Determinantal representation of outer inverses in Riemannian space

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Abstract

Starting from known determinantal representation of outer inverses we derive their determinantal representation in terms of the inner product in the Euclidean space. Subsequently, we define the double inner product of two miscellaneous tensors of rank 2 in a Riemannian space. Corresponding determinantal representation as well as the general representation of outer inverses in the Riemannian space is derived. A nonzero $\{2\}$-inverse X of a given tensor A obeying $\rho(X) = s$, $1 \le s \le r = \rho(A)$ is expressed in terms of the double inner product involving compound tensors with minors of the order s, extracted from A and appropriate tensors.

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1 Introduction and preliminaries

The set of $m \times n$ matrices of rank r with entries in the set of complex numbers $\mathbb{C}$ is denoted by $\mathbb{C}_r^{m\times n}$. Also, $\mathbf{O}$ (resp. I_p) denote an appropriate zero matrix (resp. $p \times p$ identity matrix). By $\mathrm{Tr}(A)$ we denote the trace of a square matrix, evaluated by summing the diagonal components of A. Later, $|A|$ (resp. $\mathrm{adj}(A)$) denotes the determinant of A (resp. the classical adjoint matrix of A). The determinantal rank (or matrix rank) of A is denoted by $\rho(A)$ and denotes the largest $k \ge 0$ for which there exists α, β satisfying $|\alpha| = |\beta| = k$ and $\left|A_{\alpha\beta}\right| \neq 0$. For any complex matrix A consider the following equations in X

$$(1)\ AXA = A, \quad (2)\ XAX = X, \quad (3)\ (AX)^* = AX, \quad (4)\ (XA)^* = XA.$$

where $*$ denotes the conjugate and transpose. If A is a square matrix we also consider the following equations:

$$(5) \quad AX = XA, \quad (1^k) \quad A^{k+1}X = A^k.$$

For a sequence $\mathcal{S}$ of elements from the set $\{1, 2, 3, 4, 5\}$, the set of matrices satisfying the equations represented in $\mathcal{S}$ is denoted by $A\{\mathcal{S}\}$. A matrix from $A\{\mathcal{S}\}$ is called an $\mathcal{S}$-inverse of A and denoted by $A^{(\mathcal{S})}$. The *Moore-Penrose inverse* $A^\dagger$ of A is the unique $\{1, 2, 3, 4\}$-inverse of A. The *group inverse*, denoted by $A^\#$, is the unique $\{1, 2, 5\}$-inverse of A, and it exists if and only if $\mathrm{ind}(A) = \min\{k : \rho(A^{k+1}) = \rho(A^k)\} = 1$. A matrix $X = A^D$ is said to be the *Drazin inverse* of A if (1^k) (for some positive

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integer k), (2) and (5) are satisfied. The weighted Moore-Penrose inverse is the unique pseudoinverse satisfying (1), (2) together with the following two equations

$$(3M)\ (MAX)^* = MAX, \quad (4N)\ (NXA)^* = NXA,$$

where M and N are Hermitian positive definite matrices of dimensions m and n respectively, and it is represented by the general representation

$$A^{\dagger}_{M,N} = N^{-1}Q^*\left(P^*MAN^{-1}Q^*\right)^{-1}P^*M,$$

and $A = PQ$ is a full-rank factorization of A (see, for example [29]).

As usual, $\mathcal{R}(A)$ and $\mathcal{N}(A)$ denote the range and the null space of A, respectively.

If $A \in \mathbb{C}_r^{m\times n}$, T is a subspace of $\mathbb{C}^n$ of dimension $t \le r$ and S is a subspace of $\mathbb{C}^m$ of dimension $m - t$, then A has a $\{2\}$ inverse X such that $\mathcal{R}(X) = T$ and $\mathcal{N}(X) = S$ if and only if

$$AT \oplus S = \mathbb{C}^m,$$

in which case X is unique and it is denoted by $A^{(2)}_{T,S}$.

We use the following notations from [14] and [26]. Let A be an $m \times n$ matrix of matrix rank r.

Denote the collection of strictly increasing sequences of p integers chosen from $\{1,\dots,n\}$ by

$$\mathcal{Q}_{p,n} = \{\alpha :\ \alpha = (\alpha_1,\dots,\alpha_p),\ 1 \le \alpha_1 < \cdots < \alpha_p \le n\}.$$

For any integer q satisfying $1 \le q \le \rho(A)$ we define the set $\mathcal{N}_q = \mathcal{Q}_{q,m} \times \mathcal{Q}_{q,n}$. For fixed $\alpha, \beta \in \mathcal{Q}_{q,n}$, let

$$\begin{aligned} \mathcal{I}_q(\alpha) &= \{I :\ I \in \mathcal{Q}_{q,m},\ I \supseteq \alpha\}, \quad \mathcal{J}_q(\beta) = \{J :\ J \in \mathcal{Q}_{q,n},\ J \supseteq \beta\}, \\ \mathcal{N}_q(\alpha,\beta) &= \mathcal{I}_q(\alpha) \times \mathcal{J}_q(\beta). \end{aligned}$$

By A^{α}_{β} we denote the $q \times q$ submatrix of A determined by the entries in rows indexed by α and in columns indexed by β. If A is a square matrix, then the coefficient of $|A^{\alpha}_{\beta}|$ in the Laplace expansion of $|A|$ is denoted by $\frac{\partial}{\partial |A^{\alpha}_{\beta}|}|A|$. In the particular case $\alpha = \{i\}$, $\beta = \{j\}$, we have the cofactor $\frac{\partial}{\partial a_{ij}}|A|$ of a_{ij}. By $C_q(A)$ we denote the qth compound matrix of A with rows indexed by q-element subsets of $\{1,\dots,m\}$, columns indexed by q-element subsets of $\{1,\dots,n\}$, and the (α,β) entry defined by $|A^{\alpha}_{\beta}|$.

As it is observed in [8], the determinantal representation for matrix generalized inverse and extension of Cramer rule are primary subjects of generalized inverse theory, which play an important role in computing generalized inverses and finding the solution of singular linear equations. For the sake of completeness, we restate here main results concerning the determinantal representation of generalized inverses.

A determinantal formula for arbitrary non-zero outer inverse $X = (x_{ij})$ of a given complex matrix $A \in \mathbb{C}_r^{m\times n}$ is investigated in [27]. This representation is based on the usage of minors of the order $s \le r$, selected from the matrix A and the matrix product FG, where $F \in \mathbb{C}^{n\times s}$ and $G \in \mathbb{C}^{s\times m}$ are appropriate full-rank matrices:

$$x_{ij} = \left(\operatorname{Tr}(C_s(FGA))\right)^{-1} \sum_{(\alpha,\beta)\in\mathcal{N}_s(j,i)} |(FG)^{\beta}_{\alpha}| \frac{\partial}{\partial a_{ji}} |A^{\alpha}_{\beta}|. \tag{1.1}$$

Corresponding implementation is described in [28].

Recently, the determinantal and full-rank representation of the generalized inverse $A^{(2)}_{T,S}$ is investigated in the papers [8, 23, 33, 34]. For the sake of completeness, we restate the following representation from [23, 34]

$$(A^{(2)}_{T,S})_{ij} = \left(\operatorname{Tr}(C_s(GA))\right)^{-1} \sum_{(\alpha,\beta)\in\mathcal{N}_s(j,i)} |G^{\beta}_{\alpha}| \frac{\partial}{\partial a_{ji}} |A^{\alpha}_{\beta}|, \tag{1.2}$$

where it is assumed that T is a subspace of $\mathbb{C}^n$ of dimension $t \leq r$, S is a subspace of $\mathbb{C}^m$ of dimension $m-t$ and $G \in \mathbb{C}^{n\times m}$ satisfies $\mathcal{R}(G)=T$, $\mathcal{N}(G)=S$ and $\mathrm{ind}(AG)=\mathrm{ind}(GA)=1$. The (p,q) outer generalized inverse in Banach algebra is considered in [9].

In the particular case $EF=A^k$ of (1.1) as well as in the case $G=A^k$ of (1.2), $k\geq \mathrm{ind}(A)$, we get known determinantal representation of the Drazin inverse from [26]. Furthermore, representations and characterizations of $\{1,2\}$-inverses for matrices over an integral domain are investigated in [2], [14] and [25]. These results can be considered as the partial case $s=r$ of corresponding results concerning outer inverses.

In the particular cases $G=A^*$, $G=A$ and $G=(MAN^{-1})^*$ analogous representations and characterizations for the Moore-Penrose inverse, the group inverse and the weighted Moore-Penrose inverse can be derived. These results are introduced in [1, 15] and [16, 17], respectively.

The following additional characterization of $\{1,2\}$-inverses for matrices over an integral domain is restated from [14]. For $A\in\mathbb{Z}_r^{m\times n}$, the matrix $X=(x_{ij})\in\mathbb{Z}^{n\times m}$ is a reflexive g-inverse of A if and only if

$$x_{ij}=\sum_{(\alpha,\beta)\in\mathcal{N}(j,i)} b_{\beta\alpha}\frac{\partial}{\partial a_{ji}}\left|A^\alpha_\beta\right|,\quad 1\leq i\leq n,\ 1\leq j\leq m. \tag{1.3}$$

where the matrix $B=(b_{\alpha\beta})$, $(\alpha,\beta)\in\mathcal{N}$ of the order $\binom{n}{r}\times\binom{m}{r}$ satisfies the following conditions:

$$\begin{aligned}\sum_{(\alpha,\beta)\in\mathcal{N}} b_{\beta\alpha}\left|A^\alpha_\beta\right| &= \mathrm{Tr}(BC_r(A))=1,\\ \rho(B)&=1.\end{aligned} \tag{1.4}$$

In the papers [18, 19, 20] D.W. Robinson introduced the adjoint mapping for matrices over a commutative ring $\Re$ with identity 1. The adjoint mapping of A of the order s is denoted by $A_{\bullet,s}:\Re^{\binom{n}{s}\times\binom{m}{s}}\mapsto\Re^{n\times m}$ and defined by

$$A_{\bullet,s}:B\mapsto A_{B,s}=A_B=\begin{cases}\sum\limits_{\alpha\in\mathcal{Q}_{s,m}}\ \sum\limits_{\beta\in\mathcal{Q}_{s,n}} b_{\beta\alpha}\Big(T_\beta\,\mathrm{ad}(A^\alpha_\beta)\,S_\alpha\Big), & s>1\\ B, & s=1\end{cases} \tag{1.5}$$

where A is $m\times n$ matrix of rank r over $\Re$, $s\leq\min\{m,n\}$, S_α is $s\times m$ matrix with 1 in positions $(1,\alpha(1)),\cdots,(s,\alpha(s))$ and 0 elsewhere, T_β is $n\times s$ matrix with 1 in positions $(\beta(1),1),\cdots,(\beta(s),s)$ and 0 elsewhere, $B=(b_{\beta\alpha})\in\Re^{\binom{n}{s}\times\binom{m}{s}}$ and $\mathrm{ad}(A^\alpha_\beta)$ denotes the usual adjoint matrix of A^α_β. A characterization of the outer inverses of A in terms of adjoint images under A is considered in [21].

Robinson in [22] provided a historical background and extended role of the adjoint. Recently, in [13] authors investigated determinantal representation of various weighted generalized inverses using a lemma from [22].

In [10] the authors considered determinantal representation of generalized inverses for matrices whose elements are polynomials and estimated upper bounds for the degrees of the elements involved in their generalized inverses.

Determinantal representations of generalized inverses are useful for theoretical analysis, see for example [30, 31]. As applications of the determinantal representation of the generalized inverses $A^{(2)}_{T,S}$, on the one hand it is possible to derive particular representations for many common important generalized inverses; on the other hand, this representation is used to solve restricted linear equations [36].

In the present article we consider results related to the determinantal and full-rank representation as well as the characterization of generalized inverses in the Riemannian space from a new point of view. Our motivation is based on the following sources.

