulfilled. The answer to the question of whether these conditions can be useful in determination of dimensionality of the internal space needs further careful analysis.

Going back to the “old variables”, the p-adic ground state wave function for our model is

$$\Psi_p(x,y) = \Omega\left[\left|\left(1 - \frac{D}{D+3}\right)X + \frac{D}{D+3}Y\right|_p\right]\Omega\left(\left|\frac{X-Y}{D-3}\right|_p\right). \tag{46}$$

We can also write down the solutions in the variables R and a [24].

6. Kaluza-Klein (4+1) - Dimensional “Empty” Model

We start with the metric considered in [20, 25, 26] in which spacetime is of the Friedman-Robertson-Walker type, having a compactified space which is assumed to be the circle S^1. We adopt the chart $\{t, r^i, \rho\}$ with t, r^i and ρ denoting the time, the space coordinates and the compactified space coordinate, respectively

$$\mathrm{d}s^2 = -N^2\mathrm{d}t^2 + R^2(t)\frac{\mathrm{d}r^i\mathrm{d}r^i}{(1+\frac{\kappa r^2}{4})^2} + a^2\mathrm{d}\rho^2, \tag{47}$$

where $\kappa = 0, \pm1$, N is the lapse function, and $R(t)$, $a(t)$ are the scale factors of the universe and compact dimension, respectively. The integrations of the Einstein-Hilbert action for such an empty (4+1)-dimensional Kaluza-Klein universe with the cosmological constant Λ

$$S = \int \sqrt{-g}(\tilde{R} - \Lambda)\mathrm{d}t\mathrm{d}^3r\mathrm{d}\rho \tag{48}$$

($\tilde{R}$ is a curvature scalar corresponding to metric (47)) over spatial dimensions gives an effective Lagrangian in the minisuperspace (R, a)

$$L = \frac{1}{2N}Ra\dot{R}^2 + \frac{1}{2N}R^2\dot{R}\dot{a} - \frac{1}{2}N\kappa Ra + \frac{1}{6}N\Lambda R^3 a. \tag{49}$$

6.1. Commutative model over real space

By defining $\omega^2 = -\frac{2\Lambda}{3}$ $(\Lambda < 0)$ and changing variables as

$$u = \frac{1}{\sqrt{8}}\left(R^2 + Ra - \frac{3\kappa}{\Lambda}\right), \quad v = \frac{1}{\sqrt{8}}\left(R^2 - Ra - \frac{3\kappa}{\Lambda}\right) \tag{50}$$

in the "new" minisuperspace (u, v), the Lagrangian takes on the form

$$L = \frac{1}{2N}(\dot{u}^2 - N^2\omega^2 u^2) - \frac{1}{2N}(\dot{v}^2 - N^2\omega^2 v^2), \tag{51}$$

which describes an isotropic oscillator-ghost-oscillator system. Corresponding equations of motion

$$\ddot{u} + N^2\omega^2 u = 0, \quad \ddot{v} + N^2\omega^2 v = 0 \tag{52}$$

have the following solutions

$$u(t) = A\cos N\omega t + B\sin N\omega t, \quad v(t) = C\cos N\omega t + D\sin N\omega t \tag{53}$$

The corresponding classical action S and quantum propagator $\mathcal{K}$ (up to sign) have forms

$$\begin{aligned} &\bar{S}(u'', v'', N; u', v', 0) \\ &\quad = \frac{1}{2}\omega\left[(u''^2 + u'^2 - v''^2 - v'^2)\cot N\omega + (v'v'' - u'u'')\frac{2}{\sin N\omega}\right], \end{aligned} \tag{54}$$

$$\mathcal{K}(u'', v'', N; u', v', 0) = \frac{\omega}{\sin N\omega}\ \mathrm{e}^{2\pi\mathrm{i}\bar{S}(u'', v'', N; u', v', 0)}. \tag{55}$$

To obtain the energy eigenstates and eigenvectors, we need to recast the propagator (55) in a form that permits a direct comparison with a spectral representation for the Feynman propagator given by

$$\begin{aligned} &\mathcal{K}(u'', v'', N; u', v', 0) \\ &\quad = \Theta(N)\sum_l \Phi_l^{(m_1,m_2)}(u'', v'')\Phi_l^{*(m_1,m_2)}(u', v')\mathrm{e}^{-2\psi\mathrm{i}NE_{n,m}}. \end{aligned} \tag{56}$$

A corresponding Wheeler-Dewitt equation for this model in the minisuperspace (u, v) is $(N = 1)$

$$\left(\frac{\partial^2}{\partial u^2} - \frac{\partial^2}{\partial v^2} - \omega^2 u^2 + \omega^2 v^2\right)\Psi(u, v) = 0. \tag{57}$$

