SOME ANALYTIC ITERATIVE METHODS FOR SOLVING VARIOUS CLASSES OF STOCHASTIC HEREDITARY INTEGRODIFFERENTIAL EQUATIONS

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Abstract. *The notion of hereditary phenomena is particularly convenient for studying such phenomena in continuum mechanics of materials with memories, as a version of the well-known theory of fading memory spaces. Mathematical models represent deterministic hereditary differential equations researched in manu papers and monographs. Later, this notion was appropriately used in an investigation into the effect of the Gaussian white noise, which mathematical interpretation is represented by stochastic hereditary differential equations of the Ito type.*

In the present paper we consider a general analytic iterative method for solving stochastic hereditary integrodifferential equation of the Ito type. We give sufficient conditions under which a sequence of iterations converges with probability one to the solution of the original equation. The generality of this method is in the sense that manu well-known iterative methods are its special cases, the Picard-Lindelof method of successive approximations, for example. Some other iterative methods, including linearizations of the coefficients of the original equation, are suggested.

Especially, using a concept of a random bounded integral contractor, basically introduced by Altman and Kuo, we show that the iterative procedure utilized to prove the existence and uniqueness of the solution of the stochastic hereditary integrodifferential equation, is also a special algorithm included in the considered general iterative procedure.

Keywords: Stochastic differential equation, stochastic hereditary integrodifferential equation, random integral contractor, Z-algorithm, determining sequence.

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1. Introduction

In some sense, the basic idea of the present problem goes back to papers [30, 31] by R. Zuber, treating one general analytic iterative method for solving the Cauchy problem for the ordinary differential equation

$$x' = f(t,x), \qquad x(t_0) = x_0. \tag{1.1}$$

The essence is as follows: Suppose that the functions $f : \Pi \to R$ and $f_n : \Pi \to R$ are continuous on a compact $\Pi = \{(t,x) : |t-t_0| \le a, |x-x_0| \le b\}$ and satisfy the Lipschitz condition on the last argument with the same constant L. Then, by the basic existence and uniqueness theorem it follows that there exist the unique solution $x = x(t)$ of Eq. (1.1) and the ones of the equations

$$x'_{n+1} = f_n(t, x_{n+1}), \quad x_{n+1}(t_0) = x_0, \tag{1.2}$$

all defined on the interval $[t_0 - a, t_0 + a]$. Moreover, if

$$\sum_{n=1}^{\infty} \sup_{|t-t_0|\le a} |f(t,x_n(t)) - f_n(t,x_n(t))| < \infty,$$

in [30] it is proved that there exists a constant $h,\ 0 < h \le a$, so that the sequence of the solutions $\{x_n, n \in N\}$ converges to the solution x of Eq. (1.1), uniformly on the interval $[t_0 - h, t_0 + h]$. If the choice of the functions f_n is good enough, so that the equations (1.2) can be effectively solvable, then an ε-approximation of the solution x of the original equation can be effectively found, in the sense that there exists $n = n(\varepsilon)$ so that $|x(t) - x_n(t)| < \varepsilon$ for $t \in [t_0 - h, t_0 + h]$.

This iterative method presents a general algorithm for solving ordinary differential equations, in papers [30, 31] called *the Z-algorithm*, because many well-known, historically and practically important analytic and numerical methods are its special cases: Picard method of successive approximations, Chaplygin methods of secants and tangents, Newton-Kantorovich method and some interpolation methods, as Euler one, for example. Later, this approach was appropriately extended to analyze some classes of stochastic differential and integrodifferential equations of the Ito type.

The paper is organized as follows: Since this paper is, in some sense, a summary of the investigation of the Z-algorithm in stochastic cases, Section 2 and Section 3 are devoted to earlier obtained results. In Section 2 we construct the Z-algorithm for stochastic differential equation of the Ito type and we give some algorithms representing its special cases. Especially, we prove that the well-known existence and uniqueness theorem, based on the Picard-Lindelöf method of successive approximations, is a special case of the Z-algorithm. In Section 3 we extend our consideration to stochastic hereditary integrodifferential equation, based on a past-history space.

Section 4 is a continuation of Section 3, in which we investigate the problem of iterations for a stochastic hereditary integrodifferential equation containing the coefficients which have a bounded random integral contractor of the Altman and Kuo type, instead of the usual Lipschitz condition. We show that the iterative procedure used in the proof of the existence theorem, is a special case of the Z-algorithm. This fact implies that very complicated and tiring proofs could be substantially shortened, which is one of the motivation to form the Z-algorithm.

2. The Z-algorithm for solving stochastic differential equations

Stochastic differential equations of the Ito type [11] play a mayor role in the characterization of many real phenomena in life science and engineering, and recently in financial mathematics, and arise frequently in mathematical descriptions of physical phenomena depending on the effect of a Gaussian white noise random forces (see [9], for example). Bearing in mind that a Gaussian white noise is an abstraction and not a physical process, mathematically described as a formal derivative of a Brownian motion process, all such phenomena are essentially based on stochastic differential equations of the Ito type.

For example, the behavior of any non-linear dynamical oscillator system is mathematically described by a random differential equation

$$\ddot{y} + f(t, \dot{y}) = g(t, \dot{y})\, \xi(t, \omega),$$

where $\xi(t, \omega)$ is a Gaussian stationary wide-band random process of small intensity and correlation time, with expectation equal to zero, which is treated as a Gaussian white noise excitation in mechanics and engineering practice. Since, formally, $\xi(t, \omega) = \dot{w}(t, \omega)$, where $w(t, \omega)$ is a Brownian motion, i.e. a Wiener process, this equation can be transformed into the following stochastic system

$$\begin{aligned} dy(t) &= x(t)\, dt \\ dx(t) &= -f(t, x(t)) + g(t, x(t))\, dw(t) \\ x(0) &= y(0) = c. \end{aligned}$$

where we omitted ω for simpler notation. The second equation in this system is a special case of the stochastic differential equation of the Ito type,

$$\begin{aligned} dx(t) &= a(t, x(t))\, dt + b(t, x(t))\, dw(t), \quad t \in [0, T], \\ x(0) &= x_0. \end{aligned} \tag{2.3}$$

Here $w = (w_t, t \geq 0)$ is an R^m–valued normalized Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$, with a natural filtration $\{\mathcal{F}_t, t \geq 0\}$ of nondecreasing sub σ-algebras of $\mathcal{F}$ $(\mathcal{F}_t = \sigma\{w_s, s \leq t\})$, the functions $a : [0, T] \times R^k \to R^k$

and $b : [0,T]\times R^k \to R^k \times R^m$, i.e. the drift and diffusion coefficients respectively, are assumed to be Borel measurable on their domains, and the initial condition x_0 is an R^k-valued random variable defined on the same probability space and independent of w.

The R^k-valued stochastic process $x = (x(t), t \in [0,T])$ is a strong solution of Eq. (2.3) if it is adapted to $\{\mathcal{F}_t, t \geq 0\}$, $\int_0^T |a(t,x(t))|\,dt < \infty$ and $\int_0^T |b(t,x(t))|^2\,dt < \infty$ with probability one (under these conditions the Lebesgue and Ito integrals in the integral form of Eq. (2.3) are well defined), $x(0) = x_0$ and Eq. (2.3) holds with probability one for all $t \in [0,T]$.

