CONVERGENCE IN (2m)-th MEAN FOR PERTURBED STOCHASTIC INTEGRODIFFERENTIAL EQUATIONS

Sv. Janković[1] and M. Jovanović,[2]
Department of Mathematics, University of Niš, Faculty of Philosophy
Ćirila i Metodija 2, 18000 Niš, Yugoslavia

Abstract

The goal of this paper is to study the (2m)-th asymptotic behavior for the family of stochastic processes $x^\varepsilon = (x_t^\varepsilon, t \in [t_0, \infty))$, depending on a "small" parameter $\varepsilon \in (0,1)$. We consider the case when x^ε is the solution of an Itô's stohastic integro-differential equation whose coefficients are additionally perturbed. We compare the solution x^ε with the solution of an appropriate unperturbed equation of the equal type. Sufficient conditions under which these solutions are close in the $(2m)$-th moment sense on intervals whose length tends to infinity are given.[3]

1. Introduction

In many fields of physical and engineering sciences there are large numbers of real phenomena depending on perturbations, which are mathematically modeled and described by generalized Itô type stochastic differential equations (for example, see [**3**], [**6**], [**16**]). In the present paper we consider the problems of perturbations for one of the important, very general class of these equations, for the stochastic integrodifferential equation

$$\begin{aligned} dx_t &= \left[a_1(t, x_t) + \int_{t_0}^{t} a_2(t, s, x_s)\, ds + \int_{t_0}^{t} a_3(t, s, x_s)\, dw_s \right] dt \\ &\quad + \left[b_1(t, x_t) + \int_{t_0}^{t} b_2(t, s, x_s)\, ds + \int_{t_0}^{t} b_3(t, s, x_s)\, dw_s \right] dw_t, \quad t \in [t_0, T], \qquad (1.1) \\ x_{t_0} &= x_0 \quad \text{a.s.} \end{aligned}$$

[1] svjank@archimed.filfak.ac.ni.yu
[2] mima@archimed.filfak.ac.ni.yu
[3] AMS Mathematics Subject Classification (1991):60H10

described in details in the recent work of Berger and Mizel [**2**] on general forms of Ito–Volterra stochastic integrodifferential equations. Here $w = (w_t, t \in R)$ is an R^k–valued normalized Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, P)$, with a natural filtration $\{\mathcal{F}_t, t \in R\}$ of nondecreasing sub σ–algebras of $\mathcal{F}$, x_0 is a random variable independent of w, x_t is an R^n–valued stochastic process, the functions

$$\begin{array}{ll} a_1 : [t_0, T] \times R^n \to R^n, & b_1 : [t_0, T] \times R^n \to R^n \times R^k, \\ a_2 : J \times R^n \to R^n, & b_2 : J \times R^n \to R^n \times R^k, \\ a_3 : J \times R^n \to R^n \times R^k, & b_3 : J \times R^n \to R^n \times R^k \times R^k, \end{array}$$

where $J = \{ (t,s) : t_0 \le s \le t \le T\}$, are assumed to be Borel measurable on their domains.

An R^n-valued stochastic process x_t is a (strong) *solution* of the equation (1.1) on $t \in [t_0, T]$ if:

— x_t is nonanticipating for $t \in [t_0, T]$;

— the processes

$$\begin{array}{lll} \hat{a}_1(t) = a_1(t, x_t), & \hat{a}_2(t,s) = a_2(t,s,x_s), & \hat{a}_3(t,s) = a_3(t,s,x_s), \\ \hat{b}_1(t) = b_1(t, x_t), & \hat{b}_2(t,s) = b_2(t,s,x_s), & \hat{b}_3(t,s) = b_3(t,s,x_s), \end{array}$$

are such that

$$\int_{t_0}^{T} |\hat{a}_1(t)| dt < \infty \text{ a.s.}, \quad \int_{t_0}^{T} |\hat{b}_1(t)|^2 dt < \infty \text{ a.s.}, \quad \int_{t_0}^{T}\int_{t_0}^{t} |\hat{a}_2(t,s)| ds dt < \infty \text{ a.s.},$$

and $\hat{a}_3, \hat{b}_2, \hat{b}_3$ satisfy $\int_{t_0}^{T}\int_{t_0}^{t} |f(t,s)|^2 ds dt < \infty$ a.s.;

— $x_{t_0} = x_0$ a.s.;

— the equation (1.1) holds for each $t \in [t_0, T]$.

There is a number of papers in which various, essentially diferent sufficient conditions of the existence and uniqueness of a solution of the equation (1.1) are considered (see, [**2**], [**8**], [**16**], [**17**], for example). In fact, in the paper [**2**], following the classical theory of stochastic differential equations of the Itô type (see, [**4**], [**5**], [**12**], [**13**], for example), the basic existence and uniqueness theorem is proved: If the functions $a_i, b_i,\ i = 1,2,3$, satisfy the global Lipschitz condition and the usual linear growth condition on the last argument, i.e. if there exists a constant $L > 0$ such that

$$|a_2(t,s,x) - a_2(t,s,y)| < L|x-y|, \qquad |a_2(t,s,x)| \le L\,(1 + |x|)\,, \tag{1.2}$$

for all $(t,s) \in J$, $x, y \in R^n$, and similarly for the other functions, and if $E|x_0|^2 < \infty$, then there exists a unique a.s. continuous strong solution $x_t,\ t \in [t_0, T]$, of the equation (1.1), satisfying $E\{\sup_{t\in[t_0,T]} |x_t|^2\} < \infty$. Moreover, following the procedure in the papers [**13**] and [**15**] completely, it can be proved that if $E|x_0|^{2m} < \infty$ for any fixed number $m \in N$, then $E\{\sup_{t\in[t_0,T]} |x_t|^{2m}\} < \infty$.