It has oscillator-ghost-oscillator solutions belonging to the Hilbert space $\mathcal{H}^{(m_1,m_2)}(\mathcal{L}^2)$ as

$$\Psi^{(m_1,m_2)}(u, v) = \sum_{l=0}^{\infty} c_l \Phi_l^{(m_1,m_2)}(u, v), \tag{58}$$

with $m_1, m_2 \geq 0$ and $c_l \in C$. The solutions $\Phi_l^{(m_1,m_2)}(u, v)$ are separable as

$$\Phi_l^{(m_1,m_2)}(u, v) = \alpha_{m_2+(2m_2+1)l}(u)\beta_{m_1+(2m_1+1)l}(v) \tag{59}$$

with normalized solutions

$$\alpha_n(u) = \left(\frac{\omega}{\pi}\right)^{1/4}\frac{\mathrm{e}^{-\omega u^2/2}}{\sqrt{2^n n!}}H_n(\sqrt{\omega}u), \tag{60}$$

$$\beta_n(v) = \left(\frac{\omega}{\pi}\right)^{1/4} \frac{e^{-\omega v^2/2}}{\sqrt{2^n n!}} H_n(\sqrt{\omega} v), \tag{61}$$

where $H_n(x)$ are Hermit polynomials.

6.2. Commutative model over *p*-adic space

The main relations connected with the (4+1)-dimensional model in the p-adic case are formally the same as in the real case. Let us rewrite the p-adically valued metric and action for this model

$$ds^2 = -N^2 dt^2 + R^2(t) \frac{dr^i dr^i}{(1 + \frac{\kappa r^2}{4})^2} + a^2 d\rho^2, \tag{62}$$

$$S = \int \sqrt{-g}(\tilde{R} - \Lambda) dt d^3 r d\rho. \tag{63}$$

The effective p-adic Lagrangian in the minisuperspace (R, a) is

$$L = \frac{1}{2N} Ra\dot{R}^2 + \frac{1}{2N} R^2 \dot{R}\dot{a} - \frac{1}{2} N\kappa Ra + \frac{1}{6} N\Lambda R^3 a. \tag{64}$$

By defining $\omega^2 = -\frac{2\Lambda}{3}$ $(\Lambda < 0)$ and changing variables as

$$u = \frac{1}{\sqrt{8}}\left(R^2 + Ra - \frac{3\kappa}{\Lambda}\right), \quad v = \frac{1}{\sqrt{8}}\left(R^2 - Ra - \frac{3\kappa}{\Lambda}\right), \tag{65}$$

in the "new" minisuperspace (u, v), the Lagrangian takes the form

$$L = \frac{1}{2N}(\dot{u}^2 - N^2\omega^2 u^2) - \frac{1}{2N}(\dot{v}^2 - N^2\omega^2 v^2). \tag{66}$$

For the corresponding equations of motion

$$\ddot{u} + N^2\omega^2 u = 0, \quad \ddot{v} + N^2\omega^2 v = 0 \tag{67}$$

their solutions are written down as

$$u(t) = A\cos N\omega t + B\sin N\omega t, \quad v(t) = C\cos N\omega t + D\sin N\omega t. \tag{68}$$

The p-adic classical action is

$$\bar{S}_p(u'', v'', N; u', v', 0) = \frac{1}{2}\omega\left[(u''^2 + u'^2 - v''^2 - v'^2)\cot N\omega + (v'v'' - u'u'')\frac{2}{\sin N\omega}\right]. \tag{69}$$

Being quadratic with respect to u and v, this action leads directly [22] to the propagator

$$\mathcal{K}_p(v'', u'', N; v', u', 0) = \frac{1}{|N|_p}\chi_p\left[\frac{\omega(u''^2 + u'^2 - v'^2 - v''^2)}{2\tan N\omega} + \frac{\omega(v'v'' - u'u'')}{\sin N\omega}\right]. \tag{70}$$

In the p-adic region of convergence of analytic functions $\sin x$ and $\tan x$, which is $G_p = \{x \in Q_p : |x|_p \leq |2p|_p\}|$, we find that the simplest vacuum states $\Omega(|u|_p)\Omega(|v|_p)$, $\Omega(p^\nu|u|_p)\Omega(p^\mu|v|_p)$, $\nu, \mu = 1, 2, 3, ...$, exist, as well as

$$\Psi_p(x,y) = \begin{cases} \delta(p^\nu - |x|_p)\delta(p^\mu - |y|_p), |N|_p \leq p^{2\nu-2},\ |N|_p \leq p^{2\mu-2} \\ \delta(2^\nu - |x|_2)\delta(2^\mu - |y|_2), |N|_2 \leq 2^{2\nu-3},\ |N|_2 \leq 2^{2\mu-3}, \end{cases} \tag{71}$$

where $\mu, \nu = 0, -1, -2, \cdots$.