(1) First, we observed that the determinantal representations (1.1) and (1.2) as well as the representation (1.3), (1.4) can be expressed in terms of the inner product in the Euclidean vector space.

(2) Our intention in the present article is to investigate influence of different inner products on the full-rank and determinantal representation of outer inverses, generally, into the Riemannian space. On the basis of derived inner product representations of generalized inverses in the Euclidean vector space, we give analogous determinantal representations in the Riemannian space. The determinantal representation is so far investigated in commutative rings, over integral domains or in the set of complex matrices, where the Euclidean metric with Cartesian coordinates is actual. In this way, we replaced the underlying Euclidean space with an arbitrary Riemannian manifold and extended the notion of the determinantal representation into the Riemannian space. Consequently, we got possibility to derive outer inverses by means of symmetric invertible matrices whose elements are functions. The Euclidean traditions are too strong to be rejected, and probably few generations of mathematicians are necessary to work off its influence. Why the Euclidean geometry has lived through 2 millenniums? The reason is that the square root of a quadratic form is used to define length and vectors. The length is the fundamental concept either in theoretical or applied science. In the Riemannian spaces the length is introduced by the general definition that enables it to be defined by functions of a rather wide range of classes with minimum conditions on smoothness. We hope that our representations of generalized inverses in Riemannian space will initiate investigations from that new point of view and be helpful and relevant in future studies of generalized inverses.

(3) In the scientific literature we found a germ of our main ideas. It is known that the Moore-Penrose inverse $A^\dagger$ transforms into the weighted Moore-Penrose inverse $A^\dagger_{M,N}$ after the replacement of the usual vector inner product in $\mathbb{C}^m$ and $\mathbb{C}^n$ by the following weighted inner products

$$\langle x, y\rangle_M = y^* M x,\ x, y \in \mathbb{C}^m, \quad \langle x, y\rangle_N = y^* N x,\ x, y \in \mathbb{C}^n,$$

where M and N are Hermitian positive definite matrices (see, for example [29]). The equivalence of a weighted generalized inverse and the Moore-Penrose inverse in an indefinite inner product space is studied in [12]. The notion of the generalized weighted Moore-Penrose inverse $A^\dagger_{M,N}$ over the set $\mathbb{F}^n$ with the indefinite scalar product $[x, y] = \langle Hx, y\rangle$, where $\mathbb{F}$ is the real field $\mathbb{R}$ or the complex field $\mathbb{C}$ and H is an invertible Hermitian matrix, is introduced in [24]. The authors in [24] assume that M and N are invertible and Hermitian.

In the second section we consider the double inner product of two tensors of rank 2. The determinantal representations of outer inverses and $A^{(2)}_{T,S}$ inverse of a complex matrix are generalized in terms of the double inner product of two tensors of rank 2 acting between Riemannian spaces in Section 3. In addition, a few necessary and sufficient conditions for the existence and corresponding full-rank representations of non-zero outer inverses are derived. A nonzero $\{2\}$-inverse X of A obeying $\rho(X) = s$, $1 \le s \le r = \rho(A)$ is expressed in terms of the double inner product involving compound tensors of rank 2 whose elements are minors of the order s, extracted from the matrix A and appropriate tensors. In this way, we continue and extend results from [27]. Analogous results for the generalized inverse $A^{(2)}_{T,S}$ are also derived, as a continuation of the papers [8] and [23]. Restrictions to the set of $\{1,2\}$-inverses continue results from [1, 2, 14, 25]. Also, in a particular case we get a generalization of known results from [26], relative to the Drazin inverse. The volume associated with complex matrices as well as the image of the adjoint mapping are extended in terms of the double inner product in Riemannian space. Several illustrative examples are presented in the last section.

2 Double inner product of miscellaneous tensors of rank 2

In the rest of the paper we assume the Einstein summation convention. The repetition of an index (whether contravariant or covariant) in a term will denote a summation with respect to that index over its range.

Riemannian space R_m is a differentiable manifold in which a symmetric basic tensor $g^{ij}(x_1, ..., x_m)$ is introduced, that is in generally

$$g^{ij}(x) = g^{ji}(x), \ i = 1, \dots, m. \tag{2.1}$$

The lowering and the raising of indices one defines by the tensors g_{ij} and g^{ij} respectively, where g_{ij} is defined by the equation

$$g_{ij}g^{jk} = \delta_i^k, \ i, k = 1, \dots, m, \tag{2.2}$$

where δ_i^k is the Kronecker symbol (which is 0 for $i \neq j$ and 1 for $i = j$). Therefore, the matrix $[g_{ij}]$ is the inverse of $[g^{ij}]$.

Definition 2.1 *For given coordinates x_i, $i = 1, \dots, m$ in R_m a subspace $R_n \subset R_m$ is determined by the equations*

$$x_i = x_i(u_1, \dots, u_n), \ (n < m),$$

where

$$\rho\left(\left[\frac{\partial x_i}{\partial u_j}\right]\right) = n, \ i = 1, \dots, m, j = 1, \dots, n.$$

Assume that $\mathcal{H} = \{h^{ij}\}$, $i, j = 1, \dots, n$ is the metric tensor induced in R_n by the following system of constraints:

$$h^{ij} = x_s^i x_t^j g^{st} = h^{ji}, \ i, j = 1, \dots, n, \ s, t = 1, \dots, m,$$

where $x_s^i = \frac{\partial x_i}{\partial u_s}$. The inverse tensor for $\mathcal{H} = \{h^{ij}\}$ is equal to $\mathcal{H}^{-1} = \{h_{ij}\}$, $i, j = 1, \dots, n$, i.e.

$$h_{ij}h^{jk} = \delta_i^k, \ i, k = 1, \dots, n.$$

In Cartesian coordinates, the covariant and the contravariant components are one and the same, and the fundamental tensor is merely the identity tensor.

Definition 2.2 *A system of objects whose elements are defined in the Riemannian subspace $R_n \subset R_m$, where certain indices are related to R_m and another to R_n and where transformations of these objects are defined by the tensor rules separately in R_n and in R_m*

$$a_j^i(x_1(u_1, \dots, u_n), \dots, x_n(u_1, \dots, u_n)) \equiv a_j^i(u_1, \dots, u_n) \equiv a_j^i, \ i = 1, \dots, m, j = 1, \dots, n,$$

is called the miscellaneous tensor.

In this way, it is possible to use miscellaneous tensors of rank 2 and dimensions $m \times n$, $m \neq n$. The standard conventions are assumed. In the case when a tensor is of the type $(1, 1)$, then an upper index denotes row and the lower index corresponds to a column. If a tensor is of type $(0, 2)$ or $(2, 0)$ then the first index determines row and the second one determines column.

The inner product of two 2nd rank tensors corresponds to the usual matrix multiplication while the double inner product of 2nd rank tensors is an analogy for the inner product of two matrices.

Definition 2.3 *The inner product of two 2nd rank tensors A_{kj} and B_{kj} of $(0,2)$-type with covariant components of dimensions $m\times k$ and $k\times n$, respectively, is denoted as $A\cdot B=C$, where*

$$C^i_l = A^i_j B^j_l = \delta^{ik}A_{kj}\delta^{js}B_{sl},\ i,k=1,\dots,m, j,s=1,\dots,k, l=1,\dots,n$$
$$C^i_l = A^i_j B^j_l = g^{ik}A_{kj}h^{js}B_{sl},\ i,k=1,\dots,m, j,s=1,\dots,k, l=1,\dots,n$$

are corresponding definitions in the Euclidean and Riemannian spaces, respectively.

The double inner product of $A=[a_{ij}]$ and $B=[b_{ij}]$, denoted by $A{:}B$, in the Euclidean space is a 0th rank tensor C defined as

$$C = A^i_j (B^T)^j_i = \delta^{ik}A_{kj}\delta^{js}(B^T)_{si},\ i,k=1,\dots,m;\ j,s=1,\dots,n.$$

Because of visibility and connectivity with the traditional notation of the matrix inner product, we use the notation $C=\langle A,B\rangle$ instead of $A:B$ for the double inner product. The double inner product in the Euclidean space may be expressed by means of the trace of the inner product (see, for example [11]):

$$C=\langle A,B\rangle = \mathrm{Tr}(A\cdot B^T) = \mathrm{Tr}(A^T\cdot B). \tag{2.3}$$

In the Riemannian space we have (see, for example [32])

$$A^i_j = g^{ik}A_{kj},\quad (B^T)^j_i = h^{js}(B^T)_{si},\ i,k=1,\dots,m;\ j,s=1,\dots,n, \tag{2.4}$$

where $A=[a_{ij}(x)]$, $B=[b_{ij}(x)]$, $x\in R_n\subset R_m$. Taking into account previous considerations, in the following definition we introduce the double inner product between two miscellaneous tensors of the rank 2 in the Riemannian subspace $R_n\subset R_m$.

Definition 2.4 *The double inner product of two tensors $A=[a_{ij}(x)]$, $B=[b_{ij}(x)]$, $x\in R_n\subset R_m$ of rank 2 and dimensions $m\times n$, denoted as $\langle A,B\rangle_{\mathcal{G},\mathcal{H}}$, can be defined in the following manner:*

$$\langle A,B\rangle_{\mathcal{G},\mathcal{H}} = g^{ik}A_{kj}h^{js}B_{is},\ i,k=1,\dots,m;\ j,s=1,\dots,n. \tag{2.5}$$

where $\mathcal{G}$ is given metric tensor in R_m and $\mathcal{H}$ is the metric tensor induced in R_n.

The following property of the double inner product will be useful in deriving the determinantal representation.

Lemma 2.1 *The double inner product of second rank tensors $A=[a_{ij}(x)]$, $B=[b_{ij}(x)]$, $x\in R_n\subset R_m$, defined by* (2.5)*, satisfies the following properties:*

(a) *The usual inner product of two matrices $A=[a_{ij}]$, $B=[b_{ij}]$, of the order $m\times n$ in the Euclidean space can be represented as a particular case of the double inner product:*

$$\langle A,B\rangle = \langle A,B\rangle_{I_m,I_n}. \tag{2.6}$$

(b) *If $\mathcal{G}$ is given metric matrix in R_m and $\mathcal{H}$ is the metric tensor induced in R_n, we have*

$$\langle A,B\rangle_{\mathcal{G},\mathcal{H}} = \langle \mathcal{G}\cdot A\cdot\mathcal{H}, B\rangle = \langle \mathcal{G}\cdot A, B\cdot\mathcal{H}\rangle. \tag{2.7}$$

(c) *If $\mathcal{G}$ is given metric matrix in R_m and $\mathcal{H}$ is the metric tensor induced in R_n the following holds*

$$\langle A,B\rangle_{\mathcal{G},\mathcal{H}} = h^{kj}A_{ik}g^{si}B_{sj},\ i,s=1,\dots,m;\ j,k=1,\dots,n. \tag{2.8}$$

Proof. Part (a) is evident. Taking into account (2.3) the part (b) follows from

$$\begin{aligned} & g^{ik}A_{kj}h^{js}B_{is},\ i,k=1,\dots,m;\ j,s=1,\dots,n \\ &= (\mathcal{G}\cdot A\cdot\mathcal{H})^{is}(B^T)_{si},\ i=1,\dots,m;\ s=1,\dots,n \\ &= \mathrm{Tr}((\mathcal{G}\cdot A\cdot\mathcal{H})\cdot B^T) = \langle \mathcal{G}\cdot A\cdot\mathcal{H}, B\rangle \end{aligned}$$

and

$$\begin{aligned} & g^{ik}A_{kj}h^{js}B_{is} = g^{ik}A_{kj}h^{js}(B^T)_{si},\ i,k=1,\dots,m;\ j,s=1,\dots,n \\ &= (\mathcal{G}\cdot A)^i_j(\mathcal{H}\cdot B^T)^j_i,\ i=1,\dots,m;\ j=1,\dots,n \\ &= \mathrm{Tr}((\mathcal{G}\cdot A)\cdot(\mathcal{H}\cdot B^T)) = \langle \mathcal{G}\cdot A, B\cdot\mathcal{H}\rangle. \end{aligned}$$

Part (c) is derived from (2.3) and

$$\begin{aligned} \mathrm{Tr}((\mathcal{G}\cdot A\cdot\mathcal{H})\cdot B^T) &= \mathrm{Tr}((\mathcal{G}\cdot A\cdot\mathcal{H})^T\cdot B) = \mathrm{Tr}(\mathcal{H}\cdot A^T\cdot\mathcal{G}\cdot B) \\ &= h^{kj}(A^T)_{ki}g^{si}B_{sj},\ i,s=1,\dots,m;\ j,k=1,\dots,n. \end{aligned} \tag{2.9}$$

□

Definition 2.5 *Magnitude (or the norm) of given tensor $A=[a_{ij}(x)]$, $x\in R_n\subset R_m$ of rank 2 and of dimensions $m\times n$ is equal to*

$$\begin{aligned} ||A||_{\mathcal{G},\mathcal{H}} &= \sqrt{\langle A,A\rangle_{\mathcal{G},\mathcal{H}}} = \sqrt{g^{ik}A_{kj}h^{js}A_{is}} = \sqrt{g^{ik}A_{kj}A_{is}h^{sj}},\ i,k=1,\dots,m;\ j,s=1,\dots,n, \\ &= \sqrt{h^{kj}A_{ik}g^{si}A_{sj}},\ i,s=1,\dots,m;\ j,k=1,\dots,n, \end{aligned} \tag{2.10}$$

where $\mathcal{G}$ and $\mathcal{H}$ are the metric tensors in R_m and R_n, respectively.