As we saw above, our main purpose in the present paper is to study the stochastic integrodifferential equation (1.1) with "small" perturbations, by comparing its solution with the solution of the corresponding unperturbed equation of the equal type.

More precisely, we shall give conditions ensuring the closeness in $(2m)$-th moment sense for these solutions on fixed finite intervals or on intervals whose length goes to infinity. Note that the form of perturations is motivated by the one from the paper [**7**] and, also, from the basic paper [**19**], but the treatment used in our analysis is completely different from the one used in the mentioned papers. Moreover, we generalize the results of the paper [**19**] which could be treated here as illustrative examples. Remember, also, that the problems treating stochastic perturbed equations have been studied by several authors in the past years, in the papers and books [**9**], [**10**], [**11**], [**14**], [**18**], [**20**], for example.

In the sequel we shall apply the following version of the Gronwall–Bellman inequality [**1**, p. 12]: Let $u(t)$ be a continuous function in $[t_0,T]$, $b(t)$ a nonnegative continuous function in $[t_0,T]$, $k(t,s)$ a nonnegative continuous function for $t_0 \le s \le t \le T$ and

$$u(t) \le a(t) + b(t)\int_{t_0}^{t} k(t,s)u(s)\,ds, \quad t \in [t_0,T]. \tag{1.3}$$

Then

$$u(t) \le A(t)e^{B(t)\int_{t_0}^{t} K(t,s)\,ds}, \quad t \in [t_0,T],$$

where $A(t) = \sup\limits_{s\in[t_0,t]} a(s)$, $B(t) = \sup\limits_{s\in[t_0,t]} b(s)$, $K(t,s) = \sup\limits_{r\in[s,t]} k(r,s)$.

2. Main results

Along with the equation (1.1) in integral form, i.e.

$$\begin{aligned} x_t = x_0 &+ \int_{t_0}^{t}\left[a_1(s,x_s) + \int_{t_0}^{s} a_2(s,r,x_r)\,dr + \int_{t_0}^{s} a_3(s,r,x_r)\,dw_r\right]ds \\ &+ \int_{t_0}^{t}\left[b_1(s,x_s) + \int_{t_0}^{s} b_2(s,r,x_r)\,dr + \int_{t_0}^{s} b_3(s,r,x_r)\,dw_r\right]dw_s, \end{aligned} \tag{2.1}$$

we estabilish the following equation

$$\begin{aligned} x_t^\varepsilon = x_0^\varepsilon &+ \int_{t_0}^{t}\left[\tilde a_1(s,x_s^\varepsilon,\varepsilon) + \int_{t_0}^{s} \tilde a_2(s,r,x_r^\varepsilon,\varepsilon)\,dr + \int_{t_0}^{s} \tilde a_3(s,r,x_r^\varepsilon,\varepsilon)\,dw_r\right]ds \\ &+ \int_{t_0}^{t}\left[\tilde b_1(s,x_s^\varepsilon,\varepsilon) + \int_{t_0}^{s} \tilde b_2(s,r,x_r^\varepsilon,\varepsilon)\,dr + \int_{t_0}^{s} \tilde b_3(s,r,x_r^\varepsilon,\varepsilon)\,dw_r\right]dw_s, \end{aligned} \tag{2.2}$$

where $t \in [t_0, T]$, ε is *a small parameter* from the interval $(0,1)$, the initial condition x_0^ε and the functions $\tilde{a}_i, \tilde{b}_i$, $i = 1,2,3$ are defined as for the equation (1.1), and w is the same Brownian motion.

Inspired by the paper [**19**], we suppose that there exist the nonrandom functions $\alpha_i(\cdot), \beta_i(\cdot), i = 1,2,3$, defined as $a_i, b_i, i = 1,2,3$, respectively, and depending on the small parameter ε, such that for $(t,s) \in J$, $x \in R^n$

$$\begin{aligned}
&\tilde{a}_1(t,x,\varepsilon) = a_1(t,x) + \alpha_1(t,x,\varepsilon),\\
&\tilde{b}_1(t,x,\varepsilon) = b_1(t,x) + \beta_1(t,x,\varepsilon),\\
&\tilde{a}_i(t,s,x,\varepsilon) = a_i(t,s,x) + \alpha_i(t,s,x,\varepsilon), \quad i = 2,3,\\
&\tilde{b}_i(t,s,x,\varepsilon) = b_i(t,s,x) + \beta_i(t,s,x,\varepsilon), \quad i = 2,3.
\end{aligned}$$

The terms α_i and β_i are called *the perturbations* of the coefficients a_i and b_i, respectively. Because of that, the equation (2.2) is naturaly called *the perturbed equation*, while the name *the unperturbed equation* is kept for (2.1). Likewise, we introduce the following necessary assumptions:

Let there exist a natural number m, the nonrandom value $\delta_0(\varepsilon)$ and the onedimensional nonnegative bounded functions $\delta_i(\cdot), \gamma_i(\cdot)$, $i = 1,2,3$, defined on J and depending on ε, such that

$$E|x_0|^{2m} < \infty, \quad E|x_0^\varepsilon|^{2m} < \infty, \quad E|x_0 - x_0^\varepsilon|^{2m} \le \delta_0(\varepsilon), \tag{2.3}$$

$$\begin{aligned}
&\sup_{x \in R^n} |\alpha_1(t,x,\varepsilon)| \le \delta_1(t,\varepsilon), \quad \sup_{x \in R^n} |\beta_1(t,x,\varepsilon)| \le \gamma_1(t,\varepsilon),\\
&\sup_{x \in R^n} |\alpha_i(t,s,x,\varepsilon)| \le \delta_i(t,s,\varepsilon), \quad \sup_{x \in R^n} |\beta_i(t,s,x,\varepsilon)| \le \gamma_i(t,s,\varepsilon), \quad i = 2,3.
\end{aligned} \tag{2.4}$$

In view of our previous discussion, if we suppose that they are small for small ε, then we can impose conditions under which the solution x^ε and x are close in $(2m)$-th moment sense.

In the sequel we suppose without emphasizing that the all random and nonrandom integrals employed further are well defined, as well as that à priori there exist the unique solutions of the equations (2.1) and (2.2), satisfying $E\{\sup_{t\in[t_0,T]} |x_t|^{2m}\} < \infty$ and $E\{\sup_{t\in[t_0,T]} |x_t^\varepsilon|^{2m}\} < \infty$ (under the general conditions of the existence and uniqueness theorem from [**2**], for example). Furthermore, we shall emphasize only the conditions immediately used in our consideration.

First we give the following global estimation for the $(2m)$-th moment closeness of the solutions x and x^ε, which is important for the statements in our main results.

Proposition 2.1. *Let x_t and x_t^ε be the solutions of the equations* (2.1) *and* (2.2) *respectively and let the conditions* (1.2), (2.3) *and* (2.4) *be satisfied on the finite* *interval* $[t_0, T]$. *Then*

$$E|x_t^\varepsilon - x_t|^{2m} \le a(t,\varepsilon)\, e^{\xi(t-t_0)}, \quad t \in [t_0, T], \tag{2.5}$$

where

$$\begin{aligned} a(t,\varepsilon) = A\delta_0(\varepsilon) + A(t-t_0)^{m-1} \bigg\{ &\int_{t_0}^t \Big[(t-t_0)^m \delta_1^{2m}(s,\varepsilon) + B\gamma_1^{2m}(s,\varepsilon)\Big] ds \\ &+ \int_{t_0}^t (s-t_0)^{2m-1} \int_{t_0}^s \Big[(t-t_0)^m \delta_2^{2m}(s,\varepsilon) + B\gamma_2^{2m}(s,\varepsilon)\Big] dr\, ds \\ &+ B\int_{t_0}^t (s-t_0)^{m-1} \int_{t_0}^s \Big[(t-t_0)^m \delta_3^{2m}(s,\varepsilon) + B\gamma_3^{2m}(s,\varepsilon)\Big] dr\, ds \bigg\}, \end{aligned} \tag{2.6}$$

$$\xi(t) = AL^{2m}\, t^m \left[\frac{t^{3m}}{2m+1} + \frac{B(3m+2)\, t^{2m}}{(m+1)(2m+1)} + \Big(1 + \frac{B^2}{m+1}\Big) t^m + B \right], \tag{2.7}$$

and $A = 13^{2m-1},\ B = [m(2m-1)]^m,\ L$ *is the Lipschitz constant from* (1.2).

Proof. Denote

$$z_t^\varepsilon = x_t^\varepsilon - x_t, \qquad \Delta_t^\varepsilon = E|z_t^\varepsilon|^{2m}.$$

By subtracting the equations (2.1) and (2.2) and by applying the elementary inequality $\Big(\sum_{i=1}^n a_i\Big)^s \le n^{s-1}\sum_{i=1}^n a_i^s,\ a_i \ge 0, s \in N$, we obtain, for every $t \in [t_0, T]$,

$$\begin{aligned} E|z_t^\varepsilon|^{2m} \le 13^{2m-1} \bigg\{ &E|z_{t_0}^\varepsilon|^{2m} + E\Big(\int_{t_0}^t [a_1(s,x_s^\varepsilon) - a_1(s,x_s)]\, ds\Big)^{2m} \\ &+ E\Big(\int_{t_0}^t \alpha_1(s,x_s^\varepsilon,\varepsilon)\, ds\Big)^{2m} \\ &+ E\Big(\int_{t_0}^t \int_{t_0}^s [a_2(s,r,x_r^\varepsilon) - a_2(s,r,x_r)]\, dr\, ds\Big)^{2m} \\ &+ E\Big(\int_{t_0}^t \int_{t_0}^s \alpha_2(s,r,x_r^\varepsilon,\varepsilon)\, dr\, ds\Big)^{2m} \\ &+ E\Big(\int_{t_0}^t \int_{t_0}^s [a_3(s,r,x_r^\varepsilon) - a_3(s,r,x_r)]\, dw_r\, ds\Big)^{2m} \\ &+ E\Big(\int_{t_0}^t \int_{t_0}^s \alpha_3(s,r,x_r^\varepsilon,\varepsilon)\, dw_r\, ds\Big)^{2m} \end{aligned}$$

