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Exactly solvable quantum models on nonarchimedean spaces

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Abstract

The exactly solvable models are of a great significance in any theory in physics. In this review we consider a class of exactly solvable quantum (minisuperspace cosmological) models over archimedean and nonarchimedean spaces. First of all we consider (4+D)-dimensional Kaluza-Klein cosmological model with two scaling factors. One of them corresponds to the D-dimensional internal space, and second one to the 4-dimensional universe. We present basic ideas and results concerning this model in standard cosmology and construct corresponding p-adic quantum model and explore existence of its p-adic ground state. The special attention is paid to the $4+1$ dimensional model. The corresponding propagators on real and p-adic spaces are calculated. The forms of these propagators are discussed. In brief we discuss some results for an exactly solvable model in string cosmology.

Keywords: exactly solvable models, nonarchimedean spaces, Kaluza-Klein quantum cosmology, p-adic numbers.

1 Introduction

There is a quantum gravity principal uncertainty Δx of measuring distances around the Planck length l_P (e.g. [1]),

$$\Delta x \geq l_P = \sqrt{\frac{\hbar G}{c^3}}\ 10^{-35} m, \tag{1}$$

which restricts priority of archimedean geometry based on real numbers. It could be concluded that on very short distances Archimedean axiom is not valid, i.e. space can posses ultrametric features. Geometry is always connected with a number field. Nonarchimedean geometry can be based on the field of p-adic numbers Q_p. Generally speaking, p-adic approach should be useful in describing a very early phase of the universe and processes around Planck scale [2].

A significant number of papers, motivated by [2], has been published up to now (for a review see [3, 4]). In past, we have treated many cosmological models, mainly constructed in four space-time dimensions [5, 6, 7]. In this review we are especially interested in application of p-adic numbers and analysis in multidimensional quantum cosmology [5]. The first part of this article is devoted to the formulation of multidimensional quantum cosmological model with two scaling factors and an exotic fluid. Our intentions is to use these models to consider possible p-adic structure of extra dimensions. In addition, we explore possibility to formulate a consistent real, p-adic and adelic (4+D) Kaluza-Klein model.

The structure of space-time around Planck scale is a very interesting problem. There is also a very promising approach based on noncommutative (NC) geometry. Beside noncommutative quantum mechanics [8], (mainly used to formulate some interesting toy models) NC Quantum Field Theory [9] and the NC Standard Model [10] have been considered. There have been a few attempts to formulate a NC Quantum Cosmology. For details see Refs. 11 and 12. In the second part of this article we are focussed on $(4+1)$ dimensional model, in particular Kaluza-Klein "empty" model on real and p-adic commutative spaces. We consider and calculate their corresponding quantum propagators.

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Structure of this review is as follows. After brief mathematical introduction in Section 2, a short review of p-adic and adelic quantum mechanics and cosmology is given in Section 3. Section 4 is devoted to the classical (4+D)-dimensional cosmological models filed with "exotic" fluid [13]. The corresponding p-adic model is considered in Section 5. In Section 6 we consider a particular $(4+1)$ cosmological model with spacetime metric of a Friedman-Robertson-Walker type. We calculate and consider quantum propagator in case of real and p-adic commutative space. Similarities and differences in forms of the propagators are briefly examined. In Section 7 we consider an exactly solvable string cosmological model. This paper is ended by short conclusion, including adelic generalization and suggestion for advanced research.

2 p -Adic Numbers and Adeles

Perhaps the easiest way to understand p-adic numbers is if one starts with the notion of norm. Any norm must satisfy three conditions: nonnegativity, homogeneity and triangle inequality. The completion of the field of rational numbers Q with respect to the absolute value, or standard norm $|.|_\infty$, gives the field of real numbers $R \equiv Q_\infty$. Besides this norm there are another ones which satisfy the first two conditions and the third one in a stronger way

$$\|x+y\| \leq \max(\|x\|, \|y\|), \tag{2}$$

so called strong triangle inequality. The most important of them is p-adic norm $|.|_p$ [3]. The feature (2), also called ultrametricity, is one of the most important characteristics of the p-adic norm (p denotes a prime number). The number fields obtained by completion of Q with respect to this norm are called p-adic number fields Q_p. It is known that any nonzero rational number x can be expressed as $x = \pm p^\gamma a/b$, where a γ is the integer number, and a and b are the natural numbers which are not divisible with the prime number p and have no common divisor. Then, p-adic norm of x is, by definition, $|x|_p = p^{-\gamma}$.

Metric on Q_p is defined by $d_p(x,y) = |x-y|_p$. This metric is the nonarchimedean one and the pair (Q_p, d_p) presents locally compact, topologically complete, separable and totally disconnected p-adic metric space. p-Adic ball $B_\nu(a)$, with the center at the point a and the radius p^ν is the set

$$B_\nu(a) = \{x \in Q_p : \ |x-a|_p \leq p^\nu, \ \nu \in Z\}. \tag{3}$$

p-Adic sphere $S_\nu(a)$ with the center a and the radius p^ν is

$$S_\nu(a) = \{x \in Q_p : \ |x-a|_p = p^\nu, \ \nu \in Z\}. \tag{4}$$

It holds:

$$\begin{gathered} B_\nu(a) = \bigcup_{\nu' \leq \nu} S_{\nu'}(a), \\ S_\nu(a) = B_\nu(a) \backslash B_{\nu-1}(a), \quad B_\nu(a) \subset B_{\nu'}(a), \quad \nu < \nu', \\ \bigcap_\nu B_\nu(a) = \{a\}, \quad \bigcup_\nu B_\nu(a) = \bigcup_\nu S_\nu(a) = Q_p. \end{gathered} \tag{5}$$

Elementary p-adic functions are given by the series of the same form as in the real case, e.g.