Analogy of the full-rank factorization of a non-null matrix we will denominate as full-matrix rank factorization. It is clear that if the fundamental tensors are identity matrices then the full-matrix rank factorization reduces to the full-rank factorization.

3 Outer inverses in Riemannian spaces

In this section double inner product is presented in the general form (2.5) and possesses properties from Lemma 2.1. By help of Definition 2.2 we enable that miscellaneous tensors $A=[a_{ij}(x)]$, $x\in R_n\subset R_m$ of rank 2 can be of dimensions $m\times n$ or $n\times m$. The set of tensors of determinantal rank r in the Riemannian manifold R_n we denote by R_n^r.

Definition 3.1 *Let there be given $A=[a_{ij}(x)]$, $x\in R_n^r\subset R_m$ of dimensions $m\times n$, an arbitrary integer s, $1\le s\le r$, and two integers $1\le i\le n$, $1\le j\le m$. By $\frac{\partial C_s}{\partial a_{ji}}(A)$ we denote $\binom{m}{s}\times\binom{n}{s}$ matrix, defined by*

$$\left(\frac{\partial C_s}{\partial a_{ji}}(A)\right)_{\alpha,\beta} = \frac{\partial}{\partial a_{ji}}\left|A^\alpha_\beta\right| = \left|\left(\mathrm{ad}(A^\alpha_\beta)\right)_{ij}\right| = \begin{cases} (-1)^{p_j+q_i}\left|A^{\alpha\backslash\{j\}}_{\beta\backslash\{i\}}\right|, & (\alpha,\beta)j\in\mathcal{N}_s(j,i) \\ 0, & (\alpha,\beta)j\notin\mathcal{N}_s(j,i) \end{cases} \tag{3.1}$$

for each $\alpha\in\mathcal{Q}_{s,m}$, $\beta\in\mathcal{Q}_{s,n}$, where p_j and q_i denotes indices satisfying $j=\alpha_{p_j}$ and $i=\beta_{q_i}$.

Our statements generalize known results, mainly, from the papers [1, 8, 14, 15, 16, 23, 27] into the corresponding double inner product representations in an arbitrary Riemannian manifold. The main principles used in the verifications of these assertions generalize known techniques used in the above mentioned papers.

Lemma 3.1 *Let $A = [a_{ij}(x)]$, $x \in R_n^r \subset R_m$ is a second rank tensor of dimensions $m \times n$. Assume that $1 \le s \le r$ is selected integer and let $E = [e_{ij}(x)]$, $x \in R_s \subset R_n$ and $F = [f_{ij}(x)]$, $x \in R_s \subset R_m$ are two arbitrary full-matrix rank tensors of dimensions $n \times s$ and $s \times m$, respectively. Suppose that $\mathcal{G} = [g^{ij}(x)]$, $x \in R_m$ and $\mathcal{H} = [h^{ij}(x)]$, $x \in R_n$ are two symmetric invertible tensors of dimensions m and n, such that $C_s(\mathcal{G})$ and $C_s(\mathcal{H})$ are two metric tensors in $R_{\binom{m}{s}}$ and $R_{\binom{n}{s}}$, respectively. Also, suppose that $\rho(A\cdot E\cdot F) = \rho(E\cdot F\cdot A) = s$ and $\left|F\cdot\mathcal{G}\cdot A\cdot\mathcal{H}\cdot E\right| \neq 0$. Then the following statements are valid:*

(i) *The determinant $\left|F\cdot\mathcal{G}\cdot A\cdot\mathcal{H}\cdot E\right|$ can be expressed as*

$$\begin{aligned}\left|F\cdot\mathcal{G}\cdot A\cdot\mathcal{H}\cdot E\right| &= \langle C_s(E\cdot F)^T, C_s(A)\rangle_{C_s(\mathcal{G}),C_s(\mathcal{H})}\\ &= \left|\mathcal{H}^{K\delta}\right|\,\left|(E\cdot F)_{K\gamma}\right|\left|\mathcal{G}^{S\gamma}\right|\,\left|A_{S\delta}\right|,\ (\gamma,\delta),(S,K)\in\mathcal{N}_s;\end{aligned}$$

(ii) *For arbitrary integers $1 \le i \le n$, $1 \le j \le m$ the following equality holds:*

$$\begin{aligned}(\mathcal{H}\cdot E\cdot \operatorname{adj}(F\cdot\mathcal{G}\cdot A\cdot\mathcal{H}\cdot E)\cdot F\mathcal{G})_{ij} &= \langle C_s(E\cdot F)^T, \frac{\partial C_s}{\partial a_{ji}}(A)\rangle_{C_s(\mathcal{G}),C_s(\mathcal{H})}\\ &= \left|\mathcal{H}^{K\delta}\right|\left|(E\cdot F)_{K\gamma}\right|\left|\mathcal{G}^{S\gamma}\right|\left|(\operatorname{ad}(A_{S\delta}))_{ji}\right|,\ (\gamma,\delta),(S,K)\in\mathcal{N}_s.\end{aligned}$$

(iii) *For arbitrary integers $1 \le i \le n$, $1 \le j \le m$ the following is valid:*

$$\left(\mathcal{H}\cdot E\cdot(F\cdot\mathcal{G}\cdot A\cdot\mathcal{H}\cdot E)^{-1}\cdot F\cdot\mathcal{G}\right)_{ij} = \frac{\left|\mathcal{H}^{K\delta}\right|\left|(E\cdot F)_{K\gamma}\right|\left|\mathcal{G}^{S\gamma}\right|\left|(\operatorname{ad}(A_{S\delta}))_{ij}\right|}{\left|\mathcal{H}^{K\delta}\right|\left|(E\cdot F)_{K\gamma}\right|\left|\mathcal{G}^{S\gamma}\right|\left|A_{S\delta}\right|},\ (\gamma,\delta),(S,K)\in\mathcal{N}_s. \tag{3.2}$$

Proof. **(i)** In the case $1<s\le r$, using $\rho(F\cdot\mathcal{G}\cdot A\cdot\mathcal{H}\cdot E)=s$ we conclude

$$\left|F\cdot\mathcal{G}\cdot A\cdot\mathcal{H}\cdot E\right| = \operatorname{Tr}(F\cdot\mathcal{G}\cdot A\cdot\mathcal{H}\cdot E) = \operatorname{Tr}(C_s(F\cdot\mathcal{G}\cdot A\cdot\mathcal{H}\cdot E))$$

and later

$$\begin{aligned}\left|F\cdot\mathcal{G}\cdot A\cdot\mathcal{H}\cdot E\right| &= \operatorname{Tr}(C_s(\mathcal{H}\cdot E\cdot F\cdot\mathcal{G}\cdot A)) = \operatorname{Tr}(C_s(\mathcal{H}\cdot E\cdot F\cdot\mathcal{G})\cdot C_s(A))\\ &= \left|(\mathcal{H}\cdot E\cdot F\cdot\mathcal{G})_\gamma^\delta\right|\left|A_\delta^\gamma\right|,\ (\gamma,\delta)\in\mathcal{N}_s.\end{aligned}$$

Tensors $\mathcal{G}$ and $\mathcal{H}$ are symmetric by the assumptions, so that it is not difficult to verify

$$\begin{aligned}\left|F\cdot\mathcal{G}\cdot A\cdot\mathcal{H}\cdot E\right| &= \langle C_s(\mathcal{H}\cdot E\cdot F\cdot\mathcal{G})^T, C_s(A)\rangle\\ &= \langle C_s(\mathcal{G})\cdot C_s(E\cdot F)^T\cdot C_s(\mathcal{H}), C_s(A)\rangle.\end{aligned}$$

Applying property (2.7) of the double inner product, we have that

$$\left|F\cdot\mathcal{G}\cdot A\cdot\mathcal{H}\cdot E\right| = \langle C_s(E\cdot F)^T, C_s(A)\rangle_{C_s(\mathcal{G}),C_s(\mathcal{H})}.$$

This part of the proof can be completed using the form (2.9) of the double inner product.

In the case $s = 1$ we put $E = u^T$, $F = v$, where

$$u = \{u_1,\dots,u_n\},\quad v = \{v_1,\dots,v_m\} \tag{3.3}$$

are rank 1 tensors, and obtain

$$\begin{aligned}\left|F\cdot\mathcal{G}\cdot A\cdot\mathcal{H}\cdot E\right| &= F\cdot\mathcal{G}\cdot A\cdot\mathcal{H}\cdot E\\ &= (uv)_i^j a_j^i = h^{jp}(E\cdot F)_{pi}g^{ik}A_{kj},\ i,k=1,\dots,m,\ j,p=1,\dots,n \qquad (3.4)\\ &= \langle EF, A\rangle_{\mathcal{G},\mathcal{H}} = \langle C_1(EF)^T, C_1(A)\rangle_{C_1(\mathcal{G}),C_1(\mathcal{H})}.\end{aligned}$$

(ii) The following representation can be derived generalizing known results from [23] and [27]:

$$(\mathcal{H}\cdot E\cdot \operatorname{adj}(F\cdot\mathcal{G}\cdot A\cdot\mathcal{H}\cdot E)\cdot F\cdot\mathcal{G})_{ij} = \left|(\mathcal{H}\cdot E\cdot F\cdot\mathcal{G})^{\beta}_{\alpha}\right| \frac{\partial}{\partial a_{ji}} \left|A^{\alpha}_{\beta}\right|,\ (\alpha,\beta)\in\mathcal{N}_s(j,i)$$

Since $\frac{\partial}{\partial a_{ji}}\left|A^{\alpha}_{\beta}\right| = 0$, for $j\notin\alpha$ or $i\notin\beta$, we obtain

$$\begin{aligned}(\mathcal{H}\cdot E\cdot \operatorname{adj}(F\cdot\mathcal{G}\cdot A\cdot\mathcal{H}\cdot E)\cdot F\cdot\mathcal{G})_{ij} &= \left|(\mathcal{H}\cdot E\cdot F\cdot\mathcal{G})^{\beta}_{\alpha}\right| \frac{\partial}{\partial a_{ji}} \left|A^{\alpha}_{\beta}\right|,\ (\alpha,\beta)\in\mathcal{N}_s \\ &= \left|(\mathcal{H}\cdot E\cdot F\cdot\mathcal{G})^{\beta}_{\alpha}\right| \left|\left(\operatorname{ad}(A^{\alpha}_{\beta})\right)_{ij}\right|,\ (\alpha,\beta)\in\mathcal{N}_s \\ &= \operatorname{Tr}\left(C_s(\mathcal{H}\cdot E\cdot F\cdot\mathcal{G})\cdot\frac{\partial C_s}{\partial a_{ji}}(A)\right) \\ &= \langle C_s(\mathcal{H}\cdot E\cdot F\cdot\mathcal{G})^T, \frac{\partial C_s}{\partial a_{ji}}(A)\rangle \\ &= \langle C_s(\mathcal{G})\cdot C_s(E\cdot F)^T\cdot C_s(\mathcal{H}), \frac{\partial C_s}{\partial a_{ji}}(A)\rangle.\end{aligned} \tag{3.5}$$

This part of the proof can be finished from Lemma 2.1.

