ASYMPTOTIC BEHAVIOR OF NON–LINEAR DYNAMIC SYSTEMS SUBJECTED TO PARAMETRIC AND RANDOM PERTURBATIONS[1]

Svetlana Janković
Miljana Jovanović

Faculty of Science, Department of Mathematics, University of Niš
Ćirila i Metodija 2, 18000 Niš, Yugoslavia
e-mail: svjank@archimed.filfak.ac.ni.yu., mima@archimed.filfak.ac.ni.yu,

Abstract. *In this paper we investigate the asymptotic behavior in the* $(2k)$*–th moment sense of the non–linear oscillator amplitude subjected to small perturbations and random excitations of a Gaussian white noise type. Since this problem is connected with stochastic differential equation of the Ito type, the present paper deals with the asymptotis behavior of the solution of the Ito's differential equation with small perturbations, by comparing it with the solution of the corresponding unperturbed equation. Precisely, we give conditions under which these solutions are close in the* $(2k)$*–th moment sense on intervals whose length tends to infinity.*

1. INTRODUCTION

Non–linear differential equations subjested to deterministic and random excitations have been extensively investigated both theoretically and experimentally over a long period of time. In mechanics, and much more in engineering practice, an important role is played by non–linear differential equation of the form

$$\ddot{y} + g(y)\,\dot{y} + h(t,y)\,y = 0,$$

which repersents mathematical models of elastic systems motion with one degree of freedom, or discredization of the dynamic model of an elastic body in the basic form of the dynamic

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equilibrium. The researcher's interest is concentrated on exploring the bifurcational behavior of the solution and on conditions of stability or unstability of various elastic equilibrium forms under deterministic and stochastic influences. Specially, during the last years stimulating research have been undertaken in the field of descriptions of amplitudes of non–linear oscillators subjected to random excitations of a Gaussian white noise type. Many authors, for instance Ariaratnam [****], Ibrahim [****], Kozin [*****], Kushner [****], Caughey [****], Weidenhammer [*****] and others, have studied various problems of motion of elastic oscillator systems under random excitations of a Gaussian white noise type, which is mathematically descrbed as a Gaussian stationary wide–band random process of small intensity and correlation time, with matematical expectation equal to zero. Note that white noise is, at least, a tolerable abstraction and is never a completely faithful representation of a physical noise source.

The stochastic averiging principle, introduced by Khas'minskii, encouraged several rechearchers to investigate the random behavior of dynamic systems under random parametric excitations. However, they have been used a number of techniques, basically connected with the stochastic averiging principle, for instance, the Markovian method based on the Kolmogorov– Fokker–Planck equation, the Gaussian moment function methods, the spatial correlation method, and others. Having in view that a Gaussian white noise is an abstraction and not a physical process, at least mathematically described as a formal derivative of a Brownian motion process, all the previously cited methods are essentially based on stochastic differential equations of the Itô type.

In the present paper we consider the stohastic differential equation of the Itô type with small perturbations, by comparing its solution, in the $(2k)$-th moment sense, with the solution of a simpler unperturbed equation of the equal type. Note that the form of perturbations is motivated by the one from the paper [Stoj****]. We generalize the result of the paper [Stoj****], which could be treated here as illustrative examples. The similar problems are also studied in [Jank****, Jank****]. Remember that the problems treating stochastic perturbed equations have been studied by several authors in the past years, for example, in the papers and books [Kabanov****], [Xasmin****], [Lipc.Sirj****], [Stoj.konf****], [Ibrahim****], [neko iz Ibrahima****], [Arnold****] and, clearly, in the previously cited works.

Our paper is organized as follows: In the next section, starting from the main result of the paper [Stoj****], from the global estimetion for the $(2k)$-th moment closeness of the solution of the perturbed and unperturbed equation on finite fixed time–interval, we give conditions ensuring the closeness in the $(2k)$-th moment sense on intervals whose length tends to infinity as small perturbations tend to zero. In Section 3 we illustrate the preceding results on the example of the non–linearr oscillator subjected to parametric and random excitations of a Gaussian white noise type. We give some useful conclusions and we point to possible applications of the preceding considerations.

Finally, let us suppose that all random variable and processes considered here are given on a complete probability space $(\Omega, \mathcal{F}, P)$. We should mention that we shall restrict ourselves to scalar–valued processes in this paper. For applications, extension to vector–valued process is of great importance and it is not difficult in itself, but is rather complicated in detail and involves complex notations.