Note that Eq. (2.3) can be represented in the equivalent integral form,

$$x(t) = x_0 + \int_0^t a(s,x(s))\,ds + \int_0^t b(s,x(s))\,dw(s), \quad t \in [0,T],$$

where the first integral is in the sense of Lebesgue and the second one is the Ito integral. Remember that a Brownian motion process is nowhere differentiable and its continuous sample paths are not of bounded variation on any bounded time interval, so that the Ito integral cannot be interpreted as Riemann-Stieltjes or Lebesgue-Stieltjes integral for each sample path. Because of that, it has a specific integral isometry, based on its martingale characteristics.

On the basis of classical theory of stochastic differential equations of the Ito type (see [3, 8, 10, 18, 26], for example) one can prove that if the functions $a(t,x)$ and $b(t,x)$ satisfy the global Lipschitz condition and the usual linear growth condition on the last argument, i.e. if there exists a constant $L > 0$ so that, for all $t \in [0,T]$, $x, y \in R^k$,

$$\begin{aligned} |a(t,x) - a(t,y)| + |b(t,x) - b(t,y)| &\leq L|x-y|, \\ |a(t,x)| + |b(t,x)| &\leq L(1+|x|), \end{aligned}$$

and if $E|x_0|^2 < \infty$, then there exists a unique a.s. continuous strong solution $x = (x(t), t \in [0,T])$ of Eq. (2.3) satisfying $E\{\sup_{t\in[0,T]} |x(t)|^2\} < \infty$. The proof is based on the Picard–Lindelöf method of successive approximations: for $n \in N$,

$$x_{n+1}(t) = x_0 + \int_0^t a(s,x_n(s))\,ds + \int_0^t b(s,x_n(s))\,dw(s), \quad t \in [0,T]. \tag{2.4}$$

Bearing in mind that a class of explicitly solvable such equation is, in general, very small, from theoretical point of view and from various applications, it is important to find some approximative analytic or numerical solution. One analytic approximating method will be the object of the present paper.

Following the ideas of paper [30], the analogous Z-algorithm for Eq. (2.3) was suggested in paper [12]. Let us suppose that $a_n : [0,T] \times R^k \to R^k$ and $b_n : [0,T] \times R^k \to R^k \times R^m$, $n \in N$, be deterministic functions satisfying the same conditions as the functions a and b from Eq. (2.3), and $x_{n+1} = (x_{n+1}(t), t \in [0,T])$, be the strong solution of the equation

$$dx_{n+1}(t) = a_n(t, x_{n+1}(t))\, dt + b_n(t, x_{n+1}(t))\, dw(t), \quad t \in [0,T], \qquad (2.5)$$
$$x_{n+1}(0) = x_0.$$

It is quite natural to expect that if the pair of functions (a_n, b_n) is close in some sense to (a, b), then the sequence of processes $\{x_n, n \in N\}$ will tend to x as $n \to \infty$. Of course, in addition to the requirement that $a_n(t,x) \to a(t,x)$, $b_n(t,x) \to b(t,x)$ as $n \to \infty$, uniformly in $(t,x) \in [0,\infty) \times R^k$, in accordance with paper [30] we shall also require that

$$\sum_{n=1}^{\infty} \sup_{t,x} \{|a(t,x) - a_n(t,x)| + |b(t,x) - b_n(t,x)|\} < \infty. \qquad (2.6)$$

The condition (2.6) is used essentially to prove the following assertion:

Theorem 1. *Let the functions $a, b, a_n, b_n, n \in N$ be defined as above and the condition* (2.6) *be satisfied. Then the sequence of processes $\{x_n, n \in N\}$ converges with probability one, as $n \to N$, to the solution x of Eq.* (2.3)*.*

If we denote that

$$\varepsilon_n = E\{ \sup_{t\in[0,T]} [\, |a(t, x_n(t)) - a_n(t, x_n(t))|^2 + |b(t, x_n(t)) - b_n(t, x_n(t))|^2]\},$$

then (2.6) implies that $\sum_{n=1}^{\infty} \varepsilon_n < \infty$. Through the proof, we come to the following iterative formula,

$$E\{|x(t) - x_{n+1}(t)|^2\} \le 2\alpha \int_0^t E\{|x(t) - x_n(t)|^2\}\, ds + \beta\varepsilon_n\, t$$
$$+\alpha \int_0^t \Big[2\alpha \int_0^s E\{|x(u) - x_n(u)|^2\}\, du + \beta\, \varepsilon_n\, s\Big]\, e^{\alpha(t-s)}\, ds,$$

from which, by induction, the mean square closeness between the solution x and the iteration x_n is determined so that

$$E\{ \sup_{t\in[0,T]} |x(t) - x_n(t)|^2\} \le$$
$$[c_1\, P_{n-3}(2\alpha T) + c_2\, P_{n-2}(2\alpha T) + c_3\varepsilon_{n-1}] \cdot \frac{e^{\alpha T} - 1}{\alpha},$$

where $P_n(u) = 2\alpha c\, \dfrac{u^n}{n!} + \beta \sum_{k=0}^{n} \varepsilon_{k+1}\, \dfrac{u^{n-k}}{(n-k)!}$, and $\alpha, \beta, c, c_1, c_2, c_3$ are some generic constants.

Theorem 1 can be proved under some other conditions which, in some sense, are weaker than the condition (2.6). For example, we can suppose that all the functions a, b, a_n, b_n are random and require that the Lipschitz condition, the linear growth condition and (2.6) are satisfied with probability one. Moreover, since the condition (2.6) is very strict, it could be modified and weakened with the assertion $\sum_{n=1}^{\infty} \varepsilon_n < \infty$ instead of (2.6). But, in general, it seems more difficult to verify this fact, because all iterations must be known.

Theorem 1 gives us an idea how to construct a sequence of iterations which converges with probability one to the solution of Eq. (2.3). So, by following the proof of Theorem 1, we can find an ε-approximation of the solution x, i.e. the stochastic process x_n for $n = n(\varepsilon)$, so that

$$P\{\sup_{t\in[0,T]} |x(t) - x_n(t)| < \varepsilon\} = 1.$$

At least theoretically, in order to determine an ε-approximation of the solution x, we shall find a sequence of iterations, i.e. of stochastic processes $x_1, x_2, \dots$ in the following way: Let x_1 be an arbitrary stochastic process with $x_1(0) = x_0$ with probability one and $E\{\sup_t |x_1(t)|^2\} < \infty$. Next, we choose a pair of functions (a_1, b_1), or more generally, a pair of stochastic processes depending on x_1, so that the Lipschitz and linear growth conditions be satisfied and $\sup_{t,x}\{|a(t,x) - a_1(t,x)| + |b(t,x) - b_1(t,x)|\} \le c_1 < \infty$. Then, we find a solution x_2 of the equation $dx_2(t) = a_1(t, x_2(t))\,dt + b_1(t, x_2(t))\,dw(t)$, $x_2(0) = x_0$. By induction, if we know $x_{n-1}(t)$, we choose a pair of real or random functions (a_{n-1}, b_{n-1}), so that they are Lipschitzian and satisfy the linear growth condition and $\sup_{t,x}\{|a(t,x) - a_{n-1}(t,x)| + |b(t,x) - b_{n-1}(t,x)|\} \le c_{n-1} < \infty$, where c_{n-1} is an $(n-1)$-th term of any convergent series. Now, the process x_n can be found as a solution of the equation $dx_n(t) = a_{n-1}(t, x_n(t))\,dt + b_{n-1}(t, x_n(t))\,dw(t)$, $x_n(0) = x_0$. Since $\sum_{n=1}^{\infty} \varepsilon_n \le \sum_{n=1}^{\infty} c_n < \infty$, Theorem 1 is valid for the choice of the sequence $\{(a_n, b_n), n \in N\}$.