$$\exp x = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \tag{6}$$

$$\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}, \qquad \cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}, \tag{7}$$

$$\tan x = \frac{\sin x}{\cos x}. \tag{8}$$

These functions have the same domain of convergence $G_p = \{x \in Q_p : |x|_p < |2|_p\}$. Real and p-adic numbers are unified in the form of the adeles [14]. An adele is an infinite sequence

$$a = (a_\infty, a_2, ..., a_p, ...), \tag{9}$$

where $a_\infty \in Q_\infty$, and $a_p \in Q_p$, with restriction that $a_p \in Z_p$ ($Z_p = \{x \in Q_p : |x|_p \le 1\}$) for all but a finite set S of primes p. If we introduce $\mathcal{A}(S) = Q_\infty \times \prod_{p \in S} Q_p \times \prod_{p \notin S} Z_p$ then the space of all adeles is $\mathcal{A} = \bigcup_S \mathcal{A}(S)$, which is a topological ring. Namely, $\mathcal{A}$ is a ring with respect to the componentwise addition and multiplication. A principal adele is a sequence $(r, r, ..., r, ...) \in \mathcal{A}$, where $r \in Q$. Thus, the ring of principal adeles, which is a subring of $\mathcal{A}$, is isomorphic to Q. An important function on $\mathcal{A}$ is the additive character $\chi(x)$, $x \in \mathcal{A}$, which is a continuous and complex-valued function with basic properties:

$$|\chi(x)|_\infty = 1, \quad \chi(x+y) = \chi(x)\chi(y). \tag{10}$$

This additive character may be presented as

$$\chi(x) = \prod_v \chi_v(x_v) = \exp(-2\pi i x_\infty) \prod_p \exp(2\pi i \{x_p\}_p), \tag{11}$$

where $v = \infty, 2, \cdots, p, \cdots$, and $\{x\}_p$ is the fractional part of the p-adic number x. The map $\varphi : \mathcal{A} \to C$, which has the form

$$\varphi(x) = \varphi_\infty(x_\infty) \prod_{p \in S} \varphi_p(x_p) \prod_{p \notin S} \Omega(\mid x_p \mid_p), \tag{12}$$

where $\varphi_\infty(x_\infty) \in D(Q_\infty)$ is an infinitely differentiable function on Q_∞ and falls to zero faster than any power of $\mid x_\infty \mid_\infty$ as $\mid x_\infty \mid_\infty \to \infty$, $\varphi_p(x_p) \in D(Q_p)$ is a locally constant function with compact support, and

$$\Omega(|x|_p) = \begin{cases} 1, & |x|_p \le 1, \\ 0, & |x|_p > 1, \end{cases} \tag{13}$$

is called an elementary function on Q_p. Because Q_p is a local compact commutative group the Haar measure can be introduced, which enables integration. The integrals of the Gauss type over the p-adic sphere S_ν, p-adic ball B_ν and over any Q_v are:

$$\int_{S_\nu} \chi_p\left(\alpha x^2 + \beta x\right) dx = \begin{cases} \lambda_p(\alpha)|2\alpha|_p^{-1/2} \chi_p\left(-\frac{\beta^2}{4\alpha}\right), & \left|\frac{\beta}{2\alpha}\right|_p = p^\nu, \\ 0, & \left|\frac{\beta}{2\alpha}\right|_p \neq p^\nu, \end{cases} \tag{14}$$

for $|4\alpha|_p \ge p^{2-2\nu}$,

$$\int_{B_\nu} \chi_p(\alpha x^2 + \beta x) dx = \begin{cases} p^\nu \Omega(p^\nu |\beta|_p), & |\alpha|_p p^{2\nu} \le 1, \\ \frac{\lambda_p(\alpha)}{|2\alpha|_p^{1/2}} \chi_p\left(-\frac{\beta^2}{4\alpha}\right) \Omega\left(p^{-\nu}\left|\frac{\beta}{2\alpha}\right|_p\right), & |\alpha|_p p^{2\nu} > 1, \end{cases} \tag{15}$$

$$\int_{Q_v} \chi_p(\alpha x^2 + \beta x) dx = \lambda_v(\alpha)|2\alpha|_v^{-1/2} \chi_v\left(-\frac{\beta^2}{4\alpha}\right), \quad \alpha \neq 0. \tag{16}$$

The arithmetic functions $\lambda_v(a) : \ Q_v \mapsto C$ have the following properties:

$$|\lambda_v(a)|_\infty = 1, \quad \lambda_v(0) = 1, \quad \lambda_v(ab^2) = \lambda_v(a), \tag{17}$$

$$\lambda_v(a)\lambda_v(b) = \lambda_v(a+b)\lambda_v(a^{-1} + b^{-1}), \tag{18}$$

where $a \neq 0$, $b \neq 0$. It should be noted that the real counterpart of λ function is

$$\lambda_\infty(\alpha) = \frac{1}{\sqrt{2}}(1 - i\,\mathrm{sign}\alpha), \quad \alpha \in Q_\infty. \tag{19}$$

3 p-Adic Quantum Mechanics and Cosmology

3.1 p-Adic Quantum Mechanics

In foundations of standard quantum mechanics (over R) one usually starts with a representation of the canonical commutation relation

$$[\hat{q}, \hat{k}] = i\hbar, \tag{20}$$

where q is a spatial coordinate and k is the corresponding momentum. It is well known that the procedure of quantization is not unique. In formulation of p-adic quantum mechanics [15, 16] the multiplication $\hat{q}\psi \to x\psi$ has no meaning for $x \in Q_p$ and $\psi(x) \in C$. Also, there is no possibility to define p-adic "momentum" or "Hamiltonian" operator. In the real case they are infinitesimal generators of space and time translations, but, since Q_p is disconnected field, these infinitesimal transformations become meaningless. However, finite transformations remain meaningful and the corresponding Weyl and evolution operators are p-adically well defined. Canonical commutation relation in p-adic case can be represented by the Weyl operators ($h = 1$)

