EXISTENCE AND UNIQUENESS PROBLEMS FOR NONLINEAR STOCHASTIC HEREDITARY INTEGRODIFFERENTIAL EQUATIONS

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Abstract. The present paper is a continuation of the paper [6] in which the existence and uniqueness of solutions for nonlinear stochastic hereditary integrodifferential equations of the Ito type are considered, using the concept of a random integral contractor, which includes the Lipschitz condition as a special case. Here, we introduce a notion of a modified Lipschitz condition and of a modified random integral contractor, and we give conditions for their equivalence for a special class of stochastic processes. We also give new existence and uniqueness theorems.[123]

Key words: Stochastic hereditary integrodifferential equation, Modified random integral contractor, Modified Lipschitz condition.

1. Introduction

Let us briefly give some notations and known results of the previous paper [6] and of the basic paper [8] of Mizel and Trutzer treating stochastic hereditary differential equations of the Ito type. Remember that hereditary phenomena matematically describe various problems in continuum mechanics of materials with memories, as a version of the theory of "fading memory" spaces.

Let R^n be the real n-dimensional Euclidean space and L_p^ρ $(1 \le p < \infty)$ be the usual space of classes of measurable functions, i.e.,

$$L_p^\rho = \left\{ \varphi \mid \varphi : R^+ \to R^n \therefore \int_0^\infty |\varphi(s)|^p \rho(s)\, ds < \infty \right\}.$$

Here $\rho : R^+ \to R^+$ is an influence function with relaxation property, satisfying the following conditions: ρ is summable on R^+, for every $\sigma \ge 0$ one has $\overline{K}(\sigma) = \operatorname{ess\,sup}_{s\in R^+} \dfrac{\rho(s+\sigma)}{\rho(s)} \le \overline{\overline{K}} < \infty$, $\underline{K}(\sigma) = \operatorname{ess\,sup}_{s\in R^+} \dfrac{\rho(s)}{\rho(s+\sigma)} < \infty$. It is proved in [4] that ρ is essentially bounded, essentially strictly positive and $s \cdot \rho(s) \to 0$ as $s \to \infty$.

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Consider a product space $X = R^n \times L^\rho_p$, i.e. *a past–history space* of elements $x = (\varphi(0), \varphi)$, with the norm

$$||\, x||_X = \left(|\, \varphi(0)|^p + \int\limits_0^\infty |\, \varphi(s)|^p \rho(s)\, ds \right)^{1/p} = (|\, \varphi(0)|^p + ||\, \varphi||_r^p)^{1/p} .$$

Of course, X is a Banach space.

The measurable function $x : (-\infty, T] \to R^n$, T =const$\in R$, is X*–admissible* provided that for each $t \in (-\infty, T]$ the function x^t, called its *history up* to t and defined by

$$x^t_r(s) = x(t-s), \quad s \in R^+,$$

is itself a member of X.

The following inequality, utilised in the paper [8], is needed in our subsequent discussion:

$$||x^t||^2_X \leq \tilde{k} \left[|x(t)|^2 + \overline{\overline{K}}^{2/p} ||x^{t_0}||^2_r + \left(\int\limits_{t_0}^t |x(u)|^p \rho(t-u)\, du \right)^{2/p} \right], \tag{1.1}$$

where $\tilde{k} = 3^{2/p-1} \vee 1$.

Let $w = (w_t, t \geq 0)$ be a onedimensional standard Wiener process, based on a complete probability space $(\Omega, \mathcal{F}, P)$, adapted to the usual family $(\mathcal{F}_t, t \geq 0)$ of nondecreasing sub–σ–algebras of $\mathcal{F}$.

For $t_0 \geq 0$ let $\mathcal{X}_{t_0}$ be the space of measurable random processes $x(t)$, $t \leq t_0$, such that $x^{t_0} \in X$ for a.e. ω and such that for every t, $x(t)$ is independent of $\{w_u - w_{t_0} : u \geq t_0\}$. By the structure of the space X, it follows that $x^t \in X$ for all $t \leq t_0$ a.s.

Lemma 1.1. (*Mizel, Trutzer,* [13*, p.* 5]). *Let $x(t)$, $t \in R$, be a (jointly) measurable stochastic process such that $\sigma\{x(u) : u \leq t_0\} = \mathcal{G}_{t_0}$ is independent of $w_t - w_{t_0}$, $t \geq t_0$, and such that for $t \geq t_0$, $x(\cdot)$ is continuous and $\mathcal{G}_t := \mathcal{G}_{t_0} \vee \mathcal{F}_t$–progressivelly measurable. Assume that for a.e. ω the function $x^{t_0}(\cdot, \omega) \in X$. Then for $t \geq t_0$, $x^t(\omega) \in X$ for a.e. ω and the process x^t with values in X is a.s. continuous and $\mathcal{G}_t$–progressively measurable.*

In the present paper we consider the stochastic hereditary integrodifferential equation of the Ito type ($t_0 = 0$ for simplicity)

$$\begin{aligned} dx(t) &= \left[a_1(t, x^t) + \int\limits_0^t a_2(t, s, x^s)\, ds + \int\limits_0^t a_3(t, s, x^s)\, dw(s) \right] dt \\ &\quad + \left[b_1(t, x^t) + \int\limits_0^t b_2(t, s, x^s)\, ds + \int\limits_0^t b_3(t, s, x^s)\, dw(s) \right] dw(t), \quad t \in [0, T] \\ x^0 &= \varphi^0, \end{aligned} \tag{1.2}$$

earlier studied in details in the paper [8], which presents an extension of the Ito–Volterra equations developed by Berger and Mizel ([2]). Here $w(t)$ is a onedimensional standard Wiener process defined as above and the initial value φ^0, independent on w, is assumed to belong to $\mathcal{X}_0$. We also assume that $x(t)$ is an R^n–valued stochastic process and the Borel measurable functionals $a_i, b_i, i = 1, 2, 3$ are defined in $J \times X$ with values in R^n.