$$
\begin{aligned}
&+ E\Big(\int_{t_0}^{t} [b_1(s,x_s^\varepsilon) - b_1(s,x_s)]\,dw_s\Big)^{2m}\\
&+ E\Big(\int_{t_0}^{t} \beta_1(s,x_s^\varepsilon,\varepsilon)\,dw_s\Big)^{2m}\\
&+ E\Big(\int_{t_0}^{t}\int_{t_0}^{s} [b_2(s,r,x_r^\varepsilon) - b_2(s,r,x_r)]\,dr\,dw_s\Big)^{2m}\\
&+ E\Big(\int_{t_0}^{t}\int_{t_0}^{s} \beta_2(s,r,x_r^\varepsilon,\varepsilon)\,dr\,dw_s\Big)^{2m}\\
&+ E\Big(\int_{t_0}^{t}\int_{t_0}^{s} [b_3(s,r,x_r^\varepsilon) - b_3(s,r,x_r)]\,dw_r\,dw_s\Big)^{2m}\\
&+ E\Big(\int_{t_0}^{t}\int_{t_0}^{s} \beta_3(s,r,x_r^\varepsilon,\varepsilon)\,dw_r\,dw_s\Big)^{2m}\Big\}.
\end{aligned}
$$

In order to estimate these integrals, we shall apply the usual stochastic integral isometry: the Lipschitz condition (1.2) for the functions a_i, b_i, the relations (2.3) and (2.4), the partial integration, Hölder inequality for $p = 2m$, $\frac{1}{p} + \frac{1}{q} = 1$ and the well–known integral formula for Itô integrals (see [**5**], [**12**], [**13**]):

$$
E\Big(\int_{t_0}^{t} f(s)\,dw_s\Big)^{2m} \le [m(2m-1)]^m (t-t_0)^{m-1} \int_{t_0}^{t} Ef^{2m}(s)\,ds
$$

for any measurable $\mathcal{F}_t$–adapted process $f(t)$ satisfying $\int_{t_0}^{t} Ef^{2m}(s)\,ds < \infty$. Consequently, taking $B = [m(2m-1)]^m$ we find

$$
\begin{aligned}
&E\Big(\int_{t_0}^{t} [a_1(s,x_s^\varepsilon) - a_1(s,x_s)]\,ds\Big)^{2m}\\
&\qquad \le (t-t_0)^{2m-1} \int_{t_0}^{t} E|a_1(s,x_s^\varepsilon) - a_1(s,x_s)|^{2m}\,ds\\
&\qquad \le L^{2m}(t-t_0)^{2m-1} \int_{t_0}^{t} \Delta_s^\varepsilon\,ds,\\
&E\Big(\int_{t_0}^{t} \alpha_1(s,x_s^\varepsilon,\varepsilon)\,ds\Big)^{2m} \le (t-t_0)^{2m-1} \int_{t_0}^{t} \delta_1^{2m}(s,\varepsilon)\,ds,\\
&E\Big(\int_{t_0}^{t}\int_{t_0}^{s} [a_2(s,r,x_r^\varepsilon) - a_2(s,r,x_r)]\,dr\,ds\Big)^{2m}\\
&\qquad \le L^{2m}(t-t_0)^{2m-1} \int_{t_0}^{t} (s-t_0)^{2m-1} \int_{t_0}^{s} \Delta_r^\varepsilon\,dr\,ds
\end{aligned}
$$

$$= \frac{L^{2m}}{2m}(t-t_0)^{2m-1}\int_{t_0}^{t}\left[(t-t_0)^{2m}-(s-t_0)^{2m}\right]\Delta_s^{\varepsilon}\,ds,$$

$$E\Big(\int_{t_0}^{t}\int_{t_0}^{s}[\alpha_2(s,r,x_r^{\varepsilon},\varepsilon)\,dr\,ds\Big)^{2m}$$

$$\leq (t-t_0)^{2m-1}\int_{t_0}^{t}(s-t_0)^{2m-1}\int_{t_0}^{s}\delta_2^{2m}(s,r,\varepsilon)\,dr\,ds$$

$$E\Big(\int_{t_0}^{t}\int_{t_0}^{s}[a_3(s,r,x_r^{\varepsilon})-a_3(s,r,x_r)]\,dw_r\,ds\Big)^{2m}$$

$$\leq BL^{2m}(t-t_0)^{2m-1}\int_{t_0}^{t}(s-t_0)^{m-1}\int_{t_0}^{s}\Delta_r^{\varepsilon}\,dr\,ds$$

$$= \frac{BL^{2m}}{m}(t-t_0)^{2m-1}\int_{t_0}^{t}\left[(t-t_0)^{m}-(s-t_0)^{m}\right]\Delta_s^{\varepsilon}\,ds,$$

$$E\Big(\int_{t_0}^{t}\int_{t_0}^{s}[\alpha_3(s,r,x_r^{\varepsilon},\varepsilon)\,dw_r\,ds\Big)^{2m}$$