$$\hat{Q}_p(\alpha)\psi_p(x) = \chi_p(\alpha x)\psi_p(x) \tag{21}$$

$$\hat{K}_p(\beta)\psi(x) = \psi_p(x + \beta). \tag{22}$$

Now, instead of the relation (20) we have

$$\hat{Q}_p(\alpha)\hat{K}_p(\beta) = \chi_p(\alpha\beta)\hat{K}_p(\beta)\hat{Q}_p(\alpha) \tag{23}$$

in the p-adic case. It is possible to introduce the family of unitary operators

$$\hat{W}_p(z) = \chi_p(-\frac{1}{2}qk)\hat{K}_p(\beta)\hat{Q}_p(\alpha), \quad z \in Q_p \times Q_p, \tag{24}$$

that is a unitary representation of the Heisenberg-Weyl group. Recall that this group consists of the elements (z, α) with the group product

$$(z, \alpha) \cdot (z', \alpha') = (z + z', \alpha + \alpha' + \frac{1}{2}B(z, z')), \tag{25}$$

where $B(z, z') = -kq' + qk'$ is a skew-symmetric bilinear form on the phase space. Dynamics of a p-adic quantum model is described by a unitary operator of evolution $U(t)$ without using the Hamiltonian. Instead of that, the evolution operator has been formulated in terms of its kernel $\mathcal{K}_t(x, y)$

$$U_p(t)\psi(x) = \int_{Q_p} \mathcal{K}_t(x, y)\psi(y)dy. \tag{26}$$

In this way [15] p-adic quantum mechanics is given by a triple

$$(L_2(Q_p), W_p(z_p), U_p(t_p)). \tag{27}$$

Keeping in mind that standard quantum mechanics can be also given as the corresponding triple, ordinary and p-adic quantum mechanics can be unified in the form of adelic quantum mechanics [17, 18]

$$(L_2(\mathcal{A}), W(z), U(t)). \tag{28}$$

$L_2(\mathcal{A})$ is the Hilbert space on $\mathcal{A}$, $W(z)$ is a unitary representation of the Heisenberg-Weyl group on $L_2(\mathcal{A})$ and $U(t)$ is a unitary representation of the evolution operator on $L_2(\mathcal{A})$. The evolution operator $U(t)$ is defined by

$$U(t)\psi(x) = \int_{\mathcal{A}} \mathcal{K}_t(x, y)\psi(y)dy = \prod_v \int_{Q_v} \mathcal{K}_t^{(v)}(x_v, y_v)\psi^{(v)}(y_v)dy_v. \tag{29}$$

The eigenvalue problem for $U(t)$ reads

$$U(t)\psi_{\alpha\beta}(x) = \chi(E_\alpha t)\psi_{\alpha\beta}(x), \tag{30}$$

where $\psi_{\alpha\beta}$ are adelic eigenfunctions, $E_\alpha = (E_\infty, E_2, ..., E_p, ...)$ is the corresponding energy, indices α and β denote energy levels and their degeneration. Note that any adelic eigenfunction has the form

$$\Psi(x) = \Psi_\infty(x_\infty)\prod_{p\in S}\Psi_p(x_p)\prod_{p\notin S}\Omega(\mid x_p\mid_p), \quad x\in\mathcal{A}, \tag{31}$$

where $\Psi_\infty \in L_2(R)$, $\Psi_p \in L_2(Q_p)$. A suitable way to calculate p-adic propagator $\mathcal{K}_p(x'', t''; x', t')$ is to use Feynman's path integral method, i.e.

$$\mathcal{K}(x'', t''; x', t') = \int_{(x',t')}^{(x'',t'')} \chi_p\left(-\frac{1}{h}\int_{t'}^{t''} L(\dot q, q, t)dt\right)\mathcal{D}q. \tag{32}$$

3.2 Path Integration for Quadratic Lagrangian

A general quadratic Lagrangian can be written as follows

$$L(\dot q, q, t) = L_0 + \frac{\partial L_0}{\partial q}q + \frac{\partial L_0}{\partial \dot q}\dot q + \frac{1}{2}\frac{\partial^2 L_0}{\partial q^2}q^2 + \frac{\partial^2 L_0}{\partial q\partial \dot q}q\dot q + \frac{1}{2}\frac{\partial^2 L_0}{\partial \dot q^2}\dot q^2, \tag{33}$$

where index 0 denotes the Taylor expansion of $L(\dot q, q, t)$ around $\dot q = q = 0$. A general solution of corresponding Euler-Lagrange equation describes classical trajectory

$$q = x(t) = C_1 f_1(t) + C_2 f_2(t) + \xi(t), \tag{34}$$

where $f_1(t)$ and $f_2(t)$ are two linearly independent solutions of the corresponding homogeneous equation, and $\xi(t)$ is a particular solution of the complete Euler-Lagrange equation. This solutions leads to the quadratic classical action $\bar S(x'', t'', x', t') = \int_{t'}^{t''} L(\dot x, x, t)dt$. In Refs. 19 and 20 the following theorem was proved: *The v-adic ($v = \infty, 2, 3, 5, ...$) kernel $\mathcal{K}_v(x'', t''; x', t')$ of the unitary evolution operator, evaluated as the Feynman path integral, for quadratic Lagrangians (33) (and consequently, for quadratic classical actions) has the form*

$$\mathcal{K}_v(x'', t''; x', t') = \lambda_v\left(-\frac{1}{2h}\frac{\partial^2\bar S}{\partial x''\partial x'}\right)\left|\frac{\partial^2\bar S}{\partial x''\partial x'}\right|_v^{\frac{1}{2}}\chi_v(-\frac{1}{h}\bar S(x'', t''; x', t')). \tag{35}$$

When one has a system with more then one dimension with uncoupled spatial coordinates, then the total propagator is the product of the corresponding one-dimensional propagators. As an illustration of p-adic and adelic quantum-mechanical models the following one-dimensional systems with the quadratic Lagrangians were considered: a free particle and harmonic oscillator [2, 17, 18], a free relativistic particle [6], a particle in a constant field [21] and a harmonic oscillator with time-dependent frequency [22].