Analogously to the basic paper [30], it is convenient to use the notion *the Z-algorithm* for this iterative procedure, and the sequence of functions $\{(a_n, b_n), n \in N\}$ will be called *the determining sequence for the Z-algorithm*.

Of course, the Z-algorithm can be effectively used only if the choice of the determining sequence is good enough, i.e. if the equations (2.5) can be solvable. The fact that any linear stochastic differential equation is solvable (see [3, 8, 18, 26]) leads us to the idea to linearize the functions a and b. Simple forms of such linearization by following assertions are expressed:

Corollary 1. *Let $\{\alpha_n(t), t \in [0,T], n \in N\}$ and $\{\beta_n(t), t \in [0,T], n \in N\}$ be a sequences of uniformly bounded continuous functions. Then the sequence of random functions $\{(a_n, b_n), n \in N\}$, defined by*

$$a_n(t,x) = \alpha_n(t) \cdot [x - x_n(t)] + a(t, x_n(t)),$$

$$b_n(t,x) = \beta_n(t) \cdot [x - x_n(t)] + b(t, x_n(t)),$$

is a determining sequence of the Z-algorithm for Eq. (2.3).

Indeed, it is clear that the Lipschitz condition is satisfied with probability one. It is quite more difficult to prove that the linear growth condition is also satisfied with probability one, which was proved in paper [12]. Since

$$a(t, x_n(t)) - a_n(t, x_n(t)) = b(t, x_n(t)) - b_n(t, x_n(t)) \equiv 0, \quad n \in N,$$

the condition (2.6) holds and, therefore, $\{(a_n, b_n), n \in N\}$ is the determining sequence of the Z-algorithm.

In particular, if $\alpha_n(t) = \beta_n(t) \equiv 0$ for each $n \in N$, the Z-algorithm is reduced to the usual Picard-Lindelöf method of iterations (2.4), which proves the existence of the solution x of Eq. (1.1).

The second type of the linearization is based on the Chaplygin method of secants and tangents for ordinary differential equations. This type of iterations is described in paper [13] for onedimensional case, by using some comparison theorems instead of the usual Chaplygin differential inequality theorem for deterministic case. So, we associate the sequence of the equations

$$dy_{n+1}(t) = u_n(t, y_{n+1}(t))\, dt + b(t, y_{n+1}(t))\, dw(t), \quad t \in [0,T], \quad y_0(0) = x_0,$$

$$dz_{n+1}(t) = v_n(t, z_{n+1}(t))\, dt + b(t, z_{n+1}(t))\, dw(t), \quad t \in [0,T], \quad z_0(0) = x_0,$$

to the original equation (1.1), so that the sequences of the functions $\{u_n, n \in N\}$ and $\{v_n, n \in N\}$ are determining sequences for the Z-algorithm, analogously to the Chaplygin method of secants and tangents in deterministic case. These notions by the following assertion are expressed:

Corollary 2. *Suppose the following conditions hold:*

(*i*) *$a(t,x)$ is two times differentiable in x and satisfies the linear growth condition, $a'_x(t,x)$ is bounded on $[0,T] \times R$, $a''_{xx}(t,x)$ does not change its sign on $[0,T] \times R$;*

(*ii*) *$b(t,x)$ satisfies the Lipschitz condition and the linear growth condition.*

Then the sequences of random processes $\{u_n, n \in N\}$ and $\{v_n, n \in N\}$, defined by

$$u_n(t,x) = a'_x(t, y_n(t)) \cdot [x - y_n(t)] + a(t, y_n(t)),$$

$$v_n(t,x) = \begin{cases} \dfrac{a(t, z_n(t)) - a(t, y_n(t))}{z_n(t) - y_n(t)} \cdot [x - y_n(t)] + a(t, y_n(t)), \\ \qquad\qquad\qquad\qquad z_n(t) \neq y_n(t) \quad \text{with prob. one}, \\ u_n(t,x), \qquad\qquad\quad z_n(t) = y_n(t) \quad \text{with prob. one}, \end{cases}$$

are determining sequences of the Z-algorithm for Eq. (2.3). Moreover, $y_n(t) \leq x(t) \leq z_n(t)$ or $y_n(t) \leq x(t) \leq z_n(t)$ with probability one, for all $t \in [0, T]$.

From the proof of this theorem it follows that

$$E\{\sup_{t\in[0,T]} |z_n(t) - y_n(t)|^2\} \leq K \frac{(bT)^{n-3}}{(n-3)!}, \quad n = 3, 4, \ldots,$$

where K and b are some generic constants. Clearly, the sequence of stochastic processes $\{y_n, n \in N\}$ converges with probability one to the solution x from the left side, and $\{z_n, n \in N\}$ from the right side. But, let us mention that it can not be proved, as in deterministic case, that sample pats of these sequences are monotonous with probability one.

Note that the determining sequence $\{u_n, n \in N\}$, i.e. the iterative method of tangents, is analogous to the Newton–Kantorovich method for solving ordinary differential equations. In this case it is not necessary to require that $a''_{xx}(t, x)$ is of the same sigh, and the sample paths of $\{y_n, n \in N\}$ can be located on the both sides of the sample path of the solution x.

3. The Z-algorithm for solving stochastic hereditary integrodifferential equations

The notion of hereditary phenomena are particularly useful for studying real phenomena in continuum mechanics of materials with memories, as a version of the well-known theory of "fading memory" spaces. Mathematical models in studies of viscoelasticity, represent, deterministic functional hereditary differential equations, researched elsewhere in papers [5, 6, 7, 19, 22] and in many others, in which existence, uniqueness and stability problems of solutions have been investigated for a long period of time. In particular, Mizel and Trutzer [23, 24] incorporated the effect of a Gaussian white noise on hereditary phenomena, as a random perturbation of the deterministic case, so that mathematical models were reproduced by stochastic hereditary differential and integrodifferential equations of the Ito type. In both of these papers, the existence, uniqueness and stability problems under Lipschitz and linear growth conditions for the coefficients of these equations are studied. Likewise, some applications of theoretical results to appropriate problems from continuum mechanics were described in these papers.

Let us introduce in short some notions and results, immediately used in our investigation. For more details see previously cited papers, first of all [23] and [15].

Let R^k be the real k-dimensional Euclidean space and L^ρ_p, $1 \leq p \leq \infty$, be the usual space of classes of measurable functions, i.e.,

$$L^\rho_p = \left\{ \varphi \,|\quad \varphi : R^+ \to R^k; \quad \int_0^\infty |\varphi(t)|^p \rho(t)\, dt < \infty \right\},$$

where the function $\rho : R^+ \to R^+$, called an influence function with relaxation properties, is summable on R^+ and for every $\sigma \geq 0$ one has $\overline{K}(\sigma) = \text{ess sup}_{s\in R^+} \frac{\rho(s+\sigma)}{\rho(s)} \leq \overline{\overline{K}} < \infty$, $\underline{K}(\sigma) = \text{ess sup}_{s\in R^+} \frac{\rho(s)}{\rho(s+\sigma)} < \infty$. From these conditions ρ is essentially bounded, essentially strictly positive and $s\rho(s) \to 0$ as $s \to 0$ (see [5]).

Let $X = R^k \times L_p^\rho$ be a Banach product space, i.e. *a past-history space* of elements x, $x = (\varphi(0), \varphi)$, with the norm

$$||x||_X = \left(|\varphi(0)|^p + \int_0^\infty |\,\varphi(t)|^p \rho(t)\,dt \right)^{1/p} = (|\,\varphi(0)|^p + ||\,\varphi||_r^p\,)^{1/p}.$$

In terms of the space X, one can formulate the notion of X-admissibility for measurable functions defined on any left semiaxis of R.