In the case $s=1$ the tensors E and F are of the form (3.3), and obtain

$$\begin{aligned}(\mathcal{H}\cdot E\cdot \operatorname{adj}(F\cdot\mathcal{G}\cdot A\cdot\mathcal{H}\cdot E)\cdot F\cdot\mathcal{G})_{ij} &= (\mathcal{H}\cdot E\cdot F\cdot\mathcal{G})_{ij} \\ &= \langle C_1(E\cdot F)^T, \frac{\partial C_1}{\partial a_{ji}}(A)\rangle_{C_1(\mathcal{G}),C_1(\mathcal{H})}.\end{aligned} \tag{3.6}$$

Part **(iii)** follows from **(i)** and **(ii)**. □

In the following theorem we introduce the main result. This theorem gives the answer to the following question: given $m\times n$ second rank tensor A of rank $0<s\le r$ in the Riemannian subspace $R_n\subset R_m$; determine a general form of non-zero outer inverses of A and necessary and sufficient conditions for its existence in terms of the double inner product, using $s\times s$ minors of A, an appropriate $n\times m$ second rank tensor and two metric tensors of dimensions m and n.

Theorem 3.1 *Assume that $A=[a_{ij}(x)]$ is a given $m\times n$ second rank tensor of matrix rank r over $R_n\subset R_m$, s is arbitrary integer satisfying $s\le r$. Then the following statements are equivalent:*

(i) *$X=[x_{ij}(x)]$, $x\in R_n\subset R_m$ is $\{2\}$-inverse of A of rank $s>0$.*

(ii) *There exist two full-matrix rank second rank tensors $E=[e_{ij}(x)]$, $x\in R_s\subset R_n$, $F=[f_{ij}(x)]$, $x\in R_s\subset R_m$ of dimensions $n\times s$ and $s\times m$, respectively, and two symmetric invertible tensors $\mathcal{G}=[g^{ij}(x)]$, $x\in R_m$ and $\mathcal{H}=[h^{ij}(x)]$, $x\in R_n$ of dimensions $m\times m$ and $n\times n$, such that $C_s(\mathcal{G})$ and $C_s(\mathcal{H})$ are two metric tensors in $R_{\binom{m}{s}}$ and $R_{\binom{n}{s}}$, respectively, such that*

$$\langle C_s(E\cdot F)^T, C_s(A)\rangle_{C_s(\mathcal{G}),C_s(\mathcal{H})} = \left|\mathcal{H}^{K\delta}\right| \left|(E\cdot F)_{K\gamma}\right| \left|\mathcal{G}^{S\gamma}\right| \left|A_{S\delta}\right|,\ (\gamma,\delta),(S,K)\in\mathcal{N}_s. \tag{3.7}$$

is invertible.

and $x_{ij}\in X$, $i=1,\dots,n$, $j=1,\dots,m$ possesses the following representation:

$$\begin{aligned}x_{ij} &= \frac{\langle C_s(E\cdot F)^T, \frac{\partial C_s}{\partial a_{ji}}(A)\rangle_{C_s(\mathcal{G}),C_s(\mathcal{H})}}{\langle C_s(E\cdot F)^T, C_s(A)\rangle_{C_s(\mathcal{G}),C_s(\mathcal{H})}} \\ &= \frac{\left|\mathcal{H}^{K\delta}\right| \left|(E\cdot F)_{K\gamma}\right| \left|\mathcal{G}^{S\gamma}\right| \left|(\operatorname{ad}(A_{S\delta}))_{ij}\right|}{\left|\mathcal{H}^{K\delta}\right| \left|(E\cdot F)_{K\gamma}\right| \left|\mathcal{G}^{S\gamma}\right|\ \left|A_{S\delta}\right|},\ (\gamma,\delta),(S,K)\in\mathcal{N}_s.\end{aligned} \tag{3.8}$$

(iii) *There exists a second rank tensor $W=[w_{ij}(x)]$, $x\in R_n\subset R_m$ of matrix rank s and of dimensions $n\times m$ as well as two symmetric invertible tensors $\mathcal{G}=[g^{ij}(x)]$, $x\in R_m$ and $\mathcal{H}=[h^{ij}(x)]$, $x\in R_n$ of dimensions $m\times m$ and $n\times n$, such that $C_s(\mathcal{G})$ and $C_s(\mathcal{H})$ are two metric tensors in $R_{\binom{m}{s}}$ and $R_{\binom{n}{s}}$, respectively, satisfying*

$$\langle C_s(W)^T, C_s(A)\rangle_{\mathcal{G},\mathcal{H}} = \left|\mathcal{H}^{K\delta}\right|\left|W_{K\gamma}\right|\left|\mathcal{G}^{S\gamma}\right|\left|A_{S\delta}\right| \neq 0,\ (\gamma,\delta),(S,K)\in\mathcal{N}_s \tag{3.9}$$

and any element $x_{ij}\in X$, $i=1,\ldots,n$, $j=1,\ldots,m$ can be represented as follows:

$$\begin{aligned} x_{ij} &= \frac{\langle C_s(W)^T, \frac{\partial C_s}{\partial a_{ji}}(A)\rangle_{C_s(\mathcal{G}),C_s(\mathcal{H})}}{\langle C_s(W)^T, C_s(A)\rangle_{C_s(\mathcal{G}),C_s(\mathcal{H})}} \\ &= \frac{\left|\mathcal{H}^{K\delta}\right|\left|W_{K\gamma}\right|\left|\mathcal{G}^{S\gamma}\right|\left|(\mathrm{ad}(A_{S\delta}))_{ij}\right|}{\left|\mathcal{H}^{K\delta}\right|\left|W_{K\gamma}\right|\left|\mathcal{G}^{S\gamma}\right|\left|A_{S\delta}\right|},\ (\gamma,\delta),(S,K)\in\mathcal{N}_s. \end{aligned} \tag{3.10}$$

(iv) *There exist two second rank tensors $E=[e_{ij}(x)]$, $x\in R_s\subset R_n$, $F=[f_{ij}(x)]$, $x\in R_s\subset R_m$ of full- matrix rank and of dimensions $n\times s$ and $s\times m$, respectively, and two symmetric invertible tensors $\mathcal{G}=[g^{ij}(x)]$, $x\in R_m$, $\mathcal{H}=[h^{ij}(x)]$, $x\in R_n$ of dimensions $m\times m$ and $n\times n$, satisfying that $C_s(\mathcal{G})$ and $C_s(\mathcal{H})$ are two metric tensors in $R_{\binom{m}{s}}$ and $R_{\binom{n}{s}}$, respectively, such that*

$$\left|F\cdot\mathcal{G}\cdot A\cdot\mathcal{H}\cdot E\right| \tag{3.11}$$

is invertible, and

$$X=\mathcal{H}\cdot E\cdot(F\cdot\mathcal{G}\cdot A\cdot\mathcal{H}\cdot E)^{-1}\cdot F\cdot\mathcal{G}. \tag{3.12}$$

(v) *There exist a second rank tensor $B=[b_{\beta,\alpha}(x)]$, $x\in R_{\binom{n}{s}}\subset R_{\binom{m}{s}}$ of dimensions $\binom{n}{s}\times\binom{m}{s}$ and of the matrix rank 1, two symmetric invertible second rank tensors $\mathcal{G}=[g^{ij}(x)]$, $x\in R_m$ and $\mathcal{H}=[h^{ij}(x)]$, $x\in R_n$ of dimensions $m\times m$ and $n\times n$, satisfying that $C_s(\mathcal{G})$ and $C_s(\mathcal{H})$ are two fundamental tensors in $R_{\binom{m}{s}}$ and $R_{\binom{n}{s}}$, respectively, such that the following condition holds*

$$\begin{aligned} \langle B^T, C_s(A)\rangle_{C_s(\mathcal{G}),C_s(\mathcal{H})} &= \left|\mathcal{H}^{K\delta}\right| b_{K\gamma}\left|\mathcal{G}^{S\gamma}\right|\left|A_{S\delta}\right|,\ (\gamma,\delta),(S,K)\in\mathcal{N}_s \\ &= 1, \end{aligned} \tag{3.13}$$

and $x_{ij}\in X$, $1\le i\le n$, $1\le j\le m$ is equal to

$$\begin{aligned} x_{ij} &= \langle B^T, \frac{\partial C_s}{\partial a_{ji}}(A)\rangle_{C_s(\mathcal{G}),C_s(\mathcal{H})} \\ &= \left|\mathcal{H}^{K\delta}\right| b_{K\gamma}\left|\mathcal{G}^{S\gamma}\right|\left|(\mathrm{ad}(A_{S\delta}))_{ij}\right|,\quad (\gamma,\delta),(S,K)\in\mathcal{N}_s. \end{aligned} \tag{3.14}$$

(vi) *There exist symmetric invertible second rank tensors $\mathcal{G}=[g^{ij}(x)]$, $x\in R_m$ and $\mathcal{H}=[h^{ij}(x)]$, $x\in R_n$ of dimensions $m\times m$ and $n\times n$, satisfying that $C_s(\mathcal{G})$ and $C_s(\mathcal{H})$ are two metric tensors in $R_{\binom{m}{s}}$ and $R_{\binom{n}{s}}$, respectively, such that the following condition is satisfied*

$$\begin{aligned} 1 &= \langle C_s(X)^T, C_s(A)\rangle_{C_s(\mathcal{G}),C_s(\mathcal{H})} \\ &= \left|\mathcal{H}^{K\delta}\right|\left|X_{K\gamma}\right|\left|\mathcal{G}^{S\gamma}\right|\left|A_{S\delta}\right|,\ (\gamma,\delta),(S,K)\in\mathcal{N}_s \end{aligned} \tag{3.15}$$

and arbitrary (i,j)-th element x_{ij} from X, $1\le i\le n$, $1\le j\le m$ possesses the form

$$\begin{aligned} x_{ij} &= \langle C_s(X)^T, \frac{\partial C_s}{\partial a_{ji}}(A)\rangle_{C_s(\mathcal{G}),C_s(\mathcal{H})} \\ &= \left|\mathcal{H}^{K\delta}\right|\left|X_{K\gamma}\right|\left|\mathcal{G}^{S\gamma}\right|\left|(\mathrm{ad}(A_{S\delta}))_{ij}\right|,\quad (\gamma,\delta),(S,K)\in\mathcal{N}_s. \end{aligned} \tag{3.16}$$

Proof. (i) $\Rightarrow$ (ii). Assume that $X = [x_{ij}]$, $i = 1,\dots,n$, $j = 1,\dots,n$ is $\{2\}$-inverse of A. Let us choose $\mathcal{G} = I_m$, $\mathcal{H} = I_n$. Using known full-rank representation of $\{2\}$-inverses from [27] (which is valid in that case) we conclude that there exist tensors $E \in R_s \subset R_n$ of dimensions $n \times s$ and $F \in R_s \subset R_m$ of dimensions $s \times m$, both of the matrix rank equal to s, such that $F{\cdot}A{\cdot}E$ is invertible and $X = E{\cdot}(F{\cdot}A{\cdot}E)^{-1}{\cdot}F$ can be expressed in the form (1.1), or equivalently in the following inner product representation in the Euclidean space:

$$x_{ij} = \frac{\langle C_s(E{\cdot}F)^T, \frac{\partial C_s}{\partial a_{ji}}(A)\rangle}{\langle C_s(E{\cdot}F)^T, C_s(A)\rangle}, \quad \langle C_s(E{\cdot}F)^T, C_s(A)\rangle \neq 0.$$

Now, using (2.6) and Lemma 3.1, we conclude that the condition (3.7) is satisfied in the particular case $\mathcal{G} = I_m$, $\mathcal{H} = I_n$ and x_{ij} is represented in the form (3.8) in the same particular case.