Let us note that exactly solvable model in our approach are models with a Lagrangian in quadratic form because in that case we can get exact solution for the corresponding path integral. Such are kernels are a good starting point to calculate the wave function in real, p-adic and adelic case.

3.3 p-Adic Quantum Cosmology

The words "quantum" and "cosmology" do appear to some physicists to be inherently incompatible. We usually think of cosmology in terms of the very large structure of the universe, and of quantum phenomena in terms of the very small. However, if the hot big bang is the correct description of the universe, then the universe did start out incredibly small, and there must have been an epoch when quantum mechanics applied to the universe as a whole.

In this sense quantum cosmology [23, 24] is the application of quantum theory to the universe as a whole. Independently of any particular interaction, such a theory is needed in view of the extreme sensitivity of quantum system to their environment, that is, to other degrees of freedom. However, since gravity is the dominating interaction on cosmic scales, a quantum theory of gravity is needed as a formal prerequisite for quantum cosmology. Most work in quantum cosmology is based on the WheelerDeWitt equation of quantum geometrodynamics. The method is to restrict first the configuration space to a finite number of variables (scale factors, matter fields, . . .) and then to quantize canonically. Since the full configuration space of three-geometries is called superspace, the ensuing models are called minisuperspace models.

In quantum mechanics and quantum field theory, path integrals provide a convenient tool for a wide range of applications. In quantum gravity, a path-integral formulation would have to employ a sum over all four-metrics for a given topology,

$$Z[g] = \int \mathcal{D}g_{\mu\nu}(x) e^{iS[g_{\mu\nu}(x)]/\hbar}. \tag{36}$$

In addition, one would expect that a sum over all topologies has to be performed. Since four-manifolds are not classifiable, this is an impossible task. Attention is therefore restricted to a given topology (or to a sum over few topologies). Still, the evaluation of an expression such as this meets great mathematical and conceptual difficulties. While the full (superspace) quantum cosmological models are usually unsolvable, the minisuperspace ones can be handled with available mathematical tools.

For the minisuperspace models, we use metric in the standard 3+1 decomposition

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -N^2 dt^2 + h_{ij}dx^i dx^j, \tag{37}$$

where N is the lapse function. For this models functional integral in (36) is reduced to functional integral over three-metric and configuration of matter fields, and to another usual integral over the lapse function N. For the boundary condition $q_\alpha(t_2) = q''_\alpha$, $q_\alpha(t_1) = q'_\alpha$ in the gauge $\dot{N} = 0$, we have minisuperspace propagator

$$\langle q''_\alpha ; q'_\alpha \rangle = \int dN \mathcal{K}(q''_\alpha, N; q'_\alpha, 0), \tag{38}$$

where

$$\mathcal{K}(q''_\alpha, N; q'_\alpha, 0) = \int \mathcal{D}q_\alpha \chi(-S[q_\alpha]), \tag{39}$$

is an ordinary quantum-mechanical propagator between fixed minisuperspace coordinates (q'_α, q''_α) in a fixed "time" N. S is the action of the minisuperspace model, i.e.

$$S[q_\alpha] = \int_0^1 dt N \left[\frac{1}{2N^2} f_{\alpha\beta}(q)\dot{q}^\alpha \dot{q}^\beta - U(q) \right], \tag{40}$$

where $f_{\alpha\beta}$ is a minisuperspace metric ($ds_m^2 = f_{\alpha\beta}dq^\alpha dq^\beta$) with an indefinite signature $(-, +, +, \ldots)$. This metric includes spatial (gravitational) components and also matter variables for the given model. It should be noted the necessary condition for the existence of an adelic quantum model is the existence of p-adic ground state $\Omega(|q_\alpha|_p)$ defined by

$$\int_{|q_\alpha{}'|_p \leq 1} \mathcal{K}_p(q_\alpha{}'', N; q_\alpha{}', 0) dq_\alpha{}' = \Omega(|q_\alpha{}''|_p). \tag{41}$$

4 (4+D)-Dimensional Cosmological Models Over the Field of Real Numbers

The old idea that four dimensional universe, in which we exist, is just our observation of physical multidimensional space-time attracts much attention nowadays. In such models compactification of extra dimensions play the key role and in the some of them leads to the period of accelerated expansion of the universe [13, 25, 26]. This approach is supported and encouraged with the recent results of the astronomical observations. We briefly recapitulate some facts of the real multidimensional cosmological models, necessary for p-adic and adelic generalization. The metric of such Kaluza-Klein model with D-dimensional internal space can be presented in the form [13, 27]

$$ds^2 = -\tilde{N}^2(t)dt^2 + R^2(t)\frac{dr^i dr^i}{(1+\frac{kr^2}{4})^2} + a^2(t)\frac{d\rho^a d\rho^a}{(1+k'\rho^2)^2} \tag{42}$$

where $R(t)$ and $a(t)$ are the scaling factors of 4-dimensional universe and internal space, respectively; $r^2 \equiv r^i r^i (i = 1, 2, 3)$, $\rho^2 \equiv \rho^a \rho^a (a = 1, ...D)$, and $k, k' = 0, \pm 1$. The form of the energy-momentum tensor is

$$T_{AB} = diag(-\rho, p, p, p, p_D, p_D, ..., p_D), \tag{43}$$

where indices A and B run over both spacetime coordinates and the internal space dimensions. If we want the matter is to be confined to the four-dimensional universe, we set all $p_D = 0$.

Now, we examine the case for which the pressure along all the extra dimensions vanishes $p_D = 0$ (in braneworld scenarios the matter is to be confined to the four-dimensional universe), so that all components of T_{AB} are set to zero except the spacetime components [13]. We assume the energy-momentum tensor of spacetime (43) to be an exotic fluid χ with the equation of state

$$p_\chi = \left(\frac{m}{3} - 1\right)\rho_\chi, \tag{44}$$

(p_χ and ρ_χ are the pressure and the energy density of the fluid, parameter m has value between 0 and 2).