A stochastic process $(x(t), t \in (-\infty, T])$, is a (strong) *solution* of eq.(1.2) for $t \in [0, T]$ if $x(t)$ is nonanticipating for $t \leq T$, $x^t \in X$ a.s for $t \in [0, T]$, all Lebegues's and Ito's integrals in (1.2) exist, $x^0 = \varphi^0$ and eq. (1.2) holds for each $t \in [0, T]$.

Of course, if $(x(t), t \in (-\infty, T])$ is a solution of eq. (1.2), then $x^t = (x(t), x_r^t)$ for all $t \in [0, T]$, where

$$x(t) = \begin{cases} x(t), & 0 \leq t \leq T \\ \varphi(-t), & t \leq 0 \end{cases} \qquad x_r^t(s) = \begin{cases} x(t-s), & 0 \leq s \leq t \\ \varphi(s-t), & s > t. \end{cases}$$

From the structure of the solution x^t and from Lemma 1.1 it follows that if $x(t)$ is a.s. continuous, then x^t is also a.s. continuous.

In the paper [8] sufficient conditions of the existence and uniqueness of the solution for eq. (1.2) are given:

Theorem A. (*Mizel, Trutzer,* [8, *p.* 18]) *Assume that the functionals $a_i, b_i, i = 1, 2, 3$ satisfy the global Lipschitz condition and the usual condition of the restriction on growth on the last argument, i.e. assume that there exists a constant $L > 0$ such that for all $(t, s) \in J$ and $x, y \in X$,*

$$|a_3(t, s, x) - a_3(t, s, y)| \leq L\, ||x - y||_X, \tag{1.3}$$

$$|a_3(t, s, x)| \leq L\, (1 + ||x||_X), \tag{1.4}$$

and similarly for others. Assume that $\varphi^0 \in \mathcal{X}_0$ and $E||\varphi^0||_X^2 \leq \infty$. Then there exists a unique a.s. continuous solution x of eq. (1.2), satisfying $E \sup\limits_{0 \leq t \leq T} |x(t)|^2 < \infty$.

In what follows, denote by C_X a collection of R^n–valued stochastic processes, X–admissible on $(-\infty, T]$, almost surely continuous and nonanticipating with respect to the family $\{\mathcal{G}_t, t \geq 0\}$.

Following the basic ideas of Altman ([1]) and Kuo ([7]), and later of Murge and Pachpatte ([9]), Zhang and Padgett ([11]) and others, the concept of a bounded random integral contractor was introduced in the previous paper [6], to prove the existence and uniqueness of the solution of eq. (1.2). In the present paper, our intention is to investigate relations between different conditions for the existence and uniqueness of the solution of this equation. Because of that, let us briefly employ some elements of the contractor theory.

Let

$$\Gamma_1 : [0, T] \times X \to R^n \times R^n, \quad \Phi_1 : [0, T] \times X \to R^n \times R^n$$

$$\Gamma_i : J \times X \to R^n \times R^n, \qquad \Phi_i : J \times X \to R^n \times R^n, \ i = 2, 3$$

be measurable mappings, bounded in the sense that there exist positive constants $\alpha_i,\ \beta_i,\ i = 1, 2, 3$, such that for every $(t, s, x) \in J \times X,\ y \in R^n$,

$$\begin{aligned} &|\Gamma_1(t, x)y| \leq \alpha_1 |y|, && |\Phi_1(t, x)y| \leq \beta_1 |y| \\ &|\Gamma_i(t, s, x)y| \leq \alpha_i |y|, && |\Phi_i(t, s, x)y| \leq \beta_i |y|, \ i = 2, 3. \end{aligned} \tag{1.5}$$

Let, also, $((Ax)y)^t$ be an element of the space X, i.e., $((Ax)y)^t = ((Ax)y)(t), y_r^t)$, where

$$\begin{aligned}
((Ax)y)(t) &= y(t) \\
&+ \int_0^t \left[\Gamma_1(s, x^s)y(s) + \int_0^s \Gamma_2(s, r, x^r)y(r)\, dr + \int_0^s \Gamma_3(s, r, x^r)y(r)\, dw(r) \right] ds \\
&+ \int_0^t \left[\Phi_1(s, x^s)y(s) + \int_0^s \Phi_2(s, r, x^r)y(r)\, dr + \int_0^s \Phi_3(s, r, x^r)y(r)\, dw(r) \right] dw(s),
\end{aligned} \tag{1.6}$$

and y_r^t is an element on L_p^ρ.