(ii) $\Rightarrow$ (iii). Let us choose full-matrix rank tensors $E \in R_s \subset R_n$, $F \in R_s \subset R_m$ of dimensions $n \times s$ and $s \times m$, respectively, as well as two symmetric invertible tensors $\mathcal{G}$ and $\mathcal{H}$ of dimensions $m \times m$ and $n \times n$, respectively, such that conditions (3.7) and (3.8) are satisfied. Then the matrix $W = E{\cdot}F \in R_n \subset R_m$ of dimensions $n \times m$ satisfies the conditions imposed in (3.9) and (3.10).

(iii) $\Rightarrow$ (i). Assume that W satisfies conditions (3.9) and (3.10) in (iii). Also, let $W = E{\cdot}F$ is a full-rank factorization of W, where $E = [e_{ij}(x)]$, $x \in R_s \subset R_n$ and $F = [f_{ij}(x)]$, $x \in R_s \subset R_m$ are of dimensions $n \times s$ and $s \times m$, respectively, and both of the matrix rank equal to s. Then the condition (3.9) guarantees

$$\langle C_s(W)^T, C_s(A)\rangle_{C_s(\mathcal{G}),C_s(\mathcal{H})} = \left|F{\cdot}\mathcal{G}{\cdot}A{\cdot}\mathcal{H}{\cdot}E\right| \neq 0$$

and existence of the inverse $(\mathcal{G}{\cdot}F{\cdot}A{\cdot}\mathcal{H}{\cdot}E)^{-1}$. Consider the matrix $X = [x_{ij}]$, $i = 1,\dots,n,\ \ j = 1,\dots,m$ whose elements are defined by (3.10). Using the results from Lemma 3.1, X can be expressed in the following general representation:

$$X = \mathcal{H}{\cdot}E{\cdot}(F{\cdot}\mathcal{G}{\cdot}A{\cdot}\mathcal{H}{\cdot}E)^{-1}{\cdot}F{\cdot}\mathcal{G}.$$

Now, equation $X{\cdot}A{\cdot}X = X$ can be easily verified.

(ii) $\Leftrightarrow$ (iv). Follows from Lemma 3.1.

(iii) $\Rightarrow$ (v). Assume that there exists matrix W together with two metric tensors $\mathcal{G}$ and $\mathcal{H}$ which satisfy conditions from (iii). Then the matrix

$$B = \left(\langle C_s(W)^T, C_s(A)\rangle_{C_s(\mathcal{G}),C_s(\mathcal{H})}\right)^{-1} C_s(W)$$

is of dimensions $\binom{n}{s} \times \binom{m}{s}$ and satisfies (3.13) and (3.14). Further, since $\rho(W) = s$ we have $\rho(B) = \rho(W) = 1$ (see, for example [7]). Therefore, the matrix B satisfies conditions imposed in (v).

(v) $\Rightarrow$ (i). Assume that the matrix B satisfy conditions from (v). Since (3.13) and (3.14) are satisfied, we have

$$(XA)_{il} = \left|\mathcal{H}^{K\delta}\right| b_{K\gamma} \left|\mathcal{G}^{S\gamma}\right| \sum_{t=1}^{m} a_{tl} \left|(\mathrm{ad}(A_{S\delta}))_{it}\right|, \quad (\gamma,\delta),(S,K) \in \mathcal{N}_s$$

and

$$(XAX)_{ij} = \sum_{l=1}^{n} \left(\left|\mathcal{H}^{K\delta}\right| b_{K\gamma} \left|\mathcal{G}^{S\gamma}\right| \sum_{t=1}^{m} a_{tl} \left|(\mathrm{ad}(A_{S\delta}))_{it}\right| \right) \left(\left|\mathcal{H}^{K_1\delta_1}\right| b_{K_1\gamma_1} \left|\mathcal{G}^{S_1\gamma_1}\right| \left|(\mathrm{ad}(A_{S_1\delta_1}))_{lj}\right| \right),$$
$$(\gamma,\delta),(S,K) \in \mathcal{N}_s,\ (\gamma_1,\delta_1),(S_1,K_1) \in \mathcal{N}_s$$

Consider the tensor Λ defined as $\Lambda = C_s(\mathcal{H}){\cdot}B{\cdot}C_s(\mathcal{G}) = [\lambda^{\delta\gamma}]$, $(\gamma,\delta) \in \mathcal{N}_s$. It is not difficult to verify that Λ is of dimensions $\binom{n}{s} \times \binom{m}{s}$ and of the matrix rank 1. We have

$$(XAX)_{ij} = \sum_{l=1}^{n} \left(\lambda^{\delta S} \sum_{t=1}^{m} a_{tl} \left|(\mathrm{ad}(A_{S\delta}))_{it}\right| \right) \left(\lambda^{\delta_1 S_1} \left|(\mathrm{ad}(A_{S_1\delta_1}))_{lj}\right| \right),$$
$$(S,\delta) \in \mathcal{N}_s,\ (S_1,\delta_1) \in \mathcal{N}_s.$$

Since $\rho(\Lambda)=1$ using known result from [14] we get

$$\lambda^{\delta S}\lambda^{\delta_1 S_1}=\lambda^{\delta_1 S}\lambda^{\delta S_1}$$

and later

$$\begin{aligned}(XAX)_{ij}&=\lambda^{\delta_1 S}\lambda^{\delta S_1}\sum_{l\in S_1}\left(\left|A_{S\delta(i\leftarrow l)}\right|\left|(\mathrm{ad}(A_{S_1\delta_1}))_{lj}\right|\right),\quad (S,\delta_1)\in\mathcal{N}_s,\ (S_1,\delta)\in\mathcal{N}_s\\ &=\lambda^{\delta_1 S}\left(\lambda^{\delta S_1}\left|A_{S_1\delta}\right|\right)\left|(\mathrm{ad}(A_{S\delta_1}))_{ij}\right|,\quad (\delta_1,S)\in\mathcal{N}_s,\ (\delta,S_1)\in\mathcal{N}_s.\end{aligned}$$

Using

$$\begin{aligned}\lambda^{\delta S_1}\left|A_{\delta S_1}\right|&=\left|\mathcal{H}^{K\delta}\right|b_{K\gamma}\left|\mathcal{G}^{S_1\gamma}\right|\left|A_{S\delta}\right|,\ (\gamma,\delta),(S_1,K)\in\mathcal{N}_s\\ &=1,\end{aligned}$$

we obtain

$$\begin{aligned}(XAX)_{ij}&=\lambda^{\delta_1 S}\left|(\mathrm{ad}(A_{S\delta_1}))_{ij}\right|,\quad (S,\delta_1)\in\mathcal{N}_s\\ &=\left|\mathcal{H}^{K\delta_1}\right|b_{K\gamma}\left|\mathcal{G}^{S\gamma}\right|\left|(\mathrm{ad}(A_{S\delta_1}))_{ij}\right|,\quad (\gamma,\delta),(S,K)\in\mathcal{N}_s\\ &=x_{ij}.\end{aligned}$$

Therefore, (i) is satisfied.

(i) $\Rightarrow$ (vi). It suffices to use the following result from [27], in the case $\mathcal{G}=I_m$, $\mathcal{H}=I_n$:

$$X\cdot A\cdot X=X\Leftrightarrow\ \mathrm{Tr}(C_s(X\cdot A))=1,\ X=A_{C_s(X)},$$

together with

$$\mathrm{Tr}(C_s(X\cdot A))=\langle C_s(X)^T,C_s(A)\rangle=1$$

and

$$\begin{aligned}x_{ij}=\left(A_{C_s(X)}\right)_{ij}&=\langle C_s(X)^T,\frac{\partial C_s}{\partial a_{ji}}(A)\rangle\\ &=\left|X^{\delta}_{\gamma}\right|\left|(\mathrm{ad}(A^{\gamma}_{\delta}))_{ij}\right|,\ (\gamma,\delta),(S,K)\in\mathcal{N}_s.\end{aligned}$$

Therefore, conditions (3.15) and (3.16) are satisfied in the case $\mathcal{G}=I_m$, $\mathcal{H}=I_n$.

(vi) $\Rightarrow$ (i). Assume that conditions from (vi) are satisfied. Choose an arbitrary full-rank factorization $X=M\cdot N$ of X. According to (3.16) and Lemma 3.1 we get

$$X=\mathcal{H}\cdot M\cdot(N\cdot\mathcal{G}\cdot A\cdot\mathcal{H}\cdot M)^{-1}\cdot N\cdot\mathcal{G}. \tag{3.17}$$

From (3.15) we have the following conclusion

$$\left|N\cdot\mathcal{G}\cdot A\cdot\mathcal{H}\cdot M\right|=1\Rightarrow\rho(N\cdot\mathcal{G}\cdot A\cdot\mathcal{H}\cdot M)=s>0.$$

Now, equation $X\cdot A\cdot X=X$ and $\rho(X)=s$ can be easily verified from (3.17). $\square$

In the following theorem we derive an additional characterization of the set of outer inverses by means of metric tensors.

Corollary 3.1 *Assume that $A=[a_{ij}(x)]$, $x\in R_n\subset R_m$ is miscellaneous $m\times n$ tensor of rank 2 and the matrix rank equal to r. Also, suppose that $\mathcal{G}$, $\mathcal{H}$ are two symmetric invertible tensors of dimensions $m\times m$ and $n\times n$, such that $C_s(\mathcal{G})$ and $C_s(\mathcal{H})$ are two fundamental tensors in $R_{\binom{m}{s}}$ and $R_{\binom{n}{s}}$, respectively. The set of outer inverses of A possesses the following representation:*

$$A\{2\}=\{\mathcal{H}\cdot(\mathcal{G}\cdot A\cdot\mathcal{H})\{1,2\}\cdot\mathcal{G}\}. \tag{3.18}$$

Proof. Consider an arbitrary outer inverse $X \in A\{2\}$. According to Theorem 3.1, X is of the general form (3.12). Since

$$E\cdot(F\cdot\mathcal{G}\cdot A\cdot\mathcal{H}\cdot E)^{-1}\cdot F \in (\mathcal{G}\cdot A\cdot\mathcal{H})\{1,2\}$$

we get

$$X \in \{\mathcal{H}\cdot(\mathcal{G}\cdot A\cdot\mathcal{H})\,\{1,2\}\cdot\mathcal{G}\}\,.$$

On the other hand, in the case $X = \mathcal{H}\cdot(\mathcal{G}\cdot A\cdot\mathcal{H})^{(1,2)}\cdot\mathcal{G}$, for some reflexive g-inverse of $\mathcal{G}\cdot A\cdot\mathcal{H}$, the equation $X\cdot A\cdot X = X$ evidently holds. $\square$

Remark 3.1 *In some partial cases of Theorem 3.1, we immediately obtain analogous characterizations and representations of the Moore-Penrose inverse, group inverse and the weighted Moore-Penrose inverse for the set of tensors of rank 2 in the Riemannian space $R_n \subset R_m$. Selecting $\mathcal{G} = I_m$, $\mathcal{H} = I_n$, the Moore-Penrose inverse can be obtained in the case $W = A^T$ and the group inverse is selected with $W = A$, while the Drazin inverse arises in the case $W = A^k$, $k \geq \mathrm{ind}(A)$. The weighted Moore-Penrose inverse can be characterized in two cases:*

(a) $\mathcal{G} = I_m$, $\mathcal{H} = I_n$, $W = (M\cdot A\cdot N^{-1})^T$;

(b) $W = A^T$, $\mathcal{G} = M$, $\mathcal{H} = N^{-1}$.