4.1 Classical Model

Dimensionally extended Einstein-Hilbert action (without a higher-dimensional cosmological term) is

$$S = \int \sqrt{-g}\tilde{R} dt d^3 R d^D \rho + S_m = \kappa \int dt L \tag{45}$$

where κ is an irrelevant constant, $\tilde{R}$ is the scalar curvature of the metric, so we can read off the Lagrangian of the model (for flat internal space)

$$L = \frac{1}{2\tilde{N}} R a^D \dot{R}^2 + \frac{D(D-1)}{12\tilde{N}} R^3 a^{D-2}\dot{a}^2 + \frac{D}{2\tilde{N}} R^2 a^{D-1}\dot{R}\dot{a} - \frac{1}{2}k\tilde{N}Ra^D + \frac{1}{6}\tilde{N}\rho_\chi R^3 a^D. \tag{46}$$

For closed universe ($k = 1$), substitution of the equation of state (44) in the continuity equation

$$\dot{\rho}_\chi R + 3(p_\chi + \rho_\chi)\dot{R} = 0 \tag{47}$$

leads to the energy density in form

$$\rho_\chi(R) = \rho_\chi(R_0)\left(\frac{R_0}{R}\right)^m, \tag{48}$$

where R_0 is the value of scaling factor in arbitrary reference time. If we define cosmological constant as $\Lambda = \rho_\chi(R)$, Lagrangian becomes

$$L = \frac{1}{2\tilde{N}} R a^D \dot{R}^2 + \frac{D(D-1)}{12\tilde{N}} R^3 a^{D-2}\dot{a}^2 + \frac{D}{2\tilde{N}} R^2 a^{D-1}\dot{R}\dot{a} - \frac{1}{2}\tilde{N}Ra^D + \frac{1}{6}\tilde{N}\Lambda R^3 a^D. \tag{49}$$

Growth of the scaling factor R, according to (48), leads to the decrease of the cosmological constant by the relation

$$\Lambda(R) = \Lambda(R_0)\left(\frac{R_0}{R}\right)^m. \tag{50}$$

This decaying Λ term may also explain the smallness of the present value of the cosmological constant since, as the universe evolves form its small to large size, the large initial value of Λ decays to small values. If we take $m = 2$, initial condition for cosmological constant and scaling factor $\Lambda(R_0)R_0^2 = 3$, for laps function $\tilde{N}(t) = R^3(t)a^D(t)N(t)$, the Lagrangian (49) becomes

$$L = \frac{1}{2N}\frac{\dot{R}^2}{R^2} + \frac{D(D-1)}{12N}\frac{\dot{a}^2}{a^2} + \frac{D}{2N}\frac{\dot{R}\dot{a}}{Ra}. \tag{51}$$

It should be noted that there are no parameters k and Λ in the Lagrangian. That means that although they are not zero in this model, it is equivalent to a flat universe with zero cosmological term. In another words, there is not difference between four dimensional universe which looks flat and not filled with fluid, and $(4 + D)$-dimensional closed universe filled with an exotic fluid.

The solutions of corresponding equations of motion for the initial conditions $R(0) = a(0) = l_P$, and in terms of Huble parameter $H = \dot{R}/R$ [13] are

$$R(t) = l_P e^{Ht}, \tag{52}$$

$$a(t) = l_P e^{-Ht}. \tag{53}$$

Depending of the dimensionality of the internal space we have,

$$R(t) = l_P e^{Ht}, \tag{54}$$

$$a_\pm(t) = l_P e^{\frac{2Ht}{D}[-1\pm\sqrt{1-\frac{2}{3}(1-\frac{1}{D})}]^{-1}}, \tag{55}$$

for $D = 1$ and

$$R_\pm(t) = l_P e^{\frac{D\beta t}{2}[-1\pm\sqrt{1-\frac{2}{3}(1-\frac{1}{D})}]}, \tag{56}$$

$$a(t) = l_P e^{\beta t}, \tag{57}$$

for $D > 1$, where β is a constant [13]. The solution corresponding to $D = 1$ predicts an accelerating (de Sitter) universe and a contracting internal space, with exactly the same rates. In the case $D > 1$ analysis is complicated, but results are similar.

4.2 Quantum Model

Quantum solutions are obtained from the Wheeler-DeWitt equation

$$H\Psi(R, a) = 0, \tag{58}$$

where H is the Hamiltonian and Ψ is the wave function of the universe. For this model above equation is read in the new variables ($X = \ln R$ and $Y = \ln a$)

$$\left[(D-1)\frac{\partial^2}{\partial X^2} + \frac{6}{D}\frac{\partial^2}{\partial Y^2} - 6\frac{\partial}{\partial X}\frac{\partial}{\partial Y}\right]\Psi(X, Y) = 0. \tag{59}$$

By the new change $x = X\frac{3}{D+3} + Y\frac{D}{D+3}$, $y = \frac{X-Y}{D+3}$, the Wheeler-DeWitt equation takes a simple form

$$\left[-3\frac{\partial^2}{\partial x^2} + \frac{D+2}{D}\frac{\partial^2}{\partial y^2}\right]\Psi(x, y) = 0. \tag{60}$$

Equation (60) has four possible solutions [13]

$$\Psi_D^\pm(x, y) = A^\pm e^{\pm\sqrt{\frac{\gamma}{3}}x\pm\sqrt{\frac{\gamma D}{D+2}}y}, \tag{61}$$

$$\Psi_D^\pm(x, y) = B^\pm e^{\pm\sqrt{\frac{\gamma}{3}}x\mp\sqrt{\frac{\gamma D}{D+2}}y}, \tag{62}$$

where $A^\pm$ and $B^\pm$ are the normalization constants. It is possible to impose the boundary conditions to get a $\Psi_D(R, a) = 0$ [13].