Some $4(= 3 + 1)$-dimensional quantum cosmological models which in the p-adic sector look like two decoupled harmonic oscillators were analyzed in detail in Ref. [6].

6.3. Noncommutative case

The study of various physical theories from the noncommutative point of view has been of particular interest. Besides noncommutativity applied to models defined on real numbers, there have been a few attempts to introduce a noncommutative approach to p-adic [27, 28, 29] and adelic models. In the previous section we assumed that in (u, v) minisuperspace the following algebra holds

$$[u, v] = 0, \ \ [u, p_u] = [v, p_v] = \mathrm{i}\hbar, \ \ [p_u, p_v] = 0, \tag{72}$$

(generalised momenta are $p_u = \dot{u}/N$ and $p_v = \dot{v}/N$). In the noncommutative case we deal with the same Lagrangian, but with a new algebra

$$[u, v] = \mathrm{i}\theta, \ \ [u, p_u] = [v, p_v] = \mathrm{i}\hbar, \ \ [p_u, p_v] = 0. \tag{73}$$

By transformation

$$u = u - \frac{\theta}{2}p_v, \ \ v = v + \frac{\theta}{2}p_u, \tag{74}$$

we can represent this model as a commuting one, but with the Lagrangian

$$L_\theta = \frac{\omega^2}{\omega_\theta^2}\left[\frac{1}{2N}\left(\dot{u}^2 - N^2\omega_\theta^2 u^2\right) - \frac{1}{2N}\left(\dot{v}^2 - N^2\omega_\theta^2 v^2\right) + \frac{1}{2N}\omega_\theta^2\theta(u\dot{v} + \dot{u}v)\right], \tag{75}$$

where

$$\omega_\theta^2 = \frac{\omega^2}{1 + \frac{\omega^2\theta^2}{4}}.$$

The equations of motion and solutions are

$$\ddot{u} + N^2\omega_\theta^2 u = 0, \quad \ddot{v} + N^2\omega_\theta^2 v = 0, \tag{76}$$

$$u(t) = A\cos N\omega_\theta t + B\sin N\omega_\theta t, \quad v(t) = C\cos N\omega_\theta t + D\sin N\omega_\theta t. \tag{77}$$

After some calculation, from the classical action

$$\begin{aligned} \bar{S}_\theta(u'', v'', N; u', v', 0) = {} & \frac{1}{2}\omega\sqrt{1 + \frac{\omega^2\theta^2}{4}} \\ & \times \Bigg[u''^2 + u'^2 - v''^2 - v'^2)\cot N\omega_\theta \\ & -(u'u'' - v'v'')\frac{2}{\sin N\omega_\theta} + \frac{\theta\omega_\theta}{N}(u''v'' - u'v')\Bigg], \end{aligned} \tag{78}$$

we get

$$\begin{vmatrix} -\frac{\partial^2 \bar{S}}{\partial u' \partial u''} & -\frac{\partial^2 \bar{S}}{\partial u' \partial v''} \\ -\frac{\partial^2 \bar{S}}{\partial v' \partial u''} & -\frac{\partial^2 \bar{S}}{\partial v' \partial v''} \end{vmatrix}^{1/2} = \sqrt{1 + \frac{\omega^2 \theta^2}{4}} \frac{\omega}{|\sin N\omega_\theta|}, \tag{79}$$

and finally the quantum propagator

$$\mathcal{K}_\theta(u'', v'', N; u', v', 0) \\ = \sqrt{1 + \frac{\omega^2 \theta^2}{4}} \sqrt{\frac{\omega^2}{\sin^2 N\omega_\theta}} \, \mathrm{e}^{2\pi \mathrm{i} \bar{S}_\theta(u'', v'', N; u', v', 0)}. \tag{80}$$

Let us note and remain that in the commutative regime (obtained from the above form putting $\theta = 0$) we have again (compare (55))

$$\mathcal{K}(u'', v'', N; u', v', 0) = \frac{\omega}{\sin N\omega} \chi_\infty \left[-\bar{S}(u'', v'', N; u', v', 0) \right]. \tag{81}$$

A p-Adic generalization is thus possible, and corresponding forms will be presented elsewhere. For noncommutative path integrals see [30] and references therein.