$$\leq B(t-t_0)^{2m-1}\int_{t_0}^{t}(s-t_0)^{m-1}\int_{t_0}^{s}\delta_3^{2m}(s,r,\varepsilon)\,dr\,ds$$

$$E\Big(\int_{t_0}^{t}[b_1(s,x_s^{\varepsilon})-b_1(s,x_s)]\,dw_s\Big)^{2m}\leq BL^{2m}(t-t_0)^{m-1}\int_{t_0}^{t}\Delta_s^{\varepsilon}\,ds,$$

$$E\Big(\int_{t_0}^{t}\beta_1(s,x_s^{\varepsilon},\varepsilon)\,dw_s\Big)^{2m}\leq B(t-t_0)^{m-1}\int_{t_0}^{t}\gamma_1^{2m}(s,\varepsilon)\,ds,$$

$$E\Big(\int_{t_0}^{t}\int_{t_0}^{s}[b_2(s,r,x_r^{\varepsilon})-b_2(s,r,x_r)]\,dr\,dw_s\Big)^{2m}$$

$$\leq \frac{BL^{2m}}{2m}(t-t_0)^{m-1}\int_{t_0}^{t}\left[(t-t_0)^{2m}-(s-t_0)^{2m}\right]\Delta_s^{\varepsilon}\,ds,$$

$$E\Big(\int_{t_0}^{t}\int_{t_0}^{s}[\beta_2(s,r,x_r^{\varepsilon},\varepsilon)\,dr\,dw_s\Big)^{2m}$$

$$\leq B(t-t_0)^{m-1}\int_{t_0}^{t}(s-t_0)^{2m-1}\int_{t_0}^{s}\gamma_2^{2m}(s,r,\varepsilon)\,dr\,ds,$$

$$E\Big(\int_{t_0}^{t}\int_{t_0}^{s}[b_3(s,r,x_r^{\varepsilon})-b_3(s,r,x_r)]\,dw_r\,dw_s\Big)^{2m}$$

$$\leq \frac{B^2L^{2m}}{m}(t-t_0)^{m-1}\int_{t_0}^{t}\left[(t-t_0)^{m}-(s-t_0)^{m}\right]\Delta_s^{\varepsilon}\,ds,$$

$$E\Big(\int_{t_0}^{t}\int_{t_0}^{s}[\beta_3(s,r,x_r^\varepsilon,\varepsilon)\,dw_r\,dw_s\Big)^{2m}$$
$$\le B^2(t-t_0)^{m-1}\int_{t_0}^{t}(s-t_0)^{m-1}\int_{t_0}^{s}\gamma_3^{2m}(s,r,\varepsilon)\,dr\,ds.$$

Finally, we come to the integral inequality of the type (1.3),

$$\Delta_t^\varepsilon \le a(t,\varepsilon)+b(t)\int_{t_0}^{t}k(t,s)\,\Delta_s^\varepsilon\,ds,\quad t\in[t_0,T], \tag{2.8}$$

where

$$\begin{aligned}a(t,\varepsilon)=A\delta_0(\varepsilon)+A(t-t_0)^{m-1}\Big\{&\int_{t_0}^{t}\Big[(t-t_0)^m\delta_1^{2m}(s,\varepsilon)+B\gamma_1^{2m}(s,\varepsilon)\Big]ds\\ &+\int_{t_0}^{t}(s-t_0)^{2m-1}\int_{t_0}^{s}\Big[(t-t_0)^m\delta_2^{2m}(s,\varepsilon)+B\gamma_2^{2m}(s,\varepsilon)\Big]dr\,ds\\ &+B\int_{t_0}^{t}(s-t_0)^{m-1}\int_{t_0}^{s}\Big[(t-t_0)^m\delta_3^{2m}(s,\varepsilon)+B\gamma_3^{2m}(s,\varepsilon)\Big]dr\,ds\Big\},\end{aligned}$$

$$\begin{aligned}b(t)&=AL^{2m}(t-t_0)^{m-1}[B+(t-t_0)^m],\\ k(t,s)&=1+\frac{1}{2m}\,[(t-t_0)^{2m}-(s-t_0)^{2m}]+\frac{B}{m}\,[(t-t_0)^m-(s-t_0)^m],\end{aligned}$$

and $A=13^{2m-1}$. Since the functions $a(t,\varepsilon)$ and $b(t)$ are increasing, $k(t,s)$ is increasing with respect to the first argument and

$$\int_{t_0}^{t}k(t,s)\,ds=(t-t_0)\Big[1+\frac{1}{2m+1}\,(t-t_0)^{2m}+\frac{B}{m+1}\,(t-t_0)^m\Big],$$

we easily come to the estimation (2.5) by applying the previous cited version of Gronwall–Bellman inequality to the inequality (2.8). □

If we start from the global estimation (2.5), taking into consideration that the size of the perturbations is limited by $\delta_0(\cdot)$, $\delta_i(\cdot)$, $\gamma_i(\cdot)$ and if we require that $\delta_0(\cdot)\to 0$, $\delta_i(\cdot)\to 0$, $\gamma_i(\cdot)\to 0$ as $\varepsilon\to 0$, it is reasonable to expect that, under some conditions, $\sup_{t\in[t_0,T]}E|x_t^\varepsilon-x_t|^{2m}\to 0$ as $\varepsilon\to 0$.