5 (4+D)-Dimensional Model Over the Field of p-Adic Numbers

Consideration of (4+D)-dimensional Kaluza-Klein model over the field Q_p starts from the Lagrangian in the form (51). All quantities in this Lagrangian will be treated as the p-adic ones. Taking again replacement $X = \ln R$ and $Y = \ln a$, it becomes

$$L = \frac{1}{2N}\dot{X}^2 + \frac{D(D-1)}{12N}\dot{Y}^2 + \frac{D}{2N}\dot{X}\dot{Y}. \tag{63}$$

The corresponding p-adic equations of motion are

$$\ddot{X} + \frac{D}{2}\ddot{Y} = 0, \quad \ddot{X} + \frac{D-1}{3}\ddot{Y} = 0. \tag{64}$$

It is not difficult to see that above system can be rewritten in the following form

$$\ddot{X} = 0, \quad \ddot{Y} = 0. \tag{65}$$

If we omit pseudoconstant solutions from consideration and concentrate to the analytical ones we get $X(t) = C_1 t + C_2$, $Y(t) = C_3 t + C_4$. The calculation of the p-adic classical action gives

$$\begin{aligned} &\bar{S}_p(X'', Y'', N; X', Y', 0) \\ &= \frac{1}{2N}(X''-X')^2 + \frac{D(D-1)}{12N}(Y''-Y')^2 + \frac{D}{2N}(X''-X')(Y''-Y'). \end{aligned} \tag{66}$$

Because this action is quadratic in respect to both variables X and Y, we can write down the kernel of p-adic operator of evolution [19, 28]

$$\begin{aligned} &\mathcal{K}_p(X'', Y'', N; X', Y', 0) \\ &= \lambda_p\left(\frac{D(D+2)}{48N^2}\right)\left|\frac{D(D+2)}{12N^2}\right|_p \chi_p(-\bar{S}_p(X'', Y'', N; X', Y', 0)). \end{aligned} \tag{67}$$

Let us again the replacements $x = X\frac{3}{D+3} + Y\frac{D}{D+3}$, $y = \frac{X-Y}{D+3}$ to separate variables and make further analyzes of this model rather simple. In these variables the classical action and the kernel of evolution operator read

$$\begin{aligned} &\bar{S}_p(x'', y'', N; x', y', 0) \\ &= \frac{1}{2N}(1 + \frac{D(D+5)}{6})(x''-x')^2 - \frac{1}{2N}D(D+3)(y''-y')^2, \end{aligned} \tag{68}$$

$$\begin{aligned} &\mathcal{K}_p(x'', y'', N; x', y', 0) = \lambda_p\left(\frac{6+D(D+5)}{6}\right)\lambda_p\left(-\frac{D(D+3)}{12N}\right) \\ &\times \left|\frac{D(D+3)}{2N^2}\left(1 + \frac{D(D+5)}{6}\right)\right|_p^{1/2} \chi_p(-\bar{S}_p(x'', y'', N; x', y', 0)). \end{aligned} \tag{69}$$

Now, we can examine when p-adic wave function has the form corresponding to the simplest ground state [18]

$$\Psi_p(x, y) = \Omega(|x|_p)\Omega(|y|_p). \tag{70}$$

Putting kernel of the operator of evolution (69) in equation (41) we get the required state exists if both conditions

$$|N|_p \leq \left|1 + \frac{D(D+5)}{6}\right|_p, \quad |N|_p \leq |D(D+3)|_p, \ p \neq 2, \tag{71}$$

are fulfilled.

Going back to the "old variables", p-adic ground state wave function for our model is

$$\Psi_p(x, y) = \Omega\left(\left|\left(1 - \frac{D}{D+3}\right)X + \frac{D}{D+3}Y\right|_p\right)\Omega\left(\left|\frac{X-Y}{D-3}\right|_p\right). \tag{72}$$

We can also write down the solutions in the variables R and a.

6 Kaluza-Klein (4+1) - dimensional "empty"model

We start with the metric considered in [27, 31, 32] in which spacetime is of a Friedman-Robertson-Walker type, having a compactified space which is assumed to be the circle S^1. We adopt the chart $\{t, r^i, \rho\}$ with t, r^i and ρ denoting the time, the space coordinates and the compactified space coordinate, respectively

$$ds^2 = -N^2dt^2 + R^2(t)\frac{dr^idr^i}{(1+\frac{\kappa r^2}{4})^2} + a^2d\rho^2 \tag{73}$$

where $\kappa = 0, \pm 1$ and $R(t)$, $a(t)$ are the scale factors of the universe and compact dimension, respectively. The integration of the Einstein-Hilbert action for the such empty (4+1) dimensional Kaluza-Klein universe with cosmological constant Λ

$$S = \int \sqrt{-g}(\tilde{R} - \Lambda)dtd^3rd\rho \tag{74}$$

($\tilde{R}$ is a curvature scalar corresponding to metric (73)) over spatial dimensions gives an effective Lagrangian in the minisuperspace (R, a)

$$L = \frac{1}{2N}Ra\dot{R}^2 + \frac{1}{2N}R^2\dot{R}\dot{a} - \frac{1}{2}N\kappa Ra + \frac{1}{6}N\Lambda R^3a. \tag{75}$$

6.1 Real Model

Let us consider this model on real numbers. By defining $\omega^2 = -\frac{2\Lambda}{3}$ $(\Lambda < 0)$ and changing variables as

$$u = \frac{1}{\sqrt{8}}[R^2 + Ra - \frac{3\kappa}{\Lambda}], \quad v = \frac{1}{\sqrt{8}}[R^2 - Ra - \frac{3\kappa}{\Lambda}] \tag{76}$$

in the "new" minisuperspace (u, v), the Lagrangian takes on the form

$$L = \frac{1}{2N}(\dot{u}^2 - N^2\omega^2u^2) - \frac{1}{2N}(\dot{v}^2 - N^2\omega^2v^2), \tag{77}$$

which describes an isotropic oscillator-ghost-oscillator system. Corresponding equations of motion

$$\ddot{u} + N^2\omega^2u = 0, \quad \ddot{v} + N^2\omega^2v = 0 \tag{78}$$

have the following solutions

$$u(t) = A\cos N\omega t + B\sin N\omega t, \quad v(t) = C\cos N\omega t + D\sin N\omega t. \tag{79}$$