7. Conclusion

In this paper, we demonstrated how a p-adic version of the quantum (4+D)-Kaluza-Klein model with an exotic fluid can be constructed. It is an exactly soluble model. From equations (35), (36) and (44), *i.e.*, (46), it is possible to construct an adelic model too, *i.e.*, a model which unifies standard and all p-adic models [12]. The investigation of its possible physical implication and discreteness of space-time deserves much more attention and space.

Let us note that adelic states for the (4+D)-dimensional Kaluza-Klein cosmological model (for any D which satisfies (45)) exist in the form

$$\Psi_S(x, y) = \Psi^{\pm}_{D,\infty}(x_\infty, y_\infty) \prod_{p \in S} \Psi_p(x_p, y_p) \prod_{p \notin S} \Omega(|x_p|_p) \Omega(|y_p|_p), \tag{82}$$

where $\Psi^{\pm}_{D,\infty}(x_\infty, y_\infty)$ are the corresponding real counterparts of the wave functions of the universe and S is a finite set of primes p. In the ground state wave functions $\Psi_p(x_p, y_p)$ are proportional to $\Omega(p^\nu |x_p|_p)\ \Omega(p^\mu |y_p|_p)$ or to $\delta(p^\nu - |x_p|)\delta(p^\mu - |y_p|)$. Adopting the usual probability interpretation of the wave function (82), we have

$$|\Psi_S(x, y)|^2_\infty = \left| \Psi^{\pm}_{D,\infty}(x_\infty, y_\infty) \right|^2_\infty \prod_{p \in S} |\Psi_p(x_p, y_p)|^2_\infty \prod_{p \notin S} \Omega(|x_p|_p) \Omega(|y_p|_p), \tag{83}$$

because $(\Omega(|x|_p))^2 = \Omega(|x|_p)$.

As a consequence of Ω-function properties, at the rational points x, y and in the (special) vacuum state ($S = \emptyset$, *i.e.*, all $\Psi_p(x_p, y_p) = \Omega(|x_p|_p)\Omega(|y_p|_p)$, we find

$$|\Psi(x, y)|^2_\infty = \begin{cases} \left| \Psi^{\pm}_{D,\infty}(x, y) \right|_\infty, & x, y \in Z, \\ 0, & x, y \in Q \backslash Z. \end{cases} \tag{84}$$

This result leads to some discretization of minisuperspace coordinates x, y. Namely, the probability to observe the universe corresponding to our minisuperspace model is nonzero only in the integer points of x and y. Keeping in mind that Ω-function is invariant with respect to the Fourier transform, this conclusion is also valid for the momentum space. Note that this kind of discreteness depends on adelic quantum state of the universe. When some p-adic states are different from $\Omega(|x|_p)\Omega(|y|_p)$ $(S \neq \emptyset)$, then the above adelic discreteness becomes less transparent.

Performing the integration in (83) over all p-adic spaces, and having in mind that eigenfunctions should be normed to unity, one recovers the standard effective model over real space. However, if the region of integration is over only some parts of p-adic spaces, then the adelic approach manifestly exhibits p-adic quantum effects. Since the Planck length is here the natural one, the adelic minisuperspace models refer to the Planck scale.

Further investigation could include determination of conditions for the existence of ground states in the form $\Omega(p^\nu|x|_p)\Omega(p^\mu|y|_p)$ and p-adic delta function. We should emphasize that investigation of dimensionality D of internal space from the conditions (45) and pseudoconstant solutions of Eq.(39) deserves attention. It could additionally contribute to better understanding of the model, especially from its p-adic sector.

In the last section we consider the (4+1)-dimensional Kaluza-Klein "empty" model and calculated corresponding quantum propagators and ground states on real and p-adic spaces in commutative regime. We also calculated the kernel of the operator of evolution for this model on noncommutaive minisuperspace. Transition from noncommutative to commutative case is simple, just putting $\theta = 0$. It would be very interesting to compute wave functions of the universe for this model in both cases (commutative-noncommutative) and compare their form. It could shed light on the very interesting early phase of the universe, when nonarchimedean and noncommutative effects played an important role.

Acknowledgements. This work is partially supported by the Ministry of Science and Environment Protection of the Republic of Serbia under Grant 1643. G. Dj is thankful to the organizers of the GAS'05 conference for invitation to take part and give talk, as well as for kind hospitality. He is thankful to S. Kar, R. Rashkov and D. Sasselov for fruitful discussion during and after the conference. Work on this paper is also supported in part by the UNESCO-ROSTE Grant No. 875.914.5, in frame of the Southeastern European Network in Mathematical and Theoretical Physics (SEENET-MTP).

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