The measurable function $x : (-\infty, T] \to R^k$, T =const$\in R$, is *X-admissible* if for each $t \in (-\infty, T]$ the function x^t, called *its history up to t* and defined by $x_r^t(s) = x(t-s)$, $s \in R^+$, is itself an element in X.

So, if x is X-admissible, then $x^t = (x(t), x_r^t) \in X$ for each $t \in (-\infty, T]$, where

$$x(t) = \begin{cases} x(t), & 0 \leq t \leq a, \\ \varphi(-t), & t < 0, \end{cases}, \quad x_r^t(s) = \begin{cases} x(t-s), & 0 \leq s \leq t, \\ \varphi(s-t), & s > t, \end{cases}. \tag{3.7}$$

From the definition of the norm on the space X, for every $x \in X$ and $0 \leq 0 \leq t \leq T$, it follows that

$$||x^t||_X^2 \leq \tilde{K} \left[|x(t)|^2 + \overline{\overline{K}}^{2/p} ||x^{t_0}||_r^2 + \left(\int_0^t |x(u)|^p \rho(t-u)\,du \right)^{2/p} \right], \tag{3.8}$$

where $\tilde{K} = 3^{2/p-1} \vee 1$ (see [23]).

The functional differential equation, called *the hereditary differential equation* ,

$$\dot{x}(t) = f(t, x^t), \quad x^0 = \varphi^0, \quad \varphi^0 \in X,$$

where $f : R \times X \to R^k$ is a given functional, is considered in papers [5, 7] and in many others. Its solution consists of a function $x : (-\infty, T] \to R^k$, T =const> 0, such that x is X-admissible on $(-\infty, T]$, $x(t)$ is differentiable for each $t \in (0, T]$, the equation holds for $t \in [0, T]$ and $x^0 = \varphi$. From the structure of x^t, the continuity of $x(t)$ on $[0, T]$ implies that x^t, $t \in [0, T]$, is also continuous with respect to the norm of the space X.

All preceding notions and definitions are appropriately used in paper [23] to analyze the following *stochastic hereditary integrodifferential equation*

$$\begin{aligned} dx(t) &= \left[a_1(t, x^t) + \int_0^t a_2(t, s, x^s)\,ds + \int_0^t a_3(t, s, x^s)\,dw(s) \right] dt \qquad (3.9) \\ &\quad + \left[b_1(t, x^t) + \int_0^t b_2(t, s, x^s)\,ds + \int_0^t b_3(t, s, x^s)\,dw(s) \right] dw(t), \quad t \in [0, T], \\ x^0 &= \varphi^0, \end{aligned}$$

for which the existence, uniqueness and stability problems are considered in details. Here w is an R^m-valued normalized Brownian motion, adapted to the family $\{\mathcal{F}_t, t \geq 0\}$ of nondecreasing sub σ-algebras of $\mathcal{F}$, $\varphi^0 \in X$ is independent of w, the functionals

$$\begin{array}{ll} a_1 : [0,T] \times X \to R^k, & b_1 : [0,T] \times X \to R^k \times R^m, \\ a_2 : J \times X \to R^k, & b_2 : J \times X \to R^k \times R^m, \\ a_3 : J \times X \to R^k \times R^m, & b_3 : J \times X \to R^k \times R^m \times R^m, \end{array}$$

where $J = \{(t,s) \in [0,T] \times [0,T]\}$, are Borel measurable on their domains.

A stochastic process $x = (x(t), t \in (-\infty, T])$ is a strong solution of Eq. (3.9) for $t \in [0,T]$, if $x(t)$ is nonanticipating for $t \leq T$, $x^t \in X$ with probability one for $t \in [0,T]$, all Lebesgue and Ito integrals in integral form of Eq. (3.9) exist and Eq. (3.9) holds with probability one for each $t \in [0,T]$. Of course, x^t has the form (3.7).

In paper [23] the following existence and uniqueness theorem is given:

Theorem 2. *Assume that there exists a constant $L > 0$ such that the Lipschitz and linear growth conditions are satisfied for the coefficients of Eq.* (3.9)*, i.e. for all $(t,s) \in J$ and $x, y \in X$,*

$$|a_3(t,s,x) - a_3(t,s,y)| \leq L\,||x-y||_X, \tag{3.10}$$

$$|a_3(t,s,x)|^2 \leq L^2\,(1 + ||x||_X^2), \tag{3.11}$$

and similarly for the other functionals. If $E||\varphi^0||_X^2 < \infty$, then there exists a unique, with probability one strong solution $x = (x(t),\ t \in (-\infty, T])$ of Eq. (3.9)*, satisfying $E\{\sup_{t\in[0,T]} |x(t)|^2\} < \infty$.*

Note that the proof of the existence of the solution is based on Picard–Lindelöf method of successive approximations:

$$\begin{aligned} x_0(t) &= \varphi(0), \quad t \in [0,T], \\ x_{n+1}(t) &= \varphi(0) \\ &\quad + \int_0^t \left[a_1(s, x_n^s) + \int_0^s a_2(s,u,x_n^u)\,du + \int_0^s a_3(s,u,x_n^u)\,dw(u) \right] ds \\ &\quad + \int_0^t \left[b_1(s, x_n^s) + \int_0^s b_2(s,u,x_n^u)\,du + \int_0^s b_3(s,u,x_n^u)\,dw(u) \right] dw(s), \\ &\qquad t \in [0,T], \quad n = 0,1,2,\dots, \\ x_{n+1}(t) &= \varphi(-t), \quad t \leq 0, \quad n = 0,1,2,\dots \\ (x_{n+1}^t)_r(s) &= \begin{cases} x_{n+1}(t-s), & 0 \leq s \leq t \leq T, \\ \varphi(s-t), & s > t, \end{cases} \qquad n = 0,1,2,\dots \end{aligned} \tag{3.12}$$

It is shown in paper [15] that this iterative method is a special case of a general iterative procedure, analogous to the Z-algorithm described in Section 2. Similarly, we associate to Eq. (3.9) the sequence of stochastic hereditary integrodifferential equations

$$
\begin{aligned}
&dx_{n+1}(t) = \qquad\qquad (3.13)\\
&\quad \left[a_{1n}(t,x_{n+1}^t) + \int_0^t a_{2n}(t,s,x_{n+1}^s)\,ds + \int_0^t a_{3n}(t,s,x_{n+1}^s)\,dw(s)\right] dt\\
&\quad + \left[b_{1n}(t,x_{n+1}^t) + \int_0^t b_{2n}(t,s,x_{n+1}^s)\,ds + \int_0^t b_{3n}(t,s,x_{n+1}^s)\,dw(s)\right] dw(t),\\
&\qquad t \in [0,T], \quad n \in N,\\
&x_{n+1}^0 = \varphi^0.
\end{aligned}
$$

We suppose that all the coefficients of these equations are defined as the ones for Eq. (3.9) and satisfy the conditions of Theorem 2, which quarantines the existence of their unique, with probability one continuous strong solutions $x_{n+1} = (x_{n+1}(t), t \in (-\infty, T])$. Our main purpose is to expose sufficient conditions for the closeness between the functionals a_{in}, b_{in}, $i = 1,2,3$, with the corresponding functionals a_i, b_i, $i = 1,2,3$, so that the sequence of processes $\{x_n, n \in N\}$ converges with probability one to the solution x of Eq. (3.9). In this sense, let us denote that

$$
\begin{aligned}
F_n(t,s,x) = {} & |a_1(t,x) - a_{1n}(t,x)| + |a_2(t,s,x) - a_{2n}(t,s,x)|\\
& + |a_3(t,s,x) - a_{3n}(t,s,x)| + |b_1(t,x) - b_{1n}(t,x)|\\
& + |b_2(t,s,x) - b_{2n}(t,s,x)| + |b_3(t,s,x) - b_{3n}(t,s,x)|.
\end{aligned}
$$

Then, the following assertion, proved in paper [15], is valid:

Theorem 3. *Let the functionals a_i, b_i, a_{in}, b_{in}, $i = 1,2,3$, $n \in N$, and $\varphi^0 \in X$ satisfy the conditions of Theorem 2 and let*

$$
\sum_{n=1}^{\infty} F_n(t,s,x) < \infty. \tag{3.14}
$$

Then the sequence of solutions $\{x_n, n \in N\}$ of the equations (3.13) *converges with probability one as $n \to \infty$, to the solution x of the equation* (3.9).