The corresponding classical action S and quantum propagator $\mathcal{K}$ have forms

$$\bar{S}(u'', v'', N; u', v', 0)$$
$$= \frac{1}{2}\omega\left[(u''^2 + u'^2 - v''^2 - v'^2)\cot N\omega + (v'v'' - u'u'')\frac{2}{\sin N\omega}\right], \tag{80}$$

$$\mathcal{K}(u'', v'', N; u', v', 0) = \frac{1}{i}\sqrt{\frac{\omega^2}{\sin N\omega}}\exp\left(2\pi i\bar{S}(u'', v'', N; u', v', 0)\right). \tag{81}$$

To obtain the energy eigenstates and eigenvectors, we need to recast the propagator (81) in a form that permits a direct comparison with the spectral representation for the Feynman propagator given by

$$\mathcal{K}(u'', v'', N; u', v', 0) = \Theta(N)\sum_l \Phi_l^{(m_1,m_2)}(u'', v'')\Phi_l^{*(m_1,m_2)}(u', v')e^{-iNE_{n,m}\hbar}. \tag{82}$$

Corresponding Wheeler-Dewitt equation for this model in the minisuperspace (u, v) is $(N = 1)$

$$\left(\frac{\partial^2}{\partial u^2} - \frac{\partial^2}{\partial v^2} - \omega^2 u^2 + \omega^2 v^2\right) \Psi(u, v) = 0. \tag{83}$$

It has oscillator-ghost-oscillator solutions belonging to the Hilbert space $\mathcal{H}^{(m_1,m_2)}(\mathcal{L}^2)$ as

$$\Psi^{(m_1,m_2)}(u, v) = \sum_{l=0}^{\infty} c_l \Phi_l^{(m_1,m_2)}(u, v), \tag{84}$$

with $m_1, m_2 \geq 0$ and $c_l \in C$. The basis solutions $\Phi_l^{(m_1,m_2)}(u, v)$ are separable as

$$\Phi_l^{(m_1,m_2)}(u, v) = \alpha_{m_2+(2m_2+1)l}(u)\beta_{m_1+(2m_1+1)l}(v) \tag{85}$$

with normalized solutions

$$\alpha_n(u) = \left(\frac{\omega}{\pi}\right)^{1/4} \frac{e^{-\omega u^2/2}}{\sqrt{2^n n!}} H_n(\sqrt{\omega} u), \tag{86}$$

$$\beta_n(v) = \left(\frac{\omega}{\pi}\right)^{1/4} \frac{e^{-\omega v^2/2}}{\sqrt{2^n n!}} H_n(\sqrt{\omega} v), \tag{87}$$

where $H_n(x)$ are the Hermite polynomials.

6.2 p-Adic Model

The main relations related to with the (4+1)-dimensional model in the p-adic case are formally the same as in the real case. The effective p-adic Lagrangian in the minisuperspace (R, a) is

$$L = \frac{1}{2N} Ra\dot{R}^2 + \frac{1}{2N} R^2 \dot{R}\dot{a} - \frac{1}{2} N\kappa Ra + \frac{1}{6} N\Lambda R^3 a. \tag{88}$$

By defining $\omega^2 = -\frac{2\Lambda}{3}$ $(\Lambda < 0)$ and changing variables as

$$u = \frac{1}{\sqrt{8}}[R^2 + Ra - \frac{3\kappa}{\Lambda}], \quad v = \frac{1}{\sqrt{8}}[R^2 - Ra - \frac{3\kappa}{\Lambda}], \tag{89}$$

in the "new" minisuperspace (u, v), the Lagrangian takes the form

$$L = \frac{1}{2N}(\dot{u}^2 - N^2\omega^2 u^2) - \frac{1}{2N}(\dot{v}^2 - N^2\omega^2 v^2) \tag{90}$$

For the corresponding equations of motion

$$\ddot{u} + N^2\omega^2 u = 0, \quad \ddot{v} + N^2\omega^2 v = 0 \tag{91}$$

their p-adic analytical solutions are write down as

$$u(t) = A\cos N\omega t + B\sin N\omega t, \quad v(t) = C\cos N\omega t + D\sin N\omega t. \tag{92}$$

The p-adic classical action is

$$\bar{S}_p(u'', v'', N; u', v', 0) = \frac{1}{2}\omega \left[(u''^2 + u'^2 - v''^2 - v'^2)\cot N\omega + (v'v'' - u'u'')\frac{2}{\sin N\omega}\right]. \tag{93}$$

Being quadratic in respect to u and v, this action leads directly [28] to the propagator

$$\mathcal{K}_p(v'', u'', N; v', u', 0) = \frac{1}{|N|_p}\chi_p\left(\frac{\omega(u''^2 + u'^2 - v'^2 - v''^2)}{2\tan N\omega} + \frac{\omega(v'v'' - u'u'')}{\sin N\omega}\right) \tag{94}$$

In the p-adic region of convergence of analytic functions $\sin x$ and $\tan x$, which is $G_p = \{x \in Q_p : |x|_p \leq |2p|_p\}|$, we find that the simplest vacuum states $\Omega(|u|_p)\Omega(|v|_p)$, $\Omega(p^\nu |u|_p) \times \Omega(p^\mu |v|_p)$, $\nu, \mu = 1, 2, 3, ,$ exist, as well as

$$\Psi_p(x,y) = \begin{cases} \delta(p^\nu - |x|_p)\delta(p^\mu - |y|_p), & |N|_p \leq p^{2\nu-2}, |N|_p \leq p^{2\mu-2} \\ \delta(2^\nu - |x|_2)\delta(2^\mu - |y|_2), & |N|_2 \leq 2^{2\nu-3}, |N|_2 \leq 2^{2\mu-3}, \end{cases} \tag{95}$$

where $\mu, \nu = 0, -1, -2, \cdots$.