2. FORMULATION OF THE PROBLEM AND MAIN RESULTS

We shall consider the following scalar stochastic differential equation of the Itô type

$$dx_t = a(t, x_t)\, dt + b(t, x_t)\, dw_t, \quad t \in [0, T], \quad x_o = \eta,$$

or, in equivalent integral form,

$$x_t = \eta + \int_0^t a(s, x_s)\, ds + \int_0^t b(s, x_s)\, dw_s, \quad t \in [0, T], \tag{2.1}$$

in which $w = (w_t, t \geq 0)$ is a scalar–valued normalized Brownian motion with a natural filtration $\{\mathcal{F}_t, t \geq 0\}$ (i.e. $\mathcal{F}_t$ is a smallest σ-field on which all random variables $w_s, s \leq t$ are $\mathcal{F}_t$–measurable), the initial condition η is a random variable independent of w, $a(t, x)$ and $b(t, x)$ are given scalar real functions satisfying $\int_0^T |a(t, x)|\, dt < \infty, \int_0^T |b(t, x)|^2\, dt < \infty$ (under these conditions Lebesgue and Itô integrals in (2.1) are well defined), and $x = (x_t, t \in [0, T])$ is a scalar stochastic process adapted to $\{\mathcal{F}_t, t \geq 0\}$.

It is well known that there is a number of papers and books in which various, essentially diferent sufficient conditions of the existence and uniqueness of a solution of Eq. (2.1) are given (see, for example, [Arnold****], [Ikeda, wat****], [LipcerSir****], [Skor, Gihm****], [Matinst****]. In fact, from the classical theory of stochastic differential equations of the Itô type, one can prove that if the functions $a(t, x)$ and $b(t, x)$ satisfy the global Lipschitz condition and the usual linear growth condition on the last argument, i.e. if there exists a constant $L > 0$ such that

$$|a(t, x) - a(t, y)| + |b(t, x) - b(t, y)| < L|x - y|, \quad |a(t, x)|^2 + |b(t, x)|^2 \leq L\left(1 + |x|^2\right), \tag{2.2}$$

for all $x, y \in R$, $t \in [0, T]$ and if $E|\eta|^2 < \infty$ ($E|\eta|^2$ is mathematical expectation of $|\eta|^2$), then there exists a unique solution $x = (x_t, t \in [t_0, T])$ of Eq. (2.1), continuous with probability one, satisfying $E\{\sup_{t\in[t_0,T]} |x_t|^2\} < \infty$. Moreover, if $E|\eta|^{2k} < \infty$ for any fixed integer $k \in N$, then $E\{\sup_{t\in[t_0,T]} |x_t|^{2k}\} < \infty$.

Together with Eq. (2.1) we consider the following equation of the equal type, depending on parameters,

$$x_t^\varepsilon = \eta^{\varepsilon_0} + \int_0^t [a(s, x_s^\varepsilon) + \alpha(s, x_s^\varepsilon, \varepsilon_1)]\, ds + \int_0^t [b(s, x_s^\varepsilon) + \beta(s, x_s^\varepsilon, \varepsilon_2)]\, dw_s, \quad t \in [0, T], \tag{2.3}$$

in which $\varepsilon_0, \varepsilon_1, \varepsilon_2$ are *small parameters* from the interval $(0,1)$. Because the solution depends on them, we adopt the shorter notational convenion, introducing the superscript ε in x_t^ε and emphasizing that ε also depends on them. The initial value η^{ε_0}, satisfying $E|\eta^{\varepsilon_0}|^{2k} < \infty$, is independent on the same Brownian motion w, and $\alpha(t,x,\varepsilon_1)$ and $\beta(t,x,\varepsilon_2)$ are given scalar real functions. In accordance with the paper [Stoj***], the functions $\alpha(t,x,\varepsilon_1)$ and $\beta(t,x,\varepsilon_2)$ are called *the perturbations*, while Eq. (2.3) is naturally called *the perturbed equation* with respect to *the unperturbed equation* (2.1).

We suppose that there exist a non–random value $\delta_0(\varepsilon)$ and bounded functions $\delta_1(t,\varepsilon_1)$ and $\delta_2(t,\varepsilon_2)$, such that

$$E|\eta^{\varepsilon_0} - \eta|^{2k} \le \delta_0(\varepsilon_0), \tag{2.4}$$

$$\sup_{x\in R} |\alpha(t,x,\varepsilon_1)| \le \delta_1(t,\varepsilon_1), \quad \sup_{x\in R} |\beta_1(t,x,\varepsilon_2)| \le \delta_2(t,\varepsilon_2). \tag{2.5}$$

In view of our previous discussion, if the conditions (2.2), (2.4) and (2.5) are satisfied, then both equations (2.1) and (2.3) have unique solutions x_t and x_t^ε, respectively, continuous with probability one and satisfying $\sup_{t\in[0,T]} E|x_t|^{2k} < \infty$, $\sup_{t\in[0,T]} E|x_t^\varepsilon|^{2k} < \infty$. Moreover, if the values $\delta(\varepsilon_0), \delta_1(t,\varepsilon_1), \delta_2(t,\varepsilon_2)$ are small for small $\varepsilon_0, \varepsilon_1, \varepsilon_2$, then we could expect that the solutions x_t and x_t^ε are close in $(2k)$–th moment sense. In connection with these requirements, the name *small perturbations* is logically kept for the perturbations $\alpha(t,x,\varepsilon_1)$ and $\beta(t,x,\varepsilon_2)$.