Theorem 2.1. *Let the conditions of Proposition* 2.1 *be satisfied and let* $\delta_0(\cdot)$, $\delta_i(\cdot)$, $\gamma_i(\cdot)$, $i=1,2,3$, *tend to zero as* ε *tends to zero, for every* $(t,s)\in J$. *Then*

$$\sup_{t\in[t_0,T]}E|x_t^\varepsilon-x_t|^{2m}\to 0\quad\text{as}\quad \varepsilon\to 0.$$

Proof. Denote

$$\begin{aligned}\overline{\delta}_1(\varepsilon) &= \sup_{t\in[t_0,T]} \delta_1(t,\varepsilon), \quad \overline{\delta}_i(\varepsilon) = \sup_{(t,s)\in J} \delta_i(t,s,\varepsilon),\ i=2,3\\ \overline{\gamma}_1(\varepsilon) &= \sup_{t\in[t_0,T]} \gamma_1(t,\varepsilon), \quad \overline{\gamma}_i(\varepsilon) = \sup_{(t,s)\in J} \gamma_i(t,s,\varepsilon),\ i=2,3.\end{aligned} \tag{2.9}$$

and

$$\phi(\varepsilon) = \max\left\{\delta_0(\varepsilon), \overline{\delta}_1^{2m}(\varepsilon), \overline{\delta}_2^{2m}(\varepsilon), \overline{\delta}_3^{2m}(\varepsilon), \overline{\gamma}_1^{2m}(\varepsilon), \overline{\gamma}_2^{2m}(\varepsilon), \overline{\gamma}_3^{2m}(\varepsilon)\right\}. \tag{2.10}$$

Clearly, $\phi(\varepsilon) \to 0$ as $\varepsilon \to 0$. From (2.6) we find

$$a(t,\varepsilon) \le \phi(\varepsilon)\, P_4((t-t_0)^m), \quad t \in [t_0,T],$$

where $P_4(v),\ v \ge 0$, is the corresponding polynomial of the degree 4. Now, from (2.5) it follows

$$E|x_t^\varepsilon - x_t|^{2m} \le \phi(\varepsilon)\, P_4((t-t_0)^m)\, e^{\xi(t-t_0)}, \quad t \in [t_0,T], \tag{2.11}$$

where $\xi(t-t_0)$ is defined as in (2.7). Because T is finite, then there exists a generic constant $C > 0$, not depending on ε, such that

$$E|x_t^\varepsilon - x_t|^{2m} \le C\,\phi(\varepsilon), \quad t \in [t_0,T].$$

Therefore, $\sup_{t\in[t_0,T]} E|x_t^\varepsilon - x_t|^{2m} \to 0$ as $\varepsilon \to 0$. □

Remark 1. The initial condition x_0^ε and the perturbations $\alpha_i(\cdot),\ \beta_i(\cdot), i = 1,2,3$, in the perturbed equation (2.2) could depend on different small parameters $\varepsilon_0, \varepsilon_i, \mu_i$, $i = 1,2,3$, respectively. Because the solution depends on them, we adopt the shorter notational convenion, introducing the superscript ε in x_t^ε and emphasizing that ε also depends on them. Clearly, the functions $\delta_0(\cdot),\ \delta_i(\cdot), \gamma_i(\cdot),\ i = 1,2,3$, from (2.3) and (2.4) also depend on them. If they are nondecreasing with respect to the small parameters, then Proposition 2.1 and Theorem 2.1 are valid with $\varepsilon = \max\{\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3, \mu_1, \mu_2, \mu_3\}$.

Note that all the previous considerations refer to any fixed finite time-interval. The logical question is: Are the analogous conclusions valid for the infinite time-interval? The following theorem, as the main result of this paper, shows that $\sup_t E|x_t^\varepsilon - x_t|^{2m} \to 0$ as $\varepsilon \to 0$ on intervals whose length goes to infinity. Likewise, note that the proof is partially similar to the appropriate proof in the paper [**7**].

Theorem 2.2. *Let the conditions of Theorem* 2.1 *be satisfied for* $t \geq t_0$. *Then, for an arbitrary number* $r \in (0,1)$ *and* ε *sufficiently small, there exists a number* $T(\varepsilon) > 0$, *where*

$$T(\varepsilon) = M\left[\big(-r\ln\phi(\varepsilon)\big)^{1/4} - K\right]^{1/m}, \tag{2.12}$$

$\phi(\varepsilon)$ *is given by* (2.10), M *and* K *are some generic positive constants, such that*

$$\sup_{t\in[t_0,t_0+T(\varepsilon)]} E|x_t^\varepsilon - x_t|^{2m} \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

Proof. Since the time-interval is infinite, Theorem 2.1 is generally not valid. Because of that we shall take $T = t_0 + T(\varepsilon)$ and effectively determine $T(\varepsilon)$ such that $\sup_{t\in[t_0,T(\varepsilon)]} E|x_t^\varepsilon - x_t|^{2m} \to 0$ as $\varepsilon \to 0$.