Suppose there exists a positive constant K such that, for any x^t, y^t, in X and $(t, s) \in J$, the following inequalities hold almost surely:

$$\begin{aligned}
&| a_1(t, x^t + ((Ax)y)^t) - a_1(t, x^t) - \Gamma_1(t, x^t)y(t)| \le K||\, y||_t \\
&| a_i(t, s, x^s + ((Ax)y)^s) - a_i(t, s, x^s) - \Gamma_i(t, s, x^s)y(s)| \le K||\, y||_s, \;\; i = 2, 3 \\
&| b_1(t, x^t + ((Ax)y)^t) - b_1(t, x^t) - \Phi_1(t, x^t)y(t)| \le K||\, y||_t \\
&| b_i(t, s, x^s + ((Ax)y)^s) - b_i(t, s, x^s) - \Phi_i(t, s, x^s)y(s)| \le K||\, y||_s, \;\; i = 2, 3,
\end{aligned} \tag{1.7}$$

where

$$||\, y||_t = \sup_{0 \le s \le t} ||\, y^s||_X .$$

Then the set of functionals $(a_1, a_2, a_3, b_1, b_2, b_3)$ has *a bounded random integral contractor*

$$\left\{ I + \int_0^t \left[\Gamma_1 + \int_0^s \Gamma_2\, dr + \int_0^s \Gamma_3\, dw(r) \right] ds + \int_0^t \left[\Phi_1 + \int_0^s \Phi_2\, dr \int_0^s \Phi_3\, dw(r) \right] dw(s) \right\} \tag{1.8}$$

A bounded random integral contractor is said to be *regular* if the linear equation

$$(Ax)y = z \tag{1.9}$$

has a solution y in X for any x and z in X.

A functional $h : [0, T] \times X \to R^n$ is said to be *stochastically closed* if for any x and x_n in X, such that $x_n \to x$ and $h(\cdot, x_n^\cdot) \to y$ in $L^2([0, T] \times \Omega)$, we have $y(t) = h(t, x^t)$ for every $t \in [0, T]$ almost surely. The stochastic closeness of a functional $h : J \times X \to R^n$ is defined analogously.

It is clear that if the functionals $a_i,\, b_i,\;\, i = 1, 2, 3$, satisfy the Lipschitz condition (1.3), then they are stochastically closed and the set of functionals $(a_1, a_2, a_3, b_1, b_2, b_3)$ has a trivial bounded random integral contractor (1.8) for $\Gamma_i = \Phi_i = 0,\;\, i = 1, 2, 3$. On the other hand, in the paper [6] it was also shown the Lipschitz condition implies the existence of a class of bounded random integral contractors (1.8) in which Γ_1 and Γ_2 are arbitrary mappings defined as in (1.5) and $\Gamma_3 = 0, \Phi_i = 0,\;\, i = 1, 2, 3$. Moreover, it was shown that eq. (1.2) could have a regular bounded random integral contractor, although the Lipschitz condition did not have to be satisfied.

Theorem B. (*Jovanović, Janković,* [5, *p.* 1068]) *Suppose that the functionals* a_i, b_i, $i = 1, 2, 3$, *are stochastically closed and have a bounded random integral contractor* (1.8). *Suppose, also, that* $\int_0^T E\big|a_1(t,\varphi^0)\big|^2 dt < \infty$, $\int_0^T E\big|b_1(t,\varphi^0)\big|^2 dt < \infty$, *and* $\int_0^T\int_0^t E\big|f(t,s,\varphi^0)\big|^2 ds\, dt < \infty$ *for* a_i, b_i, $i = 2, 3$. *Then eq.* (1.2) *has a solution* x *in* C_X.

Theorem C. *(Jovanović, Janković,* [5, *p.* 1068]*) Let the functionals* a_i, b_i, $i = 1, 2, 3$, *satisfy the assumptions of Theorem B and let the bounded random integral contractor be regular. Then the solution of eq.* (1.2) *in* C_X *is unique.*

The main purpose of the present paper is to give a class of stochastic processes and to define modifications of the Lipschitz condition and of the bounded random integral contractor in this class, such that the previously cited theorems for the existence and uniquenes are valid, as well as to estabilish relations between these conditions.

2. Main results

The following assertion, although independent of the previously mentioned problems, is closely connected with Theorem B and Theorem C.

Theorem 2.1. *Let the conditions of Theorem C be satisfied and let the initial value* $\varphi^0 \in \mathcal{X}_0$ *satisfy* $E||\varphi^0||_X^2 < \infty$. *Then eq.* (1.2) *has a solution* x *in* C_X, *satisfying*

$$E \sup_{0\le t\le T} |x(t)|^2 < \infty \tag{2.1}$$

Proof. Because from Theorem B and Theorem C it follows that eq. (1.2) has a unique solution $x \in C_X$, it remains to prove that (2.1) is valid.

Since the bounded random integral contractor (1.8) is regular, then the linear operator equation

$$((Ax)y)^t = \varphi^0 - x^t, \quad t \in [0, T], \tag{2.2}$$

has a solution $y^t \in X, t \in [0, T]$. Because $x \in C_X$, if tollows that $y \in C_X$. From (1.6) we have

$$\begin{aligned}
y(t) &+ \int_0^t \left[\Gamma_1(s, x^s)y(s) + \int_0^s \Gamma_2(s, r, x^r)y(r)\, dr + \int_0^s \Gamma_3(s, r, x^r)y(r)\, dw(r)\right] ds \\
&+ \int_0^t \left[\Phi_1(s, x^s)y(s) + \int_0^s \Phi_2(s, r, x^r)y(r)\, dr + \int_0^s \Phi_3(s, r, x^r)y(r)\, dw(r)\right] dw(s) \\
&= \varphi(0) - x(t), \\
y_r^t &= \varphi_r^0 - x_r^t.
\end{aligned} \tag{2.3}$$