Remark 3.2 *The general rule for generating the Moore-Penrose inverse is*

$$\mathcal{H}\cdot E\cdot F\cdot\mathcal{G} = A^T.$$

Similarly, the weighted Moore-Penrose inverse and the group inverse can be generated in the general case

$$\mathcal{H}\cdot E\cdot F\cdot\mathcal{G} = (M\cdot A\cdot N^{-1})^T = N^{-1}\cdot A^T\cdot M$$

and

$$\mathcal{H}\cdot E\cdot F\cdot\mathcal{G} = A,$$

respectively. In the case

$$\mathcal{H}\cdot E\cdot F\cdot\mathcal{G} = A^k,\ \ k \geq \mathrm{ind}(A)$$

we obtain an analogy of the determinantal representation of the Drazin inverse from [26] in the Riemannian space.

Remark 3.3 *In the particular case $\mathcal{H} = N^{-1}$, $\mathcal{G} = M$, $E = Q^*, F = P^*$, where $A = PQ$ is a full-rank factorization of A and M, N are symmetric invertible matrices, reducing the Riemannian space into the space $\mathbb{F}^n$ with the indefinite scalar product, the general representation* (3.12) *reduces to the general representation of the generalized weighted Moore-Penrose inverse which is given in Theorem 3.2 from* [24]*. Further, in the case when the matrices M and N are positive definite, from Theorem 3.1 we derive known general and the determinantal representation of the weighted Moore-Penrose inverse.*

In the following theorem we generalize results from [23] into the Riemannian space.

Theorem 3.2 *Assume that $A = [a_{ij}(x)]$ and $W = [w_{ij}(x)]$ are given $m \times n$ and $n \times m$ tensors of rank 2 and of matrix rank r and $s \leq r$, respectively, over $R_n \subset R_m$. Also, suppose that $W = E\cdot F$ is a full-matrix rank factorization of W, where $E \in R_s \subset R_n$ of dimensions $n \times s$ and $F \in R_s \subset R_m$ of dimensions $s \times m$ are two tensors of matrix rank equal to s. Then the following conditions are equivalent:*

(i) *There exist symmetric invertible second rank tensors $\mathcal{G} = [g^{ij}(x)]$, $x \in R_m$ and $\mathcal{H} = [h^{ij}(x)]$, $x \in R_n$ of dimensions $m \times m$ and $n \times n$, satisfying that $C_s(\mathcal{G})$ and $C_s(\mathcal{H})$ are two fundamental tensors in $R_{\binom{m}{s}}$ and $R_{\binom{n}{s}}$, respectively, such that* (3.7) *is satisfied and $x_{ij} \in X$, $i = 1,\dots,n$, $j = 1,\dots,m$ can be expressed as in* (3.8)*.*

(ii) *If A has $\{2\}$-inverse $A^{(2)}_{\mathcal{H}\cdot\mathcal{R}(E),\mathcal{G}^{-1}\cdot\mathcal{N}(F)}$, then it possesses the full-matrix rank representation*

$$A^{(2)}_{\mathcal{H}\cdot\mathcal{R}(E),\mathcal{G}^{-1}\cdot\mathcal{N}(F)} = \mathcal{H}\cdot E\cdot(F\cdot\mathcal{G}\cdot A\cdot\mathcal{H}\cdot E)^{-1}\cdot F\cdot\mathcal{G}. \tag{3.19}$$

Proof. (i) $\Rightarrow$ (ii). Assume that $A^{(2)}_{\mathcal{H}\cdot\mathcal{R}(E),\mathcal{G}^{-1}\cdot\mathcal{N}(F)}$ exists and X satisfies conditions from (i). Using the result (iii) from Lemma 3.1, it is clear that X can be represented in the form

$$X = \mathcal{H}\cdot E\cdot(F\cdot\mathcal{G}\cdot A\cdot\mathcal{H}\cdot E)^{-1}\cdot F\cdot\mathcal{G}.$$

Therefore, $X\cdot A\cdot X = X$. Since $\mathcal{G}$, $\mathcal{H}$ are invertible, E is full-column rank and F possesses full-row rank, it is easy to verify (see [3])

$$\begin{aligned}\mathcal{R}(X) &= \mathcal{R}(\mathcal{H}\cdot E\cdot(F\cdot\mathcal{G}\cdot A\cdot\mathcal{H}\cdot E)^{-1}\cdot F\cdot\mathcal{G}) = \mathcal{R}(\mathcal{H}\cdot E) = \mathcal{H}\cdot\mathcal{R}(E),\\ \mathcal{N}(X) &= \mathcal{N}(\mathcal{H}\cdot E\cdot(F\cdot\mathcal{G}\cdot A\cdot\mathcal{H}\cdot E)^{-1}\cdot F\cdot\mathcal{G}) = \mathcal{N}(F\cdot\mathcal{G}) = \mathcal{G}^{-1}\cdot\mathcal{N}(F).\end{aligned}$$

Therefore, since $A^{(2)}_{T,S}$ inverse is unique, we just verify $X = A^{(2)}_{\mathcal{H}\cdot\mathcal{R}(E),\mathcal{G}^{-1}\cdot\mathcal{N}(F)}$.

(ii) $\Rightarrow$ (i). Assume that $X = A^{(2)}_{\mathcal{H}\cdot\mathcal{R}(E),\mathcal{G}^{-1}\cdot\mathcal{N}(F)}$ exists. Consider the matrix $W = \mathcal{H}\cdot E\cdot F\cdot\mathcal{G}$. Since $\mathcal{R}(W) = \mathcal{R}(\mathcal{H}\cdot E) = \mathcal{H}\cdot\mathcal{R}(E)$ and $\mathcal{N}(W) = \mathcal{N}(F\cdot\mathcal{G}) = \mathcal{G}^{-1}\cdot\mathcal{N}(F)$ (see [3]), according to Theorem 3.1 from [23] (valid in the Euclidean space) we get $X = \mathcal{H}\cdot E\cdot(F\cdot\mathcal{G}\cdot A\cdot\mathcal{H}\cdot E)^{-1}\cdot F\cdot\mathcal{G}$. Now (i) follows from Lemma 3.1. $\square$

Remark 3.4 *Outer generalized inverse defined by* (3.12) *reduces to the usual outer inverse of the form $X = E\cdot(F\cdot A\cdot E)^{-1}\cdot F$ (investigated in [27]) in the Euclidean space, in the same way as the weighted Moore-Penrose inverse generalizes the Moore-Penrose inverse. And conversely, an arbitrary outer inverse in the Euclidean space, given in the form $X = E\cdot(F\cdot A\cdot E)^{-1}\cdot F$, translates by means of the symmetric invertible tensors $\mathcal{G}$ and $\mathcal{H}$ into the form $X = \mathcal{H}\cdot E\cdot(F\cdot\mathcal{G}\cdot A\cdot\mathcal{H}\cdot E)^{-1}\cdot F\cdot\mathcal{G}$, under the assumption that $C_s(\mathcal{G})$ and $C_s(\mathcal{H})$ are two fundamental tensors in $R_{\binom{m}{s}}$ and $R_{\binom{n}{s}}$, respectively. For this purpose, it is appropriately to use the notion* **weighted outer inverse** *of A for the matrix defined in* (3.12). *If the symmetric invertible tensors $\mathcal{G}$ and $\mathcal{H}$ are equal with identity tensors I_m and I_n, respectively, then* (3.19) *reduces to known full-rank representation of the inverse $A^{(2)}_{\mathcal{R}(E),\mathcal{N}(F)}$, originated in [23]. Therefore, it seems reasonable to use the notion* **weighted $A^{(2)}_{\mathcal{R}(E),\mathcal{N}(F)}$ inverse** *for $A^{(2)}_{\mathcal{H}\mathcal{R}(E),\mathcal{G}^{-1}\mathcal{N}(F)}$.*

In the rest of this section we introduce an additional determinantal representation of outer inverses. For this purpose we introduce a generalization of the k-volume, originated in [4], [5].

Definition 3.2 *Consider tensor $A = [a_{ij}(x)]$, $x \in R^r_n \subset R_m$ of rank 2 and dimensions $m \times n$, and choose arbitrary tensor $W = [w_{ij}(x)]$, $x \in R^s_n \subset R_m$ of rank 2 and dimensions $n \times m$. Assume that k is arbitrary integer satisfying $1 \le k \le r$. The k-volume of the matrix A associated with the matrix W and with symmetric invertible second rank tensors $\mathcal{G} = [g^{ij}(x)]$ and $\mathcal{H} = [h^{ij}(x)]$ of dimensions $m \times m$ and $n \times n$, denoted by $\mathrm{vol}_k(A, W)$, is defined as the following square root of the double inner product of the second rank tensors $C_k(W)^T$ and $C_k(A)$:*

$$\mathrm{vol}_k(A, W) = \sqrt{\langle C_k(W)^T, C_k(A)\rangle_{C_s(\mathcal{G}),C_s(\mathcal{H})}}, \tag{3.20}$$

where $C_s(\mathcal{G})$ and $C_s(\mathcal{H})$ are two metric tensors in $R_{\binom{m}{s}}$ and $R_{\binom{n}{s}}$, respectively.

Remark 3.5 *For any second rank tensor $A = [a_{ij}(x)]$, $x \in R^r_n \subset R_m$ of dimensions $m \times n$, any integer k satisfying $1 \le k \le r$ and fundamental tensors $\mathcal{G} = I_m$, $\mathcal{H} = I_n$, we obtain*

$$\mathrm{vol}_k(A, A^*) = \mathrm{vol}_k(A),$$

where $\mathrm{vol}_k(A)$ is the k-volume of A, defined in [4], [5].

Definition 3.3 *Consider the second rank tensor $A = [a_{ij}(x)]$, $x \in R_n^r \subset R_m$ of dimensions $m \times n$ and symmetric invertible second rank tensors $\mathcal{G} = [g^{ij}(x)]$, $x \in R_m$, $\mathcal{H} = [h^{ij}(x)]$, $x \in R_n$ of dimensions $m\times m$ and $n\times n$, assuming that $C_s(\mathcal{G})$ and $C_s(\mathcal{H})$ are two metric tensors in $R_{\binom{m}{s}}$ and $R_{\binom{n}{s}}$, respectively. The generalized adjoint matrix of A with respect to the generalized k-volume $\mathrm{vol}_k(A, W)$, $W \in R_s^r \subset R_m$ of dimensions $n \times m$, $s \le r$, $k \le s$, denoted by $\mathrm{gadj}(\mathrm{vol}_k^2(A, W))$, is $n\times m$ matrix whose (i, j)th element, $1 \le i \le n$, $1 \le j \le m$ is equal to*

$$\big(\mathrm{gadj}(\mathrm{vol}_k^2(A, W))\big)_{ij} = \langle C_k(W)^T, \frac{\partial C_k}{\partial a_{ji}}(A)\rangle_{C_s(\mathcal{G}),C_s(\mathcal{H})}.$$

Theorem 3.3 *The set of non-zero $\{2\}$-inverses for $A \in R_n^r \subset R_m$ of dimensions $m \times n$ is equal to*

$$\begin{aligned} A\{2\} &= \left\{ \frac{1}{\mathrm{vol}_s^2(A, W)} \mathrm{gadj}(\mathrm{vol}_s^2(A, W)),\ W \in R_n^s \subset R_m,\ \mathrm{vol}_s^2(A, W) \neq 0,\ 0 < s \le r \right\} \\ &= \left\{ \mathrm{gadj}(\mathrm{vol}_s^2(A, B)),\ B \in R_n \subset R_m \text{ of dimensions } \binom{n}{s} \times \binom{m}{s}, \rho(B) = 1,\ \mathrm{vol}_s^2(A, W) = 1,\ 0 < s \le r \right\}, \end{aligned}$$

where symmetric invertible second rank tensors $\mathcal{G} = [g^{ij}]$, $g^{ij} \in R_m$, $\mathcal{H} = [h^{ij}]$, $h^{ij} \in R_n$ are of dimensions $m \times m$ and $n \times n$, and $C_s(\mathcal{G})$ and $C_s(\mathcal{H})$ are two fundamental tensors in $R_{\binom{m}{s}}$ and $R_{\binom{n}{s}}$, respectively.