The proof is analogous to the one of Theorem 1, but it is more complicated because of the structure and norm of the space X. Similarly, if we denote that

$$
\begin{aligned}
\varepsilon_n = E\{ \sup_J [\, & |a_1(t,x_n^t) - a_{1n}(t,x_n^t)|^2 + |a_2(t,s,x_n^s) - a_{2n}(t,s,x_n^s)|^2\\
& + |a_3(t,s,x_n^s) - a_{3n}(t,s,x_n^s)|^2 + |b_1(t,x_n^t) - b_{1n}(t,x_n^t)|^2 \qquad (3.15)\\
& + |b_2(t,s,x_n^s) - b_{2n}(t,s,x_n^s)|^2 + |b_3(t,s,x_n^s) - b_{3n}(t,s,x_n^s)|^2 \,]\}, \ n \in N,
\end{aligned}
$$

then $\sum_{n=1}^{\infty}\varepsilon_n<\infty$, which makes it possible to prove that, for all large enough n,

$$\sup_{t\in[0,T]}|x(t)-x_n(t)|<\varepsilon \quad \text{with probability one.}$$

Therefore, $x_n(t)\to x(t), n\to\infty$ with probability one, uniformly in $[0,T]$. Likewise, following the proof of this assertion, we find that

$$E\{\sup_{t\in[0,T]}|x(t)-x_n(t)|^2\}\le P_{n-2}(2\alpha T)\cdot\frac{e^{\alpha T}-1}{\alpha},\quad n=2,3,\dots,$$

where $P_{n-2}(u)=2\alpha c\,\dfrac{u^{n-2}}{(n-2)!}+\beta\sum_{k=1}^{n-1}\varepsilon_k\,\dfrac{u^{n-k-1}}{(n-k-1)!}$, and α,β,c are generic constants.

Since the choice of functionals $a_{in},b_{in},i=1,2,3$ determines the $(n+1)$-th approximation of the solution of Eq. (3.9), the previous iterative method is logically called *the Z-algorithm*, while *its determining sequence* is the sequence of the set of functionals

$$\{(a_{1n},a_{2n},a_{3n},b_{1n},b_{2n},b_{3n}),\ n\in N\}. \tag{3.16}$$

Note that the determining sequence could be stochastic, by the same reason as the one from Section 2. From theoretical point of view, the choice of the determining sequence makes to be possible to investigate and, in the best case to solve effectively Eq. (3.9). Certainly, this requirement is extremely strong for the considered stochastic hereditary equation and it is almost impossible to form such an algorithm. This fact suggests us to construct simple forms of linearization of the coefficients of Eq. (3.9), as it is shown in the next examples.

Corollary 3. *Let the functionals $a_i,b_i,i=1,2,3$ satisfy the conditions of Theorem 2 and the functionals $\alpha_{in}:[0,T]\to R^k$, $\beta_{in}:[0,T]\to R^k$, $i=1,2,3$, $n\in N$ be uniformly bounded. Then the sequence of random functions* (3.16)*, defined by*

$$\begin{aligned}
&a_{1n}(t,x)=\alpha_{1n}(t)\cdot||x-x_n^t||_X+a_1(t,x_n^t),\\
&a_{in}(t,s,x)=\alpha_{in}(t)\cdot||x-x_n^t||_X+a_i(t,s,x_n^t),\quad i=2,3,\\
&b_{1n}(t,x)=\beta_{1n}(t)\cdot||x-x_n^t||_X+b_1(t,x_n^t),\\
&b_{in}(t,s,x)=\beta_{in}(t)\cdot||x-x_n^t||_X+b_i(t,s,x_n^t),\quad i=2,3,
\end{aligned}$$

describes the determining sequence of the Z-algorithm for Eq. (3.9).

Really, the functionals $a_{in},b_{in},i=1,2,3,n\in N$ satisfy the Lipschitz condition with the same Lipschitz constant. Since $\varepsilon_n=0$ for every $n\in N$, then $\sum_{n=1}^{\infty}\varepsilon_n<\infty$. By following the procedure used in paper [12] and the usual properties of stopping times, we can conclude that the linear growth condition is also satisfied. Therefore, by applying Theorem 3 it follows that this iterative method describes a version of the Z-algorithm for Eq. (3.9).

In particular, if $\alpha_{in} = \beta_{in} \equiv 0, i = 1,2,3, n \in N$, then this algorithm is reduced to the Picard–Lindelöf method of successive approximations (3.12).

Corollary 4. *Let the functionals* $a_i, b_i, i = 1,2,3$ *satisfy the conditions of Theorem 2 and the functionals* $\alpha_{in} : [0,T] \to R$, $\beta_{in} : [0,T] \to R$, $i = 1,2,3$, $n \in N$ *be uniformly bounded. Then the sequence of random functions* (3.16), *defined by*

$$\begin{aligned}
&a_{1n}(t,x) = \alpha_{1n}(t) \cdot [x - x_n(t)] + a_1(t, x_n^t),\\
&a_{in}(t,s,x) = \alpha_{in}(t) \cdot [x - x_n(t)] + a_i(t,s,x_n^t), \quad i = 2,3,\\
&b_{1n}(t,x) = \beta_{1n}(t) \cdot [x - x_n(t)] + b_1(t, x_n^t),\\
&b_{in}(t,s,x) = \beta_{in}(t) \cdot [x - x_n(t)] + b_i(t,s,x_n^t), \quad i = 2,3,
\end{aligned}$$

describes the determining sequence of the Z*-algorithm for Eq.* (3.9).

The proof is similar to the one of the preceding assertion. If $\alpha_{in} = \beta_{in} \equiv 0, i = 1,2,3, n \in N$, we also obtain the Picard–Lindelöf method of successive approximations (3.12).

4. The Z-algorithm and a contractor theory

In this section the existence and uniqueness of the solution of Eq. (3.9) is considered by using the concept of a random integral contractor, which includes the Lipschitz condition as a special case. Our main goal is to show that the iterative procedure used in the proof of the existence theorem, presents a special case of the general Z-algorithm. Note that analogous conclusions could be exposed to different types of stochastic differential equations which coefficients have bounded random integral contractors, first of all for Eq. (2.3).

As it is well known, the concept of an integral contractor was introduced by Altman [1, 2] as a useful tool for studying some classes of deterministic equations in Banach spaces. This approach was appropriately extended by Kuo [21] to analyze the existence and uniqueness of solutions for stochastic differential equations of the Ito type. Later, many authors applied the notion of random integral contractors to various classes of stochastic differential, integral and integrodifferential equations, [14, 16, 17, 21, 25, 27, 28, 29], for example.