If we substitute the right side in (2.3) with (1.2), we obtain

$$\begin{aligned} y(t) = &- \int_0^t \Big[a_1(s,x^s) + \Gamma_1(s,x^s)y(s) + \int_0^s \big[a_2(s,r,x^r) + \Gamma_2(s,r,x^r)y(r)\big]\,dr \\ &+ \int_0^s \big[a_3(s,r,x^r) + \Gamma_3(s,r,x^r)y(r)\big]\,dw(r) \Big]\,ds \\ &- \int_0^t \Big[b_1(s,x^s) + \Phi_1(s,x^s)y(s) + \int_0^s \big[b_2(s,r,x^r) + \Phi_2(s,r,x^r)y(r)\big]\,dr \\ &+ \int_0^s \big[b_3(s,r,x^r) + \Phi_3(s,r,x^r)y(r)\big]\,dw(r) \Big]\,dw(s), \quad t \in [0,T]. \end{aligned}$$

By applying the elementary inequality $\left(\sum_{i=1}^k a_i\right)^2 \le k \sum_{i=1}^k a_i^2$, for $k = 6$ we obtain

$$E \sup_{0\le u\le t} |y(u)|^2 \le 6 \sum_{i=1}^6 J_i,$$

where

$$\begin{aligned} J_1 &= E \sup_{0\le u\le t} \left| \int_0^u \big[a_1(s,x^s) + \Gamma_1(s,x^s)y(s)\big]\,ds \right|^2, \\ J_2 &= E \sup_{0\le u\le t} \left| \int_0^u \int_0^s \big[a_2(s,r,x^r) + \Gamma_2(s,r,x^r)y(r)\big]\,dr\,ds \right|^2, \\ J_3 &= E \sup_{0\le u\le t} \left| \int_0^u \int_0^s \big[a_3(s,r,x^r) + \Gamma_3(s,r,x^r)y(r)\big]\,dw(r)\,ds \right|^2, \\ J_4 &= E \sup_{0\le u\le t} \left| \int_0^u \big[b_1(s,x^s) + \Phi_1(s,x^s)y(s)\big]\,dw(s) \right|^2, \\ J_5 &= E \sup_{0\le u\le t} \left| \int_0^u \int_0^s \big[b_2(s,r,x^r) + \Phi_2(s,r,x^r)y(r)\big]\,dr\,dw(s) \right|^2, \\ J_6 &= E \sup_{0\le u\le t} \left| \int_0^u \int_0^s \big[b_3(s,r,x^r) + \Phi_3(s,r,x^r)y(r)\big]\,dw(r)\,dw(s) \right|^2. \end{aligned}$$

To estimate these integrals, we shall use (2.2), from where

$$a_1(s, x^s + ((Ax)y)^s) = a_1(s,\varphi^0) \ \text{ a.s.}, \quad b_2(s,r,x^r + ((Ax)y)^r) = b_2(s,r,\varphi^0) \ \text{ a.s.},$$

and similarly for the other functionals. By applying (1.6), the usual stochastic integral isometry, Schwarz inequality, Doob inequality and integration by parts, we come to the following estimations:

$$J_1 = E \sup_{0\le u\le t} \left| \int_0^u \left[a_1(s,x^s) + \Gamma_1(s,x^s)y(s) - a_1(s,x^s + ((Ax)y)^s)\right] ds + \int_0^u a_1(s,\varphi^0)\,ds \right|^2$$

$$\le 2TK^2 \int_0^t E||y||_s^2\,ds + 2T\int_0^T E|a_1(s,\varphi^0)|^2\,ds,$$

$$J_2 = E \sup_{0\le u\le t} \left| \int_0^u \int_0^s \left[a_2(s,r,x^r) + \Gamma_2(s,r,x^r)y(r) - a_2(s,r,x^r + ((Ax)y)^r)\right] dr\,ds + \int_0^u \int_0^s a_2(s,r,\varphi^0)\,dr\,ds \right|^2$$

$$\le T^3K^2 \int_0^t E||y||_s^2\,ds + 2T\int_0^T s\int_0^s E|a_2(s,r,\varphi^0)|^2\,dr\,ds,$$

$$J_3 = E \sup_{0\le u\le t} \left| \int_0^u \int_0^s \left[a_3(s,r,x^r) + \Gamma_3(s,r,x^r)y(r) - a_3(s,r,x^r + ((Ax)y)^r)\right] dw(r)\,ds + \int_0^u \int_0^s a_3(s,r,\varphi^0)\,dw(r)\,ds \right|^2$$

$$\le 2T^2K^2 \int_0^t E||y||_s^2\,ds + 2T\int_0^T \int_0^s E|a_3(s,r,\varphi^0)|^2\,dr\,ds,$$

$$J_4 = E \sup_{0\le u\le t} \left| \int_0^u \left[b_1(s,x^s) + \Phi_1(s,x^s)y(s) - b_1(s,x^s + ((Ax)y)^s)\right] dw(s) + \int_0^u b_1(s,\varphi^0)\,dw(s) \right|^2$$

$$\le 8K^2 \int_0^t E||y||_s^2\,ds + 8\int_0^T E|b_1(s,\varphi^0)|^2\,ds,$$

$$J_5 = E \sup_{0 \le u \le t} \left| \int_0^u \int_0^s \left[b_2(s,r,x^r) + \Phi_2(s,r,x^r)y(r) - b_2(s,r,x^r + ((Ax)y)^r) \right] dr\, dw(s) \right.$$