From (2.11) it follows

$$E|x_t^\varepsilon - x_t|^{2m} \leq \phi(\varepsilon)\, P_4\big(T^m(\varepsilon)\big)\, e^{\xi(T(\varepsilon))}, \quad t \in [t_0, T(\varepsilon)]. \tag{2.13}$$

Since $\varepsilon \to 0$ implies $\phi(\varepsilon) \to 0$, we can accept that there exists a constant ε_0, $0 < \varepsilon_0 < 1$, such that $\phi(\varepsilon) < 1$ for $\varepsilon \in (0, \varepsilon_0)$. Because we require that the limit on the right side of the inequality (2.13) tends to zero as ε tends to zero, we shall determine $T(\varepsilon)$ such that

$$\xi(T(\varepsilon)) \leq -r\ln\phi(\varepsilon)$$

for an arbitrary number $r \in (0,1)$ and for $\varepsilon \in (0, \varepsilon_0)$.

By applying the elementary inequality $a_1^4 + 4a_1^3a_2 + 6a_1^2a_3^2 + 4a_1a_4^3 \leq \Big(\sum_{i=1}^4 a_i\Big)^4$, $a_i \geq 0$, to the function $\xi(t - t_0)$ defined by (2.7), taking

$$\begin{aligned}
a_1 &= \Big(\frac{AL^{2m}}{2m+1}\Big)^{1/4}(t-t_0)^m, \qquad a_2 = \frac{B\,(3m+2)}{4\,(m+1)} \cdot \Big(\frac{AL^{2m}}{2m+1}\Big)^{1/4},\\
a_3 &= \big[AL^{2m}(2m+1)\big]^{1/4} \cdot \Big[\frac{1}{6}\Big(1 + \frac{B^2}{m+1}\Big)\Big]^{1/2},\\
a_4 &= \Big(\frac{B}{4}\Big)^{1/3} \cdot (2m+1)^{1/12} \cdot (AL^{2m})^{1/4},
\end{aligned}$$

we obtain

$$\xi(t-t_0) \leq \xi(T(\varepsilon)) \leq \left[\Big(\frac{AL^{2m}}{2m+1}\Big)^{1/4} T^m(\varepsilon) + K\right]^4, \quad t \in [t_0, T(\varepsilon)],$$

where $K = a_2 + a_3 + a_4$. Now, for an arbitrary number $r \in (0,1)$ and ε sufficiently small ($\varepsilon < \varepsilon_0$), such that $-r \ln \phi(\varepsilon) \geq K^4$, providing

$$\left[\Big(\frac{AL^{2m}}{2m+1}\Big)^{1/4} T^m(\varepsilon) + K\right]^4 = -r \ln \phi(\varepsilon),$$

we easily find the maximal number $T(\varepsilon)$ in the form (2.12), where $M = \big(\frac{2m+1}{AL^{2m}}\big)^{1/4m}$.

Clearly, $T(\varepsilon) \to \infty$ as $\varepsilon \to 0$.

Finally, from (2.12), for every $t \in [t_0, T(\varepsilon)]$, it follows

$$E|x_t^\varepsilon - x_t|^{2m} \leq \big(\phi(\varepsilon)\big)^{1-r} P_4\left(M\left[\big(-r\ln\phi(\varepsilon)\big)^{1/4} - K\right]^{1/m}\right) \to 0 \quad \text{as} \quad \varepsilon \to 0,$$

and, therefore, $\sup_{t\in[t_0,t_0+T(\varepsilon)]} E|x_t^\varepsilon - x_t|^{2m} \to 0$ as $\varepsilon \to 0$. Thus the proof is complete. □

Example. Let us indicate briefly how to apply the foregoing results to estimate the $(2m)$-th mean closeness for the solutions of any perturbed equation and the corresponding unperturbed equation. For example, motivated by the choice of perturbations in the paper [**19**], let us consider the scalar perturbed equation

$$\begin{aligned} x_t^\varepsilon = \eta + \varepsilon_0 + \int_0^t \left[a_s + b_s\, x_s^\varepsilon + \frac{\varepsilon_1}{1+|x_s^\varepsilon|} + \int_0^s \Big(\varepsilon_3 + e^{-(r+1)/\varepsilon_3}\Big) \sin x_r^\varepsilon\, dw_r\right] ds \\ + \int_0^t \left[c_s + \mu_1 + \int_0^s \sin\frac{\mu_3}{1+s+r+|x_r^\varepsilon|}\, dw_r\right] dw_s, \quad t \geq 0, \end{aligned}$$

where $a_t, b_t, c_t,\ t \geq 0$, are nonrandom, measurable and bounded functions and $\eta =$ const > 0 a.s., while

$$x_t = \eta + \int_0^t (a_s + b_s\, x_s^\varepsilon)\, ds + \int_0^t c_s\, dw_s, \quad t \geq 0,$$

is the corresponding unperturbed linear equation, which is effectively solvable and which solution is the gaussian and markovian process (see [4], [6], [13], for example). Note that the perturbations from the perturbed equation satisfy the conditions (2.1), (2.3) and (2.4). The conditions of Theorem 2.1 and of Theorem 2.2 are also satisfied and, therefore, we can determine intervals $[0, T(\varepsilon)]$ whose length tends to infinity as $\varepsilon \to 0$ and on which $\sup_{t\in[0,T(\varepsilon)]} E|x_t^\varepsilon - x_t|^{2m} \to 0$ as $\varepsilon \to 0$. From (2.9) and (2.10) it follows

$$\phi(\varepsilon) = \max\left\{\varepsilon^{2m}, \Big(\varepsilon + e^{-1/\varepsilon}\Big)^{2m}, (\sin\varepsilon)^{2m}\right\} = \Big(\varepsilon + e^{-1/\varepsilon}\Big)^{2m}.$$