By following the basic ideas of Altman and Kuo, the concept of a bounded random integral contractor was introduced in paper [16] to prove the existence and uniqueness of a solution of Eq. (3.9), while in paper [17] some relations between various conditions for the coefficients of this equations, were investigated. In order to prove that the Z-algorithm can be incorporated in this contractor theory, we shall briefly employ some its elements, adapted to Eq. (3.9).

Furthermore, all definitions and assumptions about the functionals $a_i,\ b_i, i = 1,2,3$, and the initial condition, are valid.

Let

$$\begin{array}{ll} \Gamma_1 : [0,T] \times X \to R^k \times R^k, & \Phi_1 : [0,T] \times X \to R^k \times R^k \\ \Gamma_i : J \times X \to R^k \times R^k, & \Phi_i : J \times X \to R^k \times R^k, \quad i = 2,3 \end{array}$$

be measurable mappings, bounded in the sense that there exist positive constants $\alpha_i,\ \beta_i,\ i = 1,2,3$, such that for every $(t,s,x) \in J \times X,\ y \in R^k$,

$$\begin{array}{ll} |\Gamma_1(t,x)\, y| \le \alpha_1\, |y|, & |\Phi_1(t,x)\, y| \le \beta_1\, |y| \\ |\Gamma_i(t,s,x)\, y| \le \alpha_i\, |y|, & |\Phi_i(t,s,x)\, y| \le \beta_i\, |y|, \quad i = 2,3. \end{array} \tag{4.17}$$

Let $((Ax)y)^t$ be an element of the space X, i.e., $((Ax)y)^t = ((Ax)y)(t), y_r^t)$, where

$$\begin{aligned} ((Ax)y)(t) = y(t) + \int_0^t \Big[&\Gamma_1(s,x^s)\, y(s) \\ &+ \int_0^s \Gamma_2(s,r,x^r)\, y(r)\, dr + \int_0^s \Gamma_3(s,r,x^r)\, y(r)\, dw(r) \Big]\, ds \\ + \int_0^t \Big[&\Phi_1(s,x^s)\, y(s) + \int_0^s \Phi_2(s,r,x^r)\, y(r)\, dr \\ &+ \int_0^s \Phi_3(s,r,x^r)\, y(r)\, dw(r) \Big]\, dw(s), \end{aligned} \tag{4.18}$$

and y_r^t is an element on L_p^ρ.

Suppose there exists a positive constant K such that, for any x^t, y^t, in X and $(t,s) \in J$, the following inequalities hold with probability one:

$$\begin{aligned} &|a_1(t,x^t + ((Ax)y)^t) - a_1(t,x^t) - \Gamma_1(t,x^t)\, y(t)| \le K\, ||\, y||_t, \\ &|a_i(t,s,x^s + ((Ax)y)^s) - a_i(t,s,x^s) - \Gamma_i(t,s,x^s)\, y(s)| \le K\, ||\, y||_s,\ i = 2,3, \\ &|b_1(t,x^t + ((Ax)y)^t) - b_1(t,x^t) - \Phi_1(t,x^t)\, y(t)| \le K\, ||\, y||_t, \\ &|b_i(t,s,x^s + ((Ax)y)^s) - b_i(t,s,x^s) - \Phi_i(t,s,x^s)\, y(s)| \le K\, ||\, y||_s,\ i = 2,3, \end{aligned} \tag{4.19}$$

where

$$||y||_t = \sup_{0 \le s \le t} ||y^s||_X.$$

Then *the set of functionals* $(a_1,a_2,a_3,b_1,b_2,b_3)$ *has a bounded random integral contractor*

$$\begin{aligned} \Big\{ I + \int_0^t &\Big[\Gamma_1 + \int_0^s \Gamma_2\, dr + \int_0^s \Gamma_3\, dw(r) \Big]\, ds \\ &+ \int_0^t \Big[\Phi_1 + \int_0^s \Phi_2\, dr \int_0^s \Phi_3\, dw(r) \Big]\, dw(s) \Big\}. \end{aligned} \tag{4.20}$$

Bearing in mind dimensions of elements of the bounded random integral contractor (4.20), it is clear that we can investigate only the case $m = 1$. Therefore, all the functionals $a_i, b_i, i = 1, 2, 3$, are R^k-valued and the Brownian process is one-dimensional.

A bounded random integral contractor is said to be *regular* if the linear equation

$$(Ax)y = z \tag{4.21}$$

has a solution y in X for any x and z in X.

A functional $h : [0,T] \times X \to R^k$ is said to be *stochastically closed* if for any x and x_n in X, so that $x_n \to x$ and $h(\cdot, x_n^{\cdot}) \to y$ in $L^2([0,T] \times \Omega)$, we have $y(t) = h(t, x^t)$ with probability one, for every $t \in [0,T]$. The stochastic closeness of a functional $h : J \times X \to R^k$ is defined analogously.

Of course, if the functionals $a_i,\ b_i,\ i = 1, 2, 3$, satisfy the Lipschitz condition, then they are stochastically closed and the set of functionals $(a_1, a_2, a_3, b_1, b_2, b_3)$ has a trivial bounded random integral contractor (4.20) for $\Gamma_i = \Phi_i = 0,\ i = 1, 2, 3$. It was also shown in paper [16], that the Lipschitz condition implies the existence of a class of bounded random integral contractors (4.20) in which Γ_1 and Γ_2 are arbitrary mappings defined as in (4.17) and $\Gamma_3 = 0, \Phi_i = 0,\ i = 1, 2, 3$. Moreover, it was shown that Eq. (3.9) could have a regular bounded random integral contractor, although the Lipschitz condition did not have to be satisfied.

In what follows, denote by C_X a collection of R^k-valued stochastic processes, X-admissible on $(-\infty, T]$ and continuous with probability one.

In paper [16] the following existence and uniqueness theorems are proved:

Theorem 4. *Let the functionals $a_i,\ b_i, i = 1, 2, 3$, be stochastically closed and have a bounded random integral contractor* (4.20), *and let $\int_0^T E|a_1(t, \varphi^0)|^2 dt < \infty$, $\int_0^T E|b_1(t, \varphi^0)|^2 dt < \infty$ and $\int_0^T \int_0^t E|f(t, s, \varphi^0)|^2 ds\, dt < \infty$ for $a_i, b_i,\ i = 2, 3$ instead of f. Then Eq.* (3.9) *has a solution x in C_X.*

Theorem 5. *Let the functionals $a_i,\ b_i,\ i = 1, 2, 3$, satisfy the assumptions of Theorem* 4 *and the bounded random integral contractor be regular. Then the solution of Eq.* (3.9) *in C_X is unique.*