In the paper [Stoj****] the following estimation of the $(2k)$–th moment closeness for the solutions x_t and x_t^ε is obtained: for all $t \in [0,T]$,

$$\begin{aligned} E|x_t^\varepsilon - x_t|^{2k} \le \bigg(& \delta_0^{1/k}(\varepsilon_0) \cdot \exp\Big\{ Mt + 2\int_0^t \delta_1(s,\varepsilon_1)\,ds \Big\} \\ & + \int_0^t \big[2d_1(s,\varepsilon_1) + (2k-1)\delta_2^2(s,\varepsilon_2) \big] \cdot \exp\Big\{ Mt + 2\int_s^t \delta_1(u,\varepsilon_1)\,du \Big\}\,ds \bigg)^k, \end{aligned} \tag{2.6}$$

where $M = 2L + 2(2k-1)L^2$. Starting from this estimation, some special types of perturbations are considered in the paper [Stoj***]. In the presen paper we shall observe a general case of perturbations and we shall give conditions under which these solutions are close on fixed finite intervals or on intervals whose length tends to infinity when the small parameters tend to zero.

First, following partially the ideas of the paper [Publ****], taking into consideration that the size of the perturbations is limited in the sense (2.4) and (2.5), we shall require that $\delta_0(\varepsilon_0), \delta_1(t,\varepsilon_1)$ and $\delta_2(t,\varepsilon_2)$ monotony tend to zero as small parameters tend to zero, uniformly in $[0,T]$.

Let us denote

$$\overline{\delta}_1(\varepsilon_1) = \sup_{t\in[t_0,T]} \delta_1(t,\varepsilon_1), \quad \overline{\delta}_2(\varepsilon_2) = \sup_{t\in[0,T]} \delta_2(t,\varepsilon_2)$$

and let us define

$$\phi(\varepsilon) = \max\left\{\delta_0^{1/k}(\varepsilon), \overline{\delta_1}(\varepsilon), \overline{\delta_2}^{\,2}(\varepsilon)\right\},$$

where $\varepsilon = \max\{\varepsilon_0, \varepsilon_1, \varepsilon_2\}$. Obviously, $\phi(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Now, from (2.6) it is easy to obtain

$$E|x_t^\varepsilon - x_t|^{2k} \le \phi^k(\varepsilon)\left(e^{(M+2\rho)t} + \frac{2k+1}{M+2\rho}\left(e^{(M+2\rho)t} - 1\right)\right)^k, \quad t \in [0,T].$$

where ρ is a constant such that $\overline{\delta_1}(\varepsilon) < \rho$ for $\varepsilon \in (0,1)$. If we take $C = \left(1 + \frac{2k+1}{M+2\rho}\right)^k$, we find

$$E|x_t^\varepsilon - x_t|^{2k} \le C \cdot \phi^k(\varepsilon) \cdot e^{k(M+2\rho)t}, \quad t \in [0,T].$$

If $T > 0$ is a fixed finite number, it follows immediately that

$$\sup_{t\in[0,T]} E|x_t^\varepsilon - x_t|^{2k} \le C \cdot \phi^k(\varepsilon) \cdot e^{k(M+2\rho)T} \to 0 \quad \text{as} \quad \varepsilon \to 0. \tag{2.7}$$

Therefore, for enough small parameters $\varepsilon_0, \varepsilon_1, \varepsilon_2$, the solutions x_t^ε and x_t are close in $(2k)$–th moment sense on a fixed finite time–interval $[0,T]$. But, if the time–interval is finite, i.e. $T = \infty$, then the previous assertion is generally not valid. Because of that, similarly to the paper [Publ****], our intention is to construct finite time–intervals which depend on ε and whose length go to infinity as ε goes to zero, such that the solutions x_t^ε and x_t are close in the $(2k)$–th moment sense on these intervals. Remember that in the paper [Stoj****] the construstion of such intervals is given only for special classes of perturbations.