Some $4(= 3+1)$-dimensional quantum cosmological models, which in p-adic sector looks like two decoupled harmonic oscillators, were analyzed in details in Ref. [5].

The study of various physical theories from noncommutative point of view has been of particular interest. Beside noncommutativity applied on models defined on real numbers, there have been a few attempts to introduce noncommutative approach in p-adic [12, 33, 34, 35] and adelic models.

7 String model with exponential dilaton potential

In recent years, there has been considerable amount of interest in cosmology from the string theory point of view. Here we consider tree level string effective action [36, 37] in $D = 4$

$$S = \int d^4x \sqrt{-g} \left(R_g - \frac{1}{2}(\partial\varphi)^2 - V(\varphi) \right) \tag{96}$$

where $g_{\mu\nu}$ is the metric tensor, g is determinant and R_g is the Ricci scalar derived from this metric, and φ and $V(\varphi)$ are the dilaton (field) and corresponding potential term. Let us choose metric in form

$$ds^2 = -\frac{N^2(t)}{a^2(t)}dt^2 + a^2(t)\delta_{ij}dx^i dx^j, \tag{97}$$

where $N(t)$ is the lapse function and $a(t)$ is the scale factor. After some rescaling of a and φ, we get the Lagrangian

$$L = \frac{1}{N}\left(-\frac{1}{2}a^2\dot{a}^2 + \frac{1}{2}a^4\dot{\varphi}^2\right) - N^2 a^2 V(\varphi) \tag{98}$$

where the dot denotes time derivative. For potential we choose exponential function in the form $V = V_0 e^{-2\varphi}$. A deeper analysis would show that V_0 has a meaning of a cosmological constant. If we introduce new variables $u = \frac{a^2}{2}\cosh(2\varphi)$ and $v = \frac{a^2}{2}\sinh(2\varphi)$, then the Lagrangian is

$$L = \frac{1}{2N}(\dot{v}^2 - \dot{u}^2) - \frac{N}{2}V_0(u - v). \tag{99}$$

The equations of the motion and solutions are

$$\ddot{v} - \frac{N^2 V_0}{2} = 0, \quad \ddot{u} - \frac{N^2 V_0}{2} = 0, \tag{100}$$

$$v(t) = \frac{N^2 V_0}{4}t^2 + (v'' - v' - \frac{N^2 V_0}{4})t + v', \;\; u(t) = \frac{N^2 V_0}{4}t^2 + (u'' - u' - \frac{N^2 V_0}{4})t + u'. \tag{101}$$

It is worth mentioning that above procedure is valued in both, real and p-adic cases. From v-adic classical action

$$\bar{S}_v(v'', u'', N; v', u', 0) = \frac{(v'' - v')^2 - (u'' - u')^2}{2N} + \frac{NV_0}{4}(v'' - u'' + v' - u') \tag{102}$$

we get v-adic kernel

$$\mathcal{K}_v(v'', u'', N; v', u, 0) = \frac{1}{|N|_v}\chi_v(-\bar{S}(v'', u'', N; v', u, 0)), \tag{103}$$

which is good basis for further investigations.

For instance, condition for the existence of the vacuum state in the form of p-adic Ω function restricts allowed values of parameters in the model. In particular, in $N = 1$ gauge one gets $|V_0|_p \leq 1$. Physical implication of this result will be discussed elsewhere.

8 Conclusion

In this paper, we show existence of a class of exactly solvable models in real, p-adic and adelic quantum cosmology. As first we demonstrate how a p-adic version of the quantum (4+D)-Kaluza-Klein model with exotic fluid can be constructed. Using equations (61), (62) and (70), i.e. (72), it is possible to construct the corresponding adelic model. This model unifies standard and p-adic models for different prime numbers p. The similar procedure is applied in the case of the $4+1$ "empty" model. It is important that string model, in particular exponential the "dilaton potential model" is solvable one.

Let us note that adelic states for the (4+D)-dimensional Kaluza-Klein cosmological model (for any D which satisfies (71)) exist in the form

$$\Psi_S(x, y) = \Psi^{\pm}_{D,\infty}(x_\infty, y_\infty) \prod_{p\in S} \Psi_p(x_p, y_p) \prod_{p\notin S} \Omega(|x_p|_p)\Omega(|y_p|_p), \tag{104}$$

where $\Psi^{\pm}_{D,\infty}(x_\infty, y_\infty)$ are the corresponding real counterparts of the wave functions of the universe. In the ground state wave functions $\Psi_p(x_p, y_p)$ are proportional to $\Omega(p^\nu|x_p|_p)\ \Omega(p^\mu|y_p|_p)$ or to $\delta(p^\nu - |x_p|)\delta(p^\mu - |y_p|)$.

As a consequence of Ω-function properties, at the rational points x, y and in the (special) vacuum state ($S = \emptyset$, i.e. all $\Psi_p(x_p, y_p) = \Omega(|x_p|_p)\Omega(|y_p|_p)$, we find

$$|\Psi(x, y)|^2_\infty = \begin{cases} \left|\Psi^{\pm}_{D,\infty}(x, y)\right|_\infty, & x, y \in Z, \\ 0, & x, y \in Q\backslash Z. \end{cases} \tag{105}$$

This result leads to some discretization of minisuperspace coordinates x, y. Namely, probability to observe the universe corresponding to our minisuperspace model is nonzero only in the integer points of x and y. Keeping in mind that Ω-function is invariant with respect to the Fourier transform, this conclusion is also valid for the momentum space. Since the Planck length is here the natural one, the adelic minisuperspace models refer to the Planck scale.

Further investigations could include determination of conditions for existence of the ground states in the form $\Omega(p^\nu|x|_p)\Omega(p^\mu|y|_p)$ and p-adic delta function. We should emphasize that investigation of dimensionality D of internal space from the conditions (71) and pseudoconstant solutions of the equation (65) deserves attention. Including of tachyonic potential and noncommutative background could additionally contribute to better understanding of the early universe, especially from its p-adic sector.

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