Since there exists $\varepsilon_0 \in (0,1)$, such that $\varepsilon + e^{-1/\varepsilon} < 1$ for all $\varepsilon \in (0, \varepsilon_0)$, then, for an arbitrary number $r \in (0,1)$, from (2.12) we observe

$$T(\varepsilon) = M \left[\left(-2mr \ln \left(\varepsilon + e^{-1/\varepsilon} \right) \right)^{1/4} - K \right]^{1/m},$$

where the constants M and K are defined as above and $A = 6^{2m-1}$.

Remark 2. By applying the previously used stochastic integral isometry, including the Burkholder–Davis–Gundy inequality (see [**5**], [**13**], [**15**], for example) one can estimate $\overline{\Delta}^{\varepsilon}(T) = E\{\sup_{t\in[t_0,T]} |x_t^{\varepsilon} - x_t|^{2m}\}$ as a measure of the closeness between the solutions x^e and x. Analogously to the procedure exposed in the present paper, one can find conditions under which $\overline{\Delta}^{\varepsilon}(T) \to 0$ as $\varepsilon \to 0$ on finite intervals or on intervals whose length goes to infinity.

Remark 3. If the time interval is infinite, the results of this paper could be immediately used to investigate the asymptotic stability in $(2m)$-th moment sense for the solutions of the perturbed equations, by studying the same asymptotic stability for the solution of the corresponding unperturbed equation.

References

[1] D. Bainov, P. Simeonov, *Integral Inequalities and Applications*, Kluwer Academic Publishers, Dordrecht, Netherlands, 1992.

[2] M.A. Berger, V.J. Mizel, *Volterra equations with Ito integrals I*, J. of Integral Equations, **2** (1980), 187–245.

[3] M.A. Berger, V.J. Mizel, *Volterra equations with Ito integrals II*, J. of Integral Equations, **4** (1980), 319–337.

[4] T. Gard, *Introduction to Stochastic Differential Equations*, Marcel Dekker, New York (1988).

[5] I.I. Gihman, A.V. Skorohod, *Stochastic Differential Equations and Their Applications*, Naukova Dumka, Kiev, 1982 (In Russian).

[6] A.R. Ibrahim, *Parametric Random Vibration*, Wiley, New York, 1985.

[7] Sv. Janković, M. Jovanović, *Perturbed stochastic hereditary differential equations*, (to appear)

[8] M. Jovanović, *Random integral contractor of stochastic integrodifferential equation of Itô type*, Filomat(Niš), **8** (1994), 103–114.

[9] Yu.M. Kabanov, S.M. Pergamenshchikov, *On singulary perturbed stochastic differential equations and partial differential equations*, Soviet. Math. Dokl., **311** (1990), 1039–1042. (In Russian)

[10] Yu.M. Kabanov, S.M. Pergamenshchikov, J.M. Stoyanov, *Asymptotic expansions for singulary perturbed stochastic differential equations*, New Trends in Probability and Statistics (In Honor of Yu. Prohorov) Sazanov V. and Shervashidze T., eds. VSP, Utrecht (NL), (1991), 413–435.

[11] R. Khasminskii, *On stochastic processes defined by differential equations with a small parameter*, Theory Probab. Appl., **11** (1966), 211–228.

[12] G. Ladde, V. Lakshmikantham, *Random Differential Inequalities*, Academic Press, New York, 1980.

[13] R. Liptser, A. Shiryaev, *Statistics of Random Processes I*, Springer, New York, 1977.

[14] R. Liptser, J. Stoyanov, *Stochastic version of the averaging principle for diffusion type processes*, Stochastics & Stochastics Reports, **32** (1990), 145–163.

[15] V. J. Mizel, V. Trutzer, *Asymptotic stability for stochastic hereditary equations*, Phycical Mathematics and Nonlinear Partial Differential Equations, Marcel Dekker Inc., New York and Basel, 1985.

[16] M.G. Murge, B.G. Pachpatte, *On generalized Itô type stochastic integral equation*, Yokohama Mathematical Journal, **34** (1986), 23–33.

[17] M.G. Murge, B.G. Pachpatte, *Sucessive approximations for solutions of second order stochastic integrodifferential equations of Itô type*, Indian J. pure and appl,. Math., **21** (3), (1990), 260–274.

[18] J. Picard, *Convergence in probability for perturbed stochastic integral equations*, Probab. Th. Rel. Fields, **81** (1989), 383–451.

[19] J. Stoyanov, D. Botev, *Quantitative results for perturbed stochastic differential equations*, J. Appl. Math. and Stoch. Analysis, **9** (3), (1996), 255–261.

[20] J. Stoyanov, *Regularly perturbed stochastic differential systems with an internal random noise*, Proc. 2^{nd} World Congress Nonlin. Anal. (Athens, July '96), (1996).