$$\left. + \int_0^u \int_0^s b_2(s,r,\varphi^0)\, dr\, dw(s) \right|^2$$

$$\le 8T^2K^2 \int_0^t E||y||_s^2\, ds + 8 \int_0^T s \int_0^s E|b_2(s,r,\varphi^0)|^2\, dr\, ds,$$

$$J_6 = E \sup_{0 \le u \le t} \left| \int_0^u \int_0^s \left[b_3(s,r,x^r) + \Phi_3(s,r,x^r)y(r) - b_3(s,r,x^r + ((Ax)y)^r) \right] dw(r)\, dw(s) \right.$$

$$\left. + \int_0^u \int_0^s b_3(s,r,\varphi^0)\, dw(r)\, dw(s) \right|^2$$

$$\le 8TK^2 \int_0^t E||y||_s^2\, ds + 8 \int_0^T \int_0^s E|b_3(s,r,\varphi^0)|^2\, dr\, ds.$$

Therefore,

$$E \sup_{0 \le u \le t} |y(u)|^2 \le c_1 \int_0^t E||y||_s^2\, ds + c_2, \quad t \in [0,T],$$

where c_1 and c_2 are some generic constants. Since from (2.3) we have $||y^0||_r = 0$, by using the inequality (1.1) we obtain

$$E||y||_t^2 \le \tilde{k} \left(1 + ||\rho||_{L_1}^{2/p}\right) E \sup_{0 \le u \le t} |y(u)|^2$$

$$\le \tilde{k} \left(1 + ||\rho||_{L_1}^{2/p}\right) \left[c_1 \int_0^t E||y||_s^2\, ds + c_2 \right], \quad t \in [0,T],$$

where $||\rho||_{L_1} = \int_0^\infty \rho(s)\, ds$. By applying now the usual Gronwall-Bellman inequality, we find

$$E||y||_t^2 \le \infty, \quad t \in [0,T].$$

Finally, it is easy to prove (2.1) starting from (2.3). By using the foregoing estimation, the basic property of the norm $||\cdot||_X$ from which $E \sup_{0 \le s \le t} |y(s)|^2 \le E||y||_t^2$, and the boundeness of the mappings $\Gamma_i, \Phi_i, i = 1,2,3$ in the sense (1.5), we come to the estimation:

$$
\begin{aligned}
E \sup_{0\le t\le T} |x(t)|^2 \le 8 \Bigg[& E|\varphi(0)|^2 + E \sup_{0\le t\le T} |y(t)|^2 + E \sup_{0\le t\le T} \left| \int_0^t \Gamma_1(s,x^s) y(s)\, ds \right|^2 \\
& + E \sup_{0\le t\le T} \left| \int_0^t \int_0^s \Gamma_2(s,r,x^r) y(r)\, dr\, ds \right|^2 + E \sup_{0\le t\le T} \left| \int_0^t \int_0^s \Gamma_3(s,r,x^r) y(r)\, dw(r)\, ds \right|^2 \\
& + E \sup_{0\le t\le T} \left| \int_0^t \Phi_1(s,x^s) y(s)\, dw(s) \right|^2 + E \sup_{0\le t\le T} \left| \int_0^t \int_0^s \Phi_2(s,r,x^r) y(r)\, dr\, dw(s) \right|^2 \\
& + E \sup_{0\le t\le T} \left| \int_0^t \int_0^s \Phi_3(s,r,x^r) y(r)\, dw(r)\, dw(s) \right|^2 \Bigg] \\
\le\, & 8 \left(E||\varphi^0||_X^2 + E||y||_T^2 \right) \\
& + 8 \left(\alpha_1^2 T + \alpha_2^2 \frac{T^3}{2} + \alpha_3^2 T^2 + 4\beta_1^2 + 4\beta_2^2 T^2 + 4\beta_3^2 T \right) \int_0^t E||y||_s^2\, ds \\
<\, & \infty. \quad \square
\end{aligned}
$$

Moreover, from Theorem 2.1 and from (1.1) it follows that

$$E||x||_T^2 \le \tilde{k} \left(1 + ||\rho||_{L_1}^{2/p}\right) E \sup_{0\le t\le T} |x(t)|^2 + \tilde{k}\, \overline{\overline{K}}^{2/p} ||\varphi^0||^2 < \infty.$$

This fact gives us a motivation to consider the existence and uniqueness problems for eq. (1.2) on a class $L_2(C_X)$ of stochastic processes $x \in C_X$ with the norm

$$||\, x||_*^2 \, := E||\, x||_T^2 < \infty.$$

Of course, $\left(L_2(C_X), ||\cdot||_*\right)$ is a Banach space.

Having in mind partially the ideas from papers [5] and [10], let us define the next norm on the space $L_2(C_X)$: For a fixed number $\mu > 0$ and for every $x \in L_2(C_X)$, denote

$$|||\, x|||^2 \, := \sup_{0\le t\le T} E\left\{ ||\, x||_t^2 \cdot e^{-2\mu t} \right\}. \tag{2.4}$$

Since

$$||\, x||_*^2 \cdot e^{-2\mu T} \le |||\, x|||^2 \le ||\, x||_*^2,$$

the norms $|||\cdot|||$ and $||\cdot||_*$ are equivalent. Therefore, $\left(L_2(C_X), |||\cdot|||\right)$ is also a Banach space.

We need the following assertion, essentially used to study the relations between different conditions for the existence and uniqueness of solutions of eq. (1.2).