The proof of Theorem 4 is based on the following iterative procedure, with help of two sequences $\{x_n^t, n \in N\}$ and $\{y_n^t, n \in N\}$ in X, so that for $n \ge 0$:

$$\begin{aligned} x_0(t) &= \varphi(0), \quad 0 \le t \le T, \\ x_{n+1}(t) &= x_n(t) - ((Ax_n)y_n)(t) = x_n(t) - y_n(t) \\ &\quad - \int_0^t \Big[\Gamma_1(s, x_n^s)\, y_n(s) + \int_0^s \Gamma_2(s, r, x_n^r) y_n(r) dr \\ &\quad + \int_0^s \Gamma_3(s, r, x_n^r)\, y_n(r)\, dw(r)\Big] ds \end{aligned} \tag{4.22}$$

$$
\begin{aligned}
&- \int_0^t \Big[\Phi_1(s, x_n^s)\, y_n(s) + \int_0^s \Phi_2(s, r, x_n^r)\, y_n(r)\, dr \\
&\quad + \int_0^s \Phi_3(s, r, x_n^r)\, y_n(r)\, dw(r)\Big] dw(s), \quad 0 \le t \le T, \\
&x_{n+1}(t) = \varphi(-t), \quad t \le 0, \\
&(x_{n+1}^t)_r(s) = \begin{cases} x_{n+1}(t-s), & 0 \le s \le t \le T, \\ \varphi(s-t), & s > t, \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
y_n(t) &= x_n(t) - \varphi(0) \\
&- \int_0^t \left[a_1(s, x_n^s) + \int_0^s a_2(s, r, x_n^r)\, dr + \int_0^s a_3(s, r, x^r)\, dw(r)\right] ds \\
&- \int_0^t \left[b_1(s, x_n^s) + \int_0^s b_2(s, r, x_n^r)\, dr + \int_0^s b_3(s, r, x^r)\, dw(r)\right] dw(s), \\
(y_n^t)_r &= 0, \quad t \in [0, T], \quad n = 1, 2, \dots
\end{aligned} \tag{4.23}
$$

Note that $y_n(t)$ is determined by $x_n(t)$, i.e. x_n^t.

By following the proof of this theorem, which is very long and tiring, we find that

$$
E||y_n||_t^2 \le c_1 \frac{(c_2 t)^n}{n!}, \quad t_0 \le t \le T, \quad n \in N, \tag{4.24}
$$

where c_1, c_2 are generic constants, and also $\{x_n^t, n \in N\}$ in X converges with probability one, uniformly in $[0, T]$. Finally, we prove that $\{x_n(t), n \in N\}$ converges with probability one, uniformly in $[0, T]$, to the solution $x \in C_X$ of Eq. (3.9).

Clearly, because the Lipschitz condition for the coefficients $a_i, b_i, i = 1, 2, 3$ of Eq. (3.9) in general does not hold, it is not possible to prove that the sequence of iterations (4.22) represents the Z-algorithm. In paper [17] we expose a class of stochastic processes and introduce some modifications of the Lipschitz condition and of the bounded random integral contractor, which enables to show that the iterative procedure (4.22) is a special Z-algorithm. In fact, in [17] the following assertion is closely connected with Theorem 4 and Theorem 5.

Theorem 6. *Let the conditions of Theorem* 5 *be satisfied and the initial value* φ^0 *satisfies* $E||\varphi^0||_X^2 < \infty$. *Then Eq.* (3.9) *has a unique solution* x *in* C_X, *satisfying* $E\{\sup_{0 \le t \le T} |x(t)|^2\} < \infty$.

Therefore, the condition $E||\varphi^0||_X^2 < \infty$ is imposed to restrict the class of processes C_X to its sub-class $L_2(C_X)$ of stochastic processes in C_X, satisfying

$$
||x||_*^2 = E||x||_T^2 < \infty.
$$

Of course, $(L_2(C_X), ||\cdot||_*)$ is a Banach space.

The following assertion was proved in [17]:

Lemma 1. *Let the mappings $\Gamma_i, \Phi_i,\ i = 1,2,3$ satisfy the conditions* (4.17)*. Then for every $x, z \in L_2(C_X)$ Eq.* (4.21) *has a unique solution $y \in L_2(C_X)$. Moreover, there exists a constant $\gamma > 0$, independent on x and z, such that*

$$E||y||_t^2 \leq \gamma\, E||z||_t^2, \quad t \in [0,T].$$

The proof is based on the Banach fixed point theorem. Note that this lemma gives us an important fact, that every bounded random integral contractor (4.20) is regular in the space $L_2(C_X)$.

The Lipschitz condition (3.10) and the bounded random integral contractor (4.20) for the coefficients of Eq. (3.9), can be weakened in the space $L_2(C_X)$ by introducing the following modifications:

Let there exist a constant $L_1 > 0$ such that for all $(t,s) \in J$ and $x, y \in L_2(C_X)$,

$$\begin{aligned} &E|a_1(t,x^t) - a_1(t,y^t)|^2 \leq L_1 E||x-y||_t^2, \\ &E|a_i(t,s,x^s) - a_i(t,s,y^s)|^2 \leq L_1 E||x-y||_s^2, \quad i = 2,3, \end{aligned} \tag{4.25}$$

and analogously for $b_i,\ i = 1,2,3$. Then we say that *the functionals $a_i, b_i, i = 1,2,3$ satisfy the modified Lipschitz condition* in the space $L_2(C_X)$.

Let there exist a constant $K_1 > 0$ such that for all $(t,s) \in J$ and $x, y \in L_2(C_X)$,

$$\begin{aligned} &E|a_1(t,x^t + ((Ax)y)^t) - a_1(t,x^t) - \Gamma_1(t,x^t)y(t)|^2 \leq K_1 E||y||_t^2, \\ &E|a_i(t,s,x^s + ((Ax)y)^s) - a_i(t,s,x^s) - \Gamma_i(t,s,x^s)y(s)|^2 \leq K_1 E||y||_s^2, \\ &\qquad i = 2,3, \end{aligned} \tag{4.26}$$

and analogously for $b_i,\ i = 1,2,3$. Then we say that *the set of functionals* $(a_1,a_2,a_3,b_1,b_2,b_3)$ *has the modified bounded random integral contractor* in the space $L_2(C_X)$,

$$\begin{aligned} \Big\{ I + \int_0^t \Big[\Gamma_1 + \int_0^s \Gamma_2\, dr + \int_0^s \Gamma_3\, dw(r)\Big] ds \\ + \int_0^t \Big[\Phi_1 + \int_0^s \Phi_2\, dr \int_0^s \Phi_3\, dw(r)\Big] dw(s) \Big\}_E . \end{aligned} \tag{4.27}$$

Obviously, if the functionals $a_i, b_i, i = 1,2,3$ satisfy the Lipschitz condition (3.10) in the space $L_2(C_X)$, then they satisfy the modified Lipschitz condition (4.26), while the opposite assertion does not hold. It is not difficult to conclude that the proof of Theorem 2 is valid with the conditions (3.10) instead of (4.26). Similarly, by following the proofs of Theorem 4 and Theorem 5, we can see that they are valid if the functionals $a_i, b_i, i = 1,2,3$ have the modified bounded random integral contractor (4.27) instead of the bounded random integral contractor (4.20) in the space $L_2(C_X)$.

The following assertion, proved in [17], summarizes the preceding considerations, by showing the equivalence between the modified Lipschitz condition and the modified bounded random integral contractor.

Theorem 7. *The functionals $a_i, b_i, i = 1, 2, 3$ satisfy the modified Lipschitz condition* (4.26) *if and only if they have the modified bounded random integral contractor* (4.27).

Moreover, Theorem 7 enables us to solve the main problem from the beginning of this section, to show that the iterative procedure (4.22) represents a special Z-algorithm in the space $L_2(C_X)$.

Theorem 8. *Let the functionals $a_i, b_i,\ i = 1, 2, 3$ have the modified bounded random integral contractor* (4.27) *and the initial value φ^0 satisfy $E||\varphi^0||_X^2 < \infty$, $\int_0^T E|a_1(t, \varphi^0)|^2 dt < \infty$, $\int_0^T E|b_1(t, \varphi^0)|^2 dt < \infty$, $\int_0^T \int_0^t E|f(t, s, \varphi^0)|^2 ds\, dt < \infty$ for $a_i, b_i,\ i = 2, 3$ instead of f. Then the sequence of iterations* (4.22) *describes the Z-algorithm of Eq.* (3.9).