Proof. Assume that $X \in A\{2\}$ possesses the matrix rank $s > 0$. According to Theorem 3.1, an arbitrary (i, j)th element x_{ij} of X can be expressed in the form (3.10). In view of Definition 2.5, the denominator of x_{ij} can be expressed as follows:

$$\langle C_s(W)^T, C_s(A)\rangle_{C_s(\mathcal{G}),C_s(\mathcal{H})} = \mathrm{vol}_s^2(A, W).$$

Also, according to Definition 2.3, for arbitrary integers $1 \le i \le n$, $1 \le j \le m$, the double inner product

$$\langle C_s(W)^T, \frac{\partial C_s}{\partial a_{ji}}(A)\rangle_{C_s(\mathcal{G}),C_s(\mathcal{H})}$$

is equal to (i, j)th element of the matrix $\mathrm{gadj}(\mathrm{vol}_s^2(A, W))$, which completes this part of the proof.

On the other hand, consider an arbitrary matrix $X = [x_{ij}]$, $x_{ij} \in R_n \subset R_m$, of dimensions $n \times m$, defined by

$$X = \frac{1}{\mathrm{vol}_s^2(A, W)} \mathrm{gadj}(\mathrm{vol}_s^2(A, W)), \quad W \in R_n^s \subset R_m,\ \mathrm{vol}_s^2(A, W) \neq 0,\ s > 0.$$

For arbitrary integers $1 \le i \le n$, $1 \le j \le m$, it is not difficult to express $x_{ij} \in X$ in the form (3.10). Using the part (i) $\Leftrightarrow$ (ii) of Theorem 3.1, we immediately obtain $X \in A\{2\}$ and $\rho(X) = s$.

The second equality in theorem can be verified in a similar way, using the part (i) $\Leftrightarrow$ (v) of Theorem 3.1. $\square$

4 Examples

Example 4.1 *Consider the second rank tensor*

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

of dimensions 4×4 *and of matrix rank* 2. *Also, consider the following tensor* W:

$$W=\begin{bmatrix} -1 & 0 & 2 & 0\\ 0 & 0 & -1 & 0\\ 1 & 0 & 2 & 0\\ 1 & 0 & 0 & 0\end{bmatrix}$$

of the same matrix rank. If the tensors $\mathcal{G}$ *and* $\mathcal{H}$ *are chosen as* $\mathcal{G}=\mathcal{H}=I_4$, *by means of Theorem 3.1 we get the following outer inverse of* A:

$$X=\begin{bmatrix} -1 & 0 & 0 & 0\\ \frac{1}{4} & 0 & -\frac{1}{8} & 0\\ 0 & 0 & \frac{1}{2} & 0\\ \frac{1}{2} & 0 & \frac{1}{4} & 0\end{bmatrix}.$$

In the case $W=A^T$ *we get the following Moore-Penrose inverse of* A:

$$A^{\dagger}=\begin{bmatrix} -\frac{1}{2} & 0 & 0 & \frac{1}{2}\\ 0 & 0 & 0 & 0\\ 0 & -\frac{1}{5} & \frac{2}{5} & 0\\ 0 & 0 & 0 & 0\end{bmatrix}.$$

Later, if we use $\mathcal{H}=I_4$ *and*

$$\mathcal{G}=\begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{bmatrix}.$$

In this case we have $C_2(\mathcal{G})=I_6$, *and applying Theorem 3.1 we generate the following outer inverse of* A:

$$X=\begin{bmatrix} -1 & 0 & 0 & 0\\ 0 & 0 & \frac{1}{4} & 0\\ 0 & 0 & \frac{1}{2} & 0\\ -1 & 0 & 0 & 0\end{bmatrix}.$$

Now, let us choose the following symmetric invertible tensors:

$$G=H=\begin{bmatrix} e^{2cx_1} & 0 & 0 & 0\\ 0 & e^{2cx_2} & 0 & 0\\ 0 & 0 & e^{2cx_3} & 0\\ 0 & 0 & 0 & e^{2cx_4}\end{bmatrix},$$

where $c\in\mathbb{R}$ *is a real constant and* $x_1,\dots,x_4$ *are real variables. Then* $C_2(\mathcal{G})$ *and* $C_2(\mathcal{H})$ *are of the form of metric tensors from* [6]:

$$C_2(\mathcal{G})=C_2(\mathcal{H})=\begin{bmatrix} e^{2cx1+2cx2} & 0 & 0 & 0 & 0 & 0\\ 0 & e^{2cx1+2cx3} & 0 & 0 & 0 & 0\\ 0 & 0 & e^{2cx1+2cx4} & 0 & 0 & 0\\ 0 & 0 & 0 & e^{2cx2+2cx3} & 0 & 0\\ 0 & 0 & 0 & 0 & e^{2cx2+2cx4} & 0\\ 0 & 0 & 0 & 0 & 0 & e^{2cx3+2cx4}\end{bmatrix}.$$

Applying Theorem 3.1 we generate the following outer inverse of A:

$$X=\begin{bmatrix} -1 & 0 & 0 & 0\\ \frac{1}{4}e^{2cx_2-2cx_1} & 0 & -\frac{1}{8}e^{2cx_2-2cx_3} & 0\\ 0 & 0 & \frac{1}{2} & 0\\ \frac{1}{2}e^{2cx_4-2cx_1} & 0 & \frac{1}{4}e^{2cx_4-2cx_3} & 0\end{bmatrix}.$$

Example 4.2 *Consider the following* 6×5 *tensor of matrix rank 4, which is generated by putting* $a=1$ *in the test matrix* M_3 *from* [35, p. 143]:

$$A=\begin{bmatrix} 1 & 2 & 3 & 4 & 1\\ 1 & 3 & 4 & 6 & 2\\ 2 & 3 & 4 & 5 & 3\\ 3 & 4 & 5 & 6 & 4\\ 4 & 5 & 6 & 7 & 6\\ 6 & 6 & 7 & 7 & 8\end{bmatrix}.$$

Let us choose tensors E and F as in [27]:

$$E=\begin{bmatrix}0&0\\2&1\\3&2\\5&3\\1&0\end{bmatrix},\quad F=\begin{bmatrix}0&1&0&1&0&1\\1&0&1&0&1&0\end{bmatrix}. \tag{4.1}$$

Furthermore, let us choose $\mathcal{G}=b^2I_6$, $\mathcal{H}=a^2I_5$, *where* a *and* b *are unevaluated real numbers. Since* $\rho(F{\cdot}\mathcal{G}{\cdot}A{\cdot}\mathcal{H}{\cdot}E)=2$ *and*

$$|F{\cdot}\mathcal{G}{\cdot}A{\cdot}\mathcal{H}{\cdot}E|=174a^4b^4$$

we conclude that corresponding $\{2\}$*-inverse of* A *of rank 2 exist, and it is defined as in* (3.12). *Simple calculation in accordance with* (3.10) *or* (3.12) *gives the following outer inverse* X*:*

$$X=\frac{1}{174}\begin{bmatrix}0&0&0&0&0&0\\-21&19&-21&19&-21&19\\60&-46&60&-46&60&-46\\39&-27&39&-27&39&-27\\-102&84&-102&84&-102&84\end{bmatrix},$$

which coincides with the outer inverse obtained in [27].

In the case $E{\cdot}F=A^T$, *using* $s=r=\rho(A)=4$, *from* (3.10) *or* (3.12) *we obtain the following outer inverse of* A*:*

$$X=\begin{bmatrix}-\frac{1243}{5562}&\frac{13}{618}&-\frac{1031}{5562}&\frac{329}{5562}&\frac{27}{103}\\ \frac{49}{309}&-\frac{11}{309}&\frac{34}{309}&-\frac{26}{309}&-\frac{32}{309}\\ \frac{9}{206}&\frac{145}{1854}&-\frac{91}{1854}&-\frac{3}{206}&-\frac{35}{927}\\ -\frac{581}{5562}&-\frac{95}{1854}&-\frac{187}{5562}&\frac{109}{5562}&\frac{37}{309}\\ \frac{905}{5562}&\frac{11}{1854}&\frac{571}{5562}&-\frac{301}{5562}&-\frac{167}{927}\\ -\frac{31}{309}&-\frac{1}{103}&-\frac{38}{927}&\frac{46}{927}&\frac{40}{927}\end{bmatrix}.$$

In the case $G=I_6$, $H=I_5$ *we obtain the Moore-Penrose inverse as in [27]:*

$$A^{\dagger}=\left(\mathrm{Tr}(C_r(A^T{\cdot}A))\right)^{-1}A_{C_r(A^T)}=\frac{1}{8}\begin{bmatrix}4&-1&-8&7&-5&3\\-8&15&-36&23&-5&3\\10&-13&26&-15&1&-1\\-2&3&-2&1&1&-1\\-4&-2&12&-10&6&-2\end{bmatrix}.$$

Example 4.3 *Consider the following tensors* A *and* W *as in [23]:*

$$A=\begin{bmatrix}-1&2&1&0\\1&0&1&1\\-1&-3&1&2\end{bmatrix},\quad W=\begin{bmatrix}3&1&0\\-2&4&-2\\-5&-4&1\\0&7&-3\end{bmatrix}.$$

Let us choose $\mathcal{G}$ *and* $\mathcal{H}$ *as in the following:*

$$\mathcal{G}=\begin{bmatrix}\frac{1}{g_1^2}&0&0\\0&\frac{1}{g_2^2}&0\\0&0&\frac{1}{g_3^2}\end{bmatrix},\quad \mathcal{H}=\begin{bmatrix}\frac{1}{h_1^2}&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}.$$

Then $C_2(\mathcal{G})$ *and* $C_2(\mathcal{H})$ *are equal to*

$$C_2(\mathcal{G})=\begin{bmatrix}\frac{1}{g_1^2g_2^2}&0&0\\0&\frac{1}{g_1^2g_3^2}&0\\0&0&\frac{1}{g_2^2g_3^2}\end{bmatrix},\quad C_2(\mathcal{H})=\begin{bmatrix}\frac{1}{h_1^2}&0&0&0&0&0\\0&\frac{1}{h_1^2}&0&0&0&0\\0&0&\frac{1}{h_1^2}&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix},$$

and both are of the form of the general Riemannian metric matrix of diagonal type. Corresponding $\{2\}$*-inverse of* A *is equal to*

$$X=\begin{bmatrix} \frac{-3g_2^2-14g_3^2}{(h_1^2+1)g_1^2+6g_2^2(h_1^2+2)+7g_3^2(h_1^2+5)} & \frac{21g_3^2-g_1^2}{(h_1^2+1)g_1^2+6g_2^2(h_1^2+2)+7g_3^2(h_1^2+5)} & \frac{-2g_1^2-9g_2^2}{(h_1^2+1)g_1^2+6g_2^2(h_1^2+2)+7g_3^2(h_1^2+5)} \\ \frac{2(3g_2^2-7g_3^2(h_1^2-1))}{(h_1^2+1)g_1^2+6g_2^2(h_1^2+2)+7g_3^2(h_1^2+5)} & \frac{2(g_1^2+7g_3^2(2h_1^2+1))}{(h_1^2+1)g_1^2+6g_2^2(h_1^2+2)+7g_3^2(h_1^2+5)} & -\frac{2((h_1^2-1)g_1^2+3g_2^2(2h_1^2+1))}{(h_1^2+1)g_1^2+6g_2^2(h_1^2+2)+7g_3^2(h_1^2+5)} \\ \frac{(6h_1^2-3)g_2^2+7g_3^2(5h_1^2-1)}{(h_1^2+1)g_1^2+6g_2^2(h_1^2+2)+7g_3^2(h_1^2+5)} & \frac{g_1^2(2h_1^2-1)-7g_3^2(8h_1^2+1)}{(h_1^2+1)g_1^2+6g_2^2(h_1^2+2)+7g_3^2(h_1^2+5)} & \frac{(5h_1^2-1)g_1^2+3g_2^2(8h_1^2+1)}{(h_1^2+1)g_1^2+6g_2^2(h_1^2+2)+7g_3^2(h_1^2+5)} \\ \frac{7g_3^2(3-5h_1^2)-3g_2^2(h_1^2-3)}{(h_1^2+1)g_1^2+6g_2^2(h_1^2+2)+7g_3^2(h_1^2+5)} & \frac{21g_3^2(3h_1^2+1)-g_1^2(h_1^2-3)}{(h_1^2+1)g_1^2+6g_2^2(h_1^2+2)+7g_3^2(h_1^2+5)} & \frac{g_1^2(3-5h_1^2)-9g_2^2(3h_1^2+1)}{(h_1^2+1)g_1^2+6g_2^2(h_1^2+2)+7g_3^2(h_1^2+5)} \end{bmatrix}.$$