Lemma 2.1. *Let the mappings $\Gamma_i, \Phi_i,\ i=1,2,3$ satisfy the conditions* (1.5)*. Then for every $x,z \in L_2(C_X)$ eq.* (1.9) *has a unique solution $y \in L_2(C_X)$. Moreover, there exists a constant $\gamma > 0$, independent on x and z, such that*

$$E||y||_t^2 \le \gamma E||z||_t^2, \quad t \in [0,T]. \tag{2.5}$$

Proof. Let us define an operator S on the space $L_2(C_X)$ in the following way: For fixed $x,z \in L_2(C_X)$ and arbitrary $y \in L_2(C_X)$, let

$$\begin{aligned}
(Sy)(t) \, : &= z(t) \\
&- \int_0^t \left[\Gamma_1(s,x^s)y(s) + \int_0^s \Gamma_2(s,r,x^r)y(r)\,dr + \int_0^s \Gamma_3(s,r,x^r)y(r)\,dw(r) \right] ds \\
&- \int_0^t \left[\Phi_1(s,x^s)y(s) + \int_0^s \Phi_2(s,r,x^r)y(r)\,dr + \int_0^s \Phi_3(s,r,x^r)y(r)\,dw(r) \right] dw(s), \\
(Sy)_r^t &= z_r^t\,.
\end{aligned} \tag{2.6}$$

Clearly, from the basic properties of Lebesgue and Ito integrals, it follows that $Sy \in C_X$. Using the inequality (1.1), for $0 \le s \le t \le T$ we obtain

$$\begin{aligned}
||(Sy)^s||_X^2 &\le \tilde{k} \left[|(Sy)(s)|^2 + \overline{\overline{K}}^{2/p} ||z^0||_r^2 + \left(\int_0^s |(Sy)(u)|^p \rho(s-u)\,du \right)^{2/p} \right] \\
&\le \tilde{k} \left(1 + ||\rho||_{L_1}^2 \right) \sup_{0 \le s \le t} |(Sy)(s)|^2 + \tilde{k}\, \overline{\overline{K}}^{2/p} ||z||_s^2.
\end{aligned} \tag{2.7}$$

Denote $B = \tilde{k} \left(1 + ||\rho||_{L_1}^{2/p} \right)$ and $M = \max\{\alpha_i, \beta_i, i=1,2,3\}$. By applying the previously used inequalities and stochastic integral isometry on (2.6), we find

$$
\begin{aligned}
E||Sy||_t^2 &= E \sup_{0\le s\le t} ||(Sy)^s||_X^2 \\
&\le \tilde{k}\,\overline{\overline{K}}^{2/p} ||z||_t^2 + 7B \Bigg[E||z||_t^2 + t\int_0^t E|\Gamma_1(s,x^s)y(s)|^2\,ds \\
&\quad + t\int_0^t s\int_0^s E|\Gamma_2(s,u,x^u)y(u)|^2\,du\,ds + t\int_0^t\int_0^s E|\Gamma_3(s,u,x^u)y(u)|^2\,du\,ds \\
&\quad + 4\int_0^t E|\Phi_1(s,x^s)y(s)|^2\,ds + 4\int_0^t s\int_0^s E|\Phi_2(s,u,x^u)y(u)|^2\,du\,ds \\
&\quad +4\int_0^t\int_0^s E|\Phi_3(s,u,x^u)y(u)|^2\,du\,ds \Bigg] \\
&\le \left(\tilde{k}\,\overline{\overline{K}}^{2/p} + 7B\right) E||z||_t^2 \\
&\quad + 7BM^2(T+4) \left[\int_0^t E||y^s||_X^2\,ds + (T+1)\int_0^t\int_0^s E||y^u||_X^2\,du\,ds \right] \\
&\le \left(\tilde{k}\,\overline{\overline{K}}^{2/p} + 7B\right) E||z||_t^2 + N\int_0^t E||y^s||_X^2\,ds, \quad t\in[0,T],
\end{aligned}
\tag{2.8}
$$

where N is a constant depending on B, M and T. Finally,

$$
||Sy||_*^2 \le \left(\tilde{k}\,\overline{\overline{K}}^{2/p} + 7B\right) ||z||_*^2 + NT\,||y||_*^2 < \infty.
$$

Therefore, $Sy \in L_2(C_X)$, i.e. $S : L_2(C_X) \to L_2(C_X)$.

Next, we shall see that there exists a constant $\mu > 0$ for which the operator S is a contraction. For arbitrary $y_1, y_2 \in L_2(C_X)$, from (2.6) it follows that $(Sy_1)_r^t - (Sy_2)_r^t = 0$ for $t \in [0,T]$. Then from the estimation (2.7) we get

$$
||(Sy_1)^s - (Sy_2)^s||_X^2 \le B \sup_{0\le s\le t} |(Sy_1)(s) - (Sy_2)(s)|^2, \quad 0\le s\le t\le T.
$$

Similarly, by repeating the procedures used in (2.8), we find

$$
E||Sy_1 - Sy_2||_t^2 \le N_1 \int_0^t E||y_1^s - y_2^s||_X^2\,ds, \quad t\in[0,T],
$$

where $N_1 = 6BN$. Thus for a number $\mu > 0$ we obtain

$$\begin{aligned} E||Sy_1 - Sy_2||_t^2 &\leq N_1 \int\limits_0^t E \sup_{0 \leq u \leq s} ||y_1^u - y_2^u||_X^2 \cdot e^{-2\mu s} \cdot e^{2\mu s}\, ds \\ &\leq N_1\, |||y_1 - y_2|||^2 \int\limits_0^t e^{2\mu s}\, ds \\ &\leq \frac{N_1}{2\mu} \cdot e^{2\mu t} \cdot |||y_1 - y_2|||^2, \quad 0 \leq t \leq T. \end{aligned}$$

So, for $\mu > 2N_1 + 1$ we have

$$|||Sy_1 - Sy_2|||^2 \leq \sup_{0 \leq t \leq T} E||Sy_1 - Sy_2||_t^2 \cdot e^{-2\mu t} < \frac{1}{4}\, |||y_1 - y_2|||^2,$$

and, therefore, S is a contraction. By the Banach fixed point theorem it follows that eq. (1.2) has a unique solution $y \in L_2(C_X)$, which completes the proof of the first part of Lemma 2.1.