Indeed, from (4.22) and (4.23) we find

$$
\begin{aligned}
x_{n+1}(t) = \varphi(0) &+ \int_0^t \Big\{ a_1(s, x_n^s) - \Gamma_1(s, x_n^s)\, y_n(s) \\
&+ \int_0^s [a_2(s, r, x_n^r) - \Gamma_2(s, r, x_n^r)\, y_n(r)]\, dr \\
&+ \int_0^s [a_3(s, r, x_n^r) - \Gamma_3(s, r, x_n^r)\, y_n(r)]\, dw(r) \Big\} ds \\
&+ \int_0^t \Big\{ b_1(s, x_n^s) - \Phi_1(s, x_n^s)\, y_n(s) \\
&+ \int_0^s [b_2(s, r, x_n^r) - \Phi_2(s, r, x_n^r)\, y_n(r)]\, dr \\
&+ \int_0^s [b_3(s, r, x_n^r) - \Phi_3(s, r, x_n^r)\, y_n(r)]\, dw(r) \Big\} dw(s), \ \ 0 \le t \le T, \\
x_{n+1}(t) = \varphi(-t), &\quad t \le 0, \\
(x_{n+1}^t)_r(s) = &\begin{cases} x_{n+1}(t - s), & 0 \le s \le t \le T, \\ \varphi(s - t), & s > t, \end{cases}
\end{aligned}
$$

Let us denote that

$$
\begin{aligned}
f_{1n}(t, x) &= a_1(t, x_n^t) - \Gamma_1(t, x_n^t)\, y_n(t) \\
f_{in}(t, s, x) &= a_i(t, s, x_n^s) - \Gamma_i(t, s, x_n^s)\, y_n(s), \quad i = 2, 3, \\
g_{1n}(t, x) &= b_1(t, x_n^t) - \Phi_1(t, x_n^t)\, y_n(t) \\
g_{in}(t, s, x) &= b_i(t, s, x_n^s) - \Phi_i(t, s, x_n^s)\, y_n(s), \quad i = 2, 3,
\end{aligned}
$$

where $y_n(t)$ is expressed by $x_n(t)$ with help of (4.23). Then we obtain the following sequence of equations,

$$
\begin{aligned}
&dx_{n+1}(t) = \\
&\quad \left[f_{1n}(t, x^t_{n+1}) + \int_0^t f_{2n}(t, s, x^s_{n+1})\, ds + \int_0^t f_{3n}(t, s, x^s_{n+1})\, dw(s) \right] dt \\
&+ \left[g_{1n}(t, x^t_{n+1}) + \int_0^t g_{2n}(t, s, x^s_{n+1})\, ds + \int_0^t g_{3n}(t, s, x^s_{n+1})\, dw(s) \right] dw(t), \\
&\qquad t \in [0, T], \quad n \in N, \\
&\ x^0_{n+1} = \varphi^0.
\end{aligned}
$$

Clearly, the random functionals $f_{in}, g_{in}, i = 1, 2, 3$, satisfy the Lipschitz condition and the linear growth condition with probability one. Moreover, from Theorem 7 we find that the functionals $a_i, b_i, i = 1, 2,$, satisfy the modified Lipschitz condition (4.26) with the same constant $L_1 = 2\,(K + M^2)\gamma$, where γ is a constant from Lemma 1 and $M = \max\{\alpha_i, \beta_i, i = 1, 2, 3\}$. Then, from (3.15), for every $n = 0, 1, 2, \ldots$ we find

$$
\begin{aligned}
\varepsilon_n = E\{ \sup_J [\, &|a_1(t, x^t_n) - f_{1n}(t, x^t_n)|^2 + |a_2(t, s, x^s_n) - f_{2n}(t, s, x^s_n)|^2 \\
&+ |a_3(t, s, x^s_n) - f_{3n}(t, s, x^s_n)|^2 + |b_1(t, x^t_n) - g_{1n}(t, x^t_n)|^2 \\
&+ |b_2(t, s, x^s_n) - b_{2n}(t, s, x^s_n)|^2 + |b_3(t, s, x^s_n) - b_{3n}(t, s, x^s_n)|^2 \,]\} \\
\le 6M &\sup_{t \in [0,T]} E|y_n(t)|^2.
\end{aligned}
$$

Because of the norm in the space X, from which we have $|y_n(t)| \le ||y^t_n||_X$, and from the estimation (4.24), we find

$$
\varepsilon_n \le 6M \sup_{t \in [0,T]} E||y^t_n||_X \le 6M c_1 \frac{(c_2 t)^n}{n!}, \quad n \in N,
$$

which implies $\sum_{n=1}^{\infty} \varepsilon_n < \infty$. Therefore, from the weakened version of Theorem 3, it follows that $\{(f_{1n}, f_{2n}, f_{3n}, g_{1n}, g_{2n}, g_{3n}),\ n \in N\}$ represents a Z-algorithm for Eq. (3.9). It means that the sequence of iterations $\{x_n, n \in N\}$ converges with probability one, uniformly in $[0, T]$, to the solution $x \in L_2(C_X)$ of this equation, which is in accordance with the conclusion of the proof of Theorem 4.

Conclusion:

In general, it could be very interesting to study how the speed of convergence of the sequence of iterations $\{x_n, n \in N\}$ to the solution x of the original equation, depends on a choice of a determining sequence, and, also, how to choose the best one. Our further intention is to construct some other determining sequences and special Z-algorithms for different classes of stochastic differential equations and to choose the best ones, in the sense that the iterative equations could be effectively solvable or suitable for numerical treatments, with the fastest convergence of their solutions to the solution of the original equation.

Likewise, the iterative method presented in this paper, could be appropriately extended to stochastic differential equations including martingales and martingale measures instead of the Brownian motion process.

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NEKE ANALITIČKE ITERATIVNE METODE ZA REŠAVANJE RAZLIČITIH KLASA STOHASTIČKIH NASLEDNIH INTEGRODIFERENCIJALNIH JEDNAČINA

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Nasledni fenomeni su posebno pogodni za proučavanje fenomena u mehanici kontinuuma materijala sa memorijom. Matematički modeli takvih pojava se opisuju determinističkim naslednim diferencijalnim jednačinama, proučavanim u mnogim radovima i monografijama. Kasnije, ovi pojmovi su adekvatno prošireni na istraživanja pod uticajem Gaussovog belog

šuma, sa matematičkom interpretacijom stohastičkim naslednim diferencijalnim jednačinama tipa Itoa.

U ovom radu se razmatra opšta analitička iterativna metoda za rešavanje stohastičih naslednih integrodiferencijalnih jednačina tipa Itoa. Daju se dovoljni uslovi pri kojima niz iteracija konvergira u verovatnoći ka rešenju originalne jednačine. Ova metoda je opšta, u smislu da su mnoge poznate iterativne metode njeni specijalni slučajevi, na primer metoda sukcesivnih aproksimacija Picard-Lindelofa. Prikazane su i neke druge iterativne metode sa linearizacijom koeficijenata originalne jednačine.

Specijalno, koristeći koncept ograničenog slučajnog integralnog kontraktora u smislu Altmana i Kuoa, pokazuje se da je iterativna metoda primenjena u dokazu teoreme egzistencije i jedinstvenosti rešenja stohastičke nasledne integrodiferencijalne jednačine takodje specijalan slučaj prethodno opisane opšte iterativne metode.