Example 4.4 *In the following example we consider tensors* A *and* W *whose elements are transcendental functions, for example*

$$A=\begin{bmatrix} e^{2x} & \sin(x) & 0 \\ e^{2x} & \sin(x) & 0 \\ 1+e^{3x} & \cos(x) & 0 \\ \sin(x) & \cos(x) & 0 \end{bmatrix}, \quad W=\begin{bmatrix} 1 & 1 & 0 & 0 \\ 3x & 0 & 1 & 0 \\ x & 0 & \frac{1}{3} & 0 \end{bmatrix}.$$

Further, let us chose

$$\mathcal{G}=\begin{bmatrix} \frac{1}{g_1^2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{g_4^2} \end{bmatrix}, \quad \mathcal{H}=\begin{bmatrix} \frac{1}{h_1^2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{h_3^2} \end{bmatrix}.$$

In this case, an application of Theorem 3.1 gives the following outer inverse of A*:*

$$X=\begin{bmatrix} \frac{3x\sin(x)-\cos(x)}{((1+e^{3x})\sin(x)-e^{2x}\cos(x))(g_1^2+1)} & -\frac{\cos(x)g_1^2+3x\sin(x)}{((1+e^{3x})\sin(x)-e^{2x}\cos(x))(g_1^2+1)} & \frac{\sin(x)}{(1+e^{3x})\sin(x)-e^{2x}\cos(x)} & 0 \\ \frac{-3e^{2x}x+e^{3x}+1}{((1+e^{3x})\sin(x)-e^{2x}\cos(x))(g_1^2+1)} & \frac{(1+e^{3x})g_1^2+3e^{2x}x}{((1+e^{3x})\sin(x)-e^{2x}\cos(x))(g_1^2+1)} & -\frac{e^{2x}}{(1+e^{3x})\sin(x)-e^{2x}\cos(x)} & 0 \\ \frac{-3e^{2x}x+e^{3x}+1}{3((1+e^{3x})\sin(x)-e^{2x}\cos(x))(g_1^2+1)h_3^2} & \frac{(1+e^{3x})g_1^2+3e^{2x}x}{3((1+e^{3x})\sin(x)-e^{2x}\cos(x))(g_1^2+1)h_3^2} & -\frac{e^{2x}}{3((1+e^{3x})\sin(x)-e^{2x}\cos(x))h_3^2} & 0 \end{bmatrix}.$$

5 Conclusion

We consider the determinantal representation and characterization of generalized inverses from a new point of view.

Firstly, it is verified that the determinantal representations (1.1) and (1.2) as well as the determinantal representation (1.3)-(1.4) can be expressed in terms of the double inner product in the Euclidean vector space. For this purpose we define double inner product of two tensors of rank 2. Motivated by this fact, we continue these representations and give analogous representations in the Riemannian space in the general case.

In this way, we extended recently obtained results from [12, 24] where the weighted Moore-Penrose inverse and its generalization are investigated in an indefinite inner product space. The generalization was done in two different ways: in the present paper we consider more general class of generalized inverses in a Riemannian space, which contains any indefinite inner product space.

We also continue and extended results from [27] concerning outer inverses and results concerning $A^{(2)}_{T,S}$ inverse from [23, 33, 34]. Corresponding results from [27] and [23, 33, 34] can be derived in the case when both tensors (g^{ij}) and (h^{ij}) coincide with appropriate identity tensors.

In the partial case $s = r$ we extended analogous results relative to $\{1,2\}$-inverses, introduced in [2, 14] and [25]. As a corollaries we get extensions of corresponding results concerning the Drazin inverse from [26], the Moore-Penrose inverse from [1], the weighted Moore-Penrose inverse from [16, 17] as well as the group inverse from [15]. Analogous results for the generalized inverse $A^{(2)}_{T,S}$ are also derived, as a continuation of papers [8] and [23, 33, 34].

Of course, during the verification of our statements, we follow some of the known matrix algebra results on $\{2\}$-inverses, weighted Moore-Penrose inverses, Drazin inverses, and the k-volume. We also make use of known techniques developed in cited papers. But, the final contributions of the paper are extensions of known results. We emphasize the following results of the paper:

- We show that known determinantal representations of generalized inverses are quotients of two scalar products.

- It is possible to extend these scalar product in the Riemannian manifold. In this way, generalized inverses are available for the most general class of matrices whose elements are arbitrary functions. Moreover, the underlying geometry which is established in the definition of the best approximate solution of linear systems (related to $\{1, 3\}$-inverses), the minimal norm solution (associated with $\{1, 4\}$-inverses) as well as the best approximate solution of minimal norm (derived by means of the Moore-Penrose inverse) can be fully reconsidered through the aspect of the Riemannian geometry.

It is reliable to expect that these facts will be a motivation in further investigations of generalized inverses, and generally for all extensions from the Euclidean into the Riemannian geometry.

References

[1] R.B. Bapat, K.P.S. Bhaskara and K. Manjunatha Prasad, *Generalized inverses over integral domains*, Linear Algebra Appl. **140** (1990), 181–196.

[2] R.B. Bapat, *Generalized inverses with proportional minors*, Linear Algebra Appl. **211** (1994), 27–35.

[3] A. Ben-Israel and T.N.E. Greville, *Generalized inverses: theory and applications*, Second Ed., Springer, 2003.

[4] A. Ben-Israel, *Generalized inverses of matrices: a perspective of the work of Penrose*, Math. Proc. Camb. Phil. Soc. **100** (1986), 407–425.

[5] A. Ben-Israel, *A volume associated with $m \times n$ matrices*, Linear Algebra Appl. **167** (1992), 87–111.

[6] G. Bercu, *Pseudo-Riemannian structures with the same connection*, Balkan Journal of Geometry and Its Applications **11** (2006), 23–38.

[7] K.P.S. Bhaskara Rao, *On generalized inverses of matrices over integral domains*, Linear Algebra Appl. **49** (1983), 179–189.

[8] J. Cai, G. Chen, *On determinantal representation for the generalized inverse $A^{(2)}_{T,S}$ and its applications*, Numerical Linear algebra with applications **14** (2007), 169–182.

[9] D. Cvetkovic-Ilić, X. Liu, J. Zhong, *On the (p,q) outer generalized inverse in Banach algebra*, Appl. Math. Comput. **209** (2009), 191–196.

[10] A. Fabiano, V. Marino, *Bounds for the degree of the entries of generalized inverses of polynomial matrices*, Rend. Instit. Mat. Univ. Trieste **22** (1992), 126–135.

[11] C. Gu, *Generalized inverse matrix Padé approximation on the basis of scalar products*, Linear Algebra Appl. **322** (2001), 141-167.

[12] K. Kamaraj, K. C. Sivakumar, *Moore-Penrose inverse in an indefinite inner product space*, J. Appl. Math. & Computing **19** (2005), 297–310.

[13] X. Liu, Y. Yu, H. Wang, *Determinantal representation of weighted generalized inverses*, Appl. Math. Comput. **208** (2009), 556–563.

[14] J. Miao, *Reflexive generalized inverses and their minors*, Linear and Multilinear Algebra **35** (1993), 153–163.

[15] K.M. Prasad, K.P.S.B. Rao and R.B. Bapat, *Generalized inverses over integral domains. II. Group inverses and Drazin inverses*, Linear Algebra Appl. **146** (1991), 31–47.

[16] K.M. Prasad and R.B. Bapat, *The generalized Moore-Penrose inverse*, Linear Algebra Appl. **165** (1992), 59–69.

[17] K.M. Prasad and R.B. Bapat, *A note of the Khatri inverse inverse*, Sankhya: Indian J. Stat. **54** (1992), 291–295.

[18] D.W. Robinson, *The image of the adjoint mapping*, Linear Algebra Appl. **277** (1998), 143–148.

[19] D.W. Robinson, *The Moore idempotents of a matrix*, Linear Algebra Appl. **211** (1994), 15–26.

[20] D.W. Robinson, *The determinantal rank idempotents of a matrix*, Linear Algebra Appl. **237/238** (1996), 83–96.

[21] D.W. Robinson, *Outer inverses of matrices*, Linear Algebra Appl. **348** (2002), 247-258.

[22] D.W. Robinson, *The classical adjoint*, Linear Algebra Appl. **411** (2005), 254-276.

[23] X. Sheng, G. Chen *Full-rank representation of generalized inverse $A^{(2)}_{T,S}$ and its applications*, Comput. Math. Appl. **54** (2007), 1422–1430.

[24] X. Sheng, G. Chen *The generalized weighted Moore-Penrose inverse*, J. Appl. Math. & Computing **25** (2007), 407–413.

[25] P. S. Stanimirović, *General determinantal representation of generalized inverses over integral domains*, Publicationes Mathematicae Debrecen **54** (1999), 221–249.

[26] P. S. Stanimirović and D.S. Djordjević, *Full-rank and determinantal representation of the Drazin inverse*, Linear Algebra Appl., **311** (2000), 31–51.

[27] P. S. Stanimirović, S. Bogdanović, M. Ćirić, *Adjoint Mappings and Inverses of Matrices*, Algebra Colloquium, **13** (2006), 421–432.

[28] P. S. Stanimirović, M. B. Tasić, *Computing determinantal representation of generalized inverses*, Korean J. Comput. & Appl. Math. **9** (2002), 349–359.

[29] G.Wang, Y.Wei and S. Qiao, *Generalized Inverses: Theory and Computations*, Science Press, Beijing, 2004.

[30] M. Wei, *Supremum and stability of weighted pseudoinverses and weighted least squares problems. Analysis and computations*, Nova Science Publishers, Inc., Huntington, NY, 2001. xiv+182 pp. ISBN: 1-56072-922-8.

[31] M. Wei, A.R. De Pierro, *Upper perturbation bounds of weighted projections, weighted and constrained least squares problems*, SIAM J. Matrix Anal. Appl. **21** (2000), 931–951

[32] K. Yano, *Integral formulas in Riemannian geometry*, Marcel Dekker, New York, 1970.

[33] Y. Yaoming, Y. Wei, *Determinantal representation of the generalized inverse $A^{(2)}_{T,S}$ over integral domains and its applications*, Linear and Multilinear Algebra **57** (2009), 547–559.

[34] Y. Yaoming, G. Wang, *On the generalized inverse* $A^{(2)}_{T,S}$ *over integral domains*, The Australian Journal of Mathematical Analysis and Applications **4** (2007), 1–20.

[35] G. Zielke, *Report on test matrices for generalized inverses*, Computing **36** (1986), 105–162.

[36] S. Zhang, *A characterization and determinantal formula for the generalized inverse* $A^{(2)}_{T,S}$ *and applications*, Appl. Math. Comput. **125** (2002), 261–269.