In order to prove the second part of this lemma, we need the following version of the Gronwall – Bellman inequality (Bainov, Simeonov [2, p. 3]):

Let $u(t)$, $p(t)$ and $q(t)$ be continuous functions in $[a, b]$ and let $q(t)$ be nonnegative in $[a, b]$. Suppose

$$u(t) \leq p(t) + \int\limits_a^t q(s)u(s)\, ds, \quad t \in [a, b]. \tag{2.9}$$

Then

$$u(t) \leq p(t) + \int\limits_a^t p(s)q(s) \exp\Big(\int\limits_s^t q(u)\, du \Big) ds, \quad t \in [a, b].$$

By putting $Sy = y$ in (2.8) and by using $E||y^s||_X^2 \leq E||y||_s^2$, we obtain

$$E||y||_t^2 \leq \big(\tilde{k}\, \overline{\overline{K}}^{2/p} + 7B \big)\, E||z||_t^2 + N \int\limits_0^t E||y||_s^2\, ds, \quad t \in [0, T].$$

By applying the cited integral inequality (2.9), it is easy to arrive at the desired relation (2.5), in which γ is a generic constant depending on $B, N, T, \tilde{k}, \overline{\overline{K}}$. Thus this lemma is completely proved. $\square$

Now, it is easy to formulate the following assertion:

Theorem 2.2. *Let the conditions of Theorem B be satisfied and let the initial value* $\varphi^0 \in \mathcal{X}_0$ *satisfy* $E||\varphi^0||_X^2 < \infty$. *Then eq.* (1.2) *has a unique solution* x *in* $L_2(C_X)$.

Proof. The proof immediately holds from Theorem 2.1, since from Lemma 2.1 it follows that every bounded random integral contractor (1.8) is regular on the space $L_2(C_X)$. $\square$

Let us introduce the following version of the Lipschitz condition:

Definition 2.1. *Let there exist a constant $L_1 > 0$ such that for all $(t, s) \in J$ and $x, y \in L_2(C_X)$,*

$$\begin{aligned} &E|a_1(t, x^t) - a_1(t, y^t)|^2 \le L_1 E||x - y||_t^2, \\ &E|a_i(t, s, x^s) - a_i(t, s, y^s)|^2 \le L_1 E||x - y||_s^2, \ i = 2, 3, \end{aligned} \tag{2.10}$$

and analogously for $b_i, \ i = 1, 2, 3$. Then we say that the functionals $a_i, b_i, i = 1, 2, 3$ satisfy the modified Lipschitz condition on the space $L_2(C_X)$.

Obviously, if the functionals $a_i, b_i, i = 1, 2, 3$ satisfy the Lipschitz condition (1.3) on the space $L_2(C_X)$, then they satisfy the modified Lipschitz condition (2.10). Moreover, following the proofs of Theorem A from the paper [8], it is not difficult to conclude that this theorem is valid with the condition (2.10) instead of (1.3).

Lemma 2.1 can be applied to express a relation between the bounded random integral contractor and the modified Lipschitz condition on the space $L_2(C_X)$.

Proposition 2.1. *Let the functionals $a_i, b_i, i = 1, 2, 3$ from eq.* (1.2) *have a bounded random integral contractor* (1.8)*. Then they satisfy the modified Lipcshitz condition* (2.10) *on the space $L_2(C_X)$.*

Proof. Let (1.8) be a bounded random integral contractor for the functionals $a_i, b_i, i = 1, 2, 3$. Then from Lemma 2.1, for fixed $x, z \in L_2(C_X)$ there exists a unique solution $y \in L_2(C_X)$ of eq. (1.9). From (1.7) it follows that

$$|a_1(t, x^t + z^t) - a_1(t, x^t) - \Gamma_1(t, x^t)y(t)| \le K\,||y||_t \text{ a.s.}$$

From the properties (1.5) for the mappings $\Gamma_i, \Phi_i, i = 1, 2, 3$ we have

$$\begin{aligned} |a_1(t, x^t + z^t) - a_1(t, x^t)|^2 &\le 2|a_1(t, x^t + z^t) - a_1(t, x^t) - \Gamma_1(t, x^t)y(t)|^2 + 2|\Gamma_1(t, x^t)y(t)|^2 \\ &\le 2(K^2 + \alpha_1^2)\,||y||_t^2 \quad \text{a.s.} \end{aligned}$$

Finally, by using the estimation (2.5) from Lemma 2.1, we find

$$\begin{aligned} E|a_1(t, x^t + z^t) - a_1(t, x^t)|^2 &\le 2(K^2 + \alpha_1^2)\,E||y||_t^2 \\ &\le 2(K^2 + \alpha_1^2)\,\gamma\,E||z||_t^2, \quad t \in [0, T], \end{aligned}$$

and analogously for the other functionals, which completes the proof. □

Since the opposite assertion with respect to Proposition 2.1 is generally not valid, we introduce the notion of a modified bounded random integral contractor on the space $L_2(C_X)$.

Definition 2.2. *Let there exists a constant $K_1 > 0$ such that for all $(t, s) \in J$ and $x, y \in L_2(C_X)$,*

$$\begin{aligned} &E|a_1(t, x^t + ((Ax)y)^t) - a_1(t, x^t) - \Gamma_1(t, x^t)y(t)|^2 \le K_1 E||y||_t^2, \\ &E|a_i(t, s, x^s + ((Ax)y)^s) - a_i(t, s, x^s) - \Gamma_i(t, s, x^s)y(s)|^2 \le K_1 E||y||_s^2, \ i = 2, 3, \end{aligned} \tag{2.11}$$

and analogously for $b_i,\ i=1,2,3$. Then we say that the set of functionals $(a_1,a_2,a_3,b_1,b_2,b_3)$ *has a modified bounded random integral contractor on the space* $L_2(C_X)$

$$\left\{ I + \int_0^t \left[\Gamma_1 + \int_0^s \Gamma_2 dr + \int_0^s \Gamma_3 dw(r) \right] ds + \int_0^t \left[\Phi_1 + \int_0^s \Phi_2 dr \int_0^s \Phi_3 dw(r) \right] dw(s) \right\}_E . \quad (2.12)$$

Let us remember that by following the proofs of Theorem B, Theorem C and Theorem 2.2, we can see that they are valid if the functionals $a_i, b_i, i=1,2,3$ have the modified bounded random integral contractor (2.12) instead of the bounded random integral contractor (1.8) on the space $L_2(C_X)$.

It is easy now to prove the equivalence between the modified Lipschitz condition (2.10) and the modified bounded random integral contractor (2.12).

Proposition 2.2. *The functionals* $a_i, b_i, i=1,2,3$ *satisfy the modified Lipschitz condition* (2.10) *if and only if they have the modified bounded random integral contractor* (2.12).

Proof. First, from $x,y\in L_2(C_X)$ it follows $(Ax)y\in L_2(C_X)$. Starting from (1.6) and applying the same reasoning as for the operator S defined with (2.6), from (2.8) we obtain

$$E||(Ax)y||_t^2 \le \overline{L}\, E||y||_t^2, \quad t\in[0,T], \quad (2.13)$$

where $\overline{L}_1$ is some generic constant.

Now, let the functionals $a_i, b_i, i=1,2,3$ satisfy the modified Lipschitz condition (2.10). Then for every $t\in[0,T]$ and $x,y\in L_2(C_X)$, by applying (2.13) we find

$$\begin{aligned} E|a_1(t,x^t+((Ax)y)^t)-a_1(t,x^t)-\Gamma_1(t,x^t)y(t)|^2 &\le 2L_1E||(Ax)y||_t^2+2\alpha_1^2\,E||y||_t^2 \\ &\le 2\big(L_1\overline{L}+\alpha_1^2\big)\,E||y||_t^2, \end{aligned}$$

and analogously for the other functionals. Since the relations (2.11) are satisfied, then the set of functionals $(a_1,a_2,a_3,b_1,b_2,b_3)$ has the modified bounded random integral contractor (2.12).

Conversely, let the set of functionals $(a_1,a_2,a_3,b_1,b_2,b_3)$ has the modified bounded random integral contractor (2.12). Then from Lemma 2.1 it follows that for every $x,z\in L_2(C_X)$ eq. $(Ax)y=z$ has a unique solution $y\in L_2(C_X)$, such that $E||y||_t^2\le\gamma\,E||z||_t^2$ for all $t\in[0,T]$. Therefore,

$$\begin{aligned} E|a_1(t,x^t+z^t)-a_1(t,x^t)|^2 &\le 2E|a_1(t,x^t+((Ax)y)^t)-a_1(t,x^t)-\Gamma_1(t,x^t)y(t)|^2 \\ &\quad + 2E|\Gamma_1(t,x^t)y(t)|^2 \\ &\le 2(K_1+\alpha_1^2)\,E||y||_t^2 \\ &\le 2(K_1+\alpha_1^2)\,\gamma\,E||z||_t^2. \end{aligned}$$

Similar proof holds for the other functionals. Thus the assertion is completely proved. □

The following theorem summarises the foregoing assertions:

Theorem 2.3. *Let the functionals a_i, b_i, $i = 1,2,3$ satisfy the modified Lipschitz condition* (2.12). *Let also the initial value $\varphi^0 \in \mathcal{X}_0$ satisfy $E||\varphi^0||_X^2 < \infty$, $\int_0^T E|a_1(t,\varphi^0)|^2 dt < \infty$, $\int_0^T E|b_1(t,\varphi^0)|^2 dt < \infty$ and $\int_0^T \int_0^t E|f(t,s,\varphi^0)|^2 ds\, dt < \infty$ for a_i, b_i, $i = 2,3$. Then eq.* (1.2) *has a unique solution x in $L_2(C_X)$.*

Proof. Remember that if the functionals $a_i, b_i, i = 1,2,3$ satisfy the modified Lipcshitz condition (2.10), then they are stochastically closed in the sense of the definition of stochastical closeness from Section 1. Furthermore, the proof follows immediately by applying Proposition 2.2 and Theorem 2.1. □

Note that this theorem is expressed without the growth condition (1.4).

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