Let us take $T = T(\varepsilon)$ and determine $T(\varepsilon)$ such that $\sup_{t\in[0,T(\varepsilon)]} E|x_t^\varepsilon - x_t|^{2k} \to 0$ as $\varepsilon \to 0$. From (2.7) we find

$$\sup_{t\in[0,T(\varepsilon)]} E|x_t^\varepsilon - x_t|^{2k} \le C \cdot \phi^k(\varepsilon) \cdot e^{k(M+2\rho)T(\varepsilon)}\,. \tag{2.8}$$

Obviously, we shall claim that the right hand side of this inequality tends to zero as ε tends to zero. Since $\phi(\varepsilon) \to 0$ as $\varepsilon \to 0$, then there exists $\overline{\varepsilon} \in (0,1)$ such that $\phi(\varepsilon) < 1$ for $\varepsilon < \overline{\varepsilon}$. Now we easily find $T(\varepsilon)$ by taking $(M+2\rho)\,T(\varepsilon) = -r\,\ln\phi(\varepsilon)$ for any number $r \in (0,1)$ and for $\varepsilon < \overline{\varepsilon}$. Thus,

$$T(\varepsilon) = -\frac{r}{M+2\rho}\ln\phi(\varepsilon) \tag{2.9}$$

and, clearly, $T(\varepsilon) \to \infty$ as $\varepsilon \to 0$. It is easy now to conclude that

$$\sup_{t\in[0,T(\varepsilon)]} E|x_t^\varepsilon - x_t|^{2k} \le C \cdot \left(\phi(\varepsilon)\right)^{k(1-r)} \to 0 \quad \text{as} \quad \varepsilon \to 0. \tag{2.10}$$

Note that the preceding relation gives us an important result, the rate of closeness of the solutions x_t^ε and x_t for a fixed ε. Namely, for a given small value $\theta > 0$, from the relation $C\big(\phi(\varepsilon)\big)^{k(1-r)} < \theta$ we can determine a limit for $\phi(\varepsilon)$ as the size of the small perturbations,

$$\phi(\varepsilon) < (\theta/C)^{1/k(1-r)}$$

and, after that, $T(\varepsilon)$ from (2.9), such that $\sup_{t\in[0,T(\varepsilon)]} E|x_t^\varepsilon - x_t|^{2k} < \theta$.

Remember, also, that we have estimated only even moments of $|x_t^\varepsilon - x_t|$, while odd ones could be estimated by using the elementary property of mathematical expectation: $\big(E|X|\big)^2 \le E|X|^2$ for any random variable X. From that,

$$\sup_{t\in[0,T(\varepsilon)]} E|x_t^\varepsilon - x_t|^k \le \sup_{t\in[0,T(\varepsilon)]} \big(E|x_t^\varepsilon - x_t|^{2k}\big)^{1/2} \le C^{1/2}\cdot\big(\phi(\varepsilon)\big)^{k(1-r)/2}.$$

In connection with previous discussion, it is clear that the method exposed here could be used to study stability properties in (k)–th moment sense for the solution of the perturbed equation, by studying stability properties in the same sense for the solution of the corresponding unperturbed equation, what will be illustrated in the next section.

3. (2k)-TH MEAN BEHAVIOR OF NON–LINEAR OSCILLATOR AMPLITUDE SUBJECTED TO SMALL PARAMETRIC PERTURBATIONS

In this section we shall apply the previous results to describe the behavior of any non–linear oscillator under parametric and random excitations, by comparing its amplitude, in $(2k)$–th moment sense, with the one of the corresponding linear oscillator. Precisely, we consider the non–linear oscillator motion, which is mathematically modeled with the following random differential equation

$$\ddot{y} + (\alpha + \beta y^2)\,\dot{y} + \big(\omega_0^2 + \gamma y^2 + f(t,\omega)\big)\,y = 0, \tag{3.1}$$

earlier studied in [Ariarat****1980]. Here $f(t,\omega)$ is a Gaussian stationary wide–band random process of small intensity and correlation time, with mathematical expectation equal to zero, which is treated as a Gaussian white noise excitation in mechanics and in engineering practice; α, β and γ are linear and non–linear damping factors, i.e. positive constants small comparing to one and of the same intensity order as the spectral density $S(2\omega_0)$ of the Gaussian random process; ω_0 is an natural frequency of the unperturbed system oscillation. Obviously, the special problem mainly arize in the specification of the infinitesimal character of the parameters involved in (3.1).

It is well known that Eq. (3.1) can be transformed in the following way (see, for instance, [Ariar80****], [Ibrahim****]: First, we introduce the coordinate transformation $y_1 = y, y_2 = \dot{y}$, such that Eq. (3.1) may be rewritten into the system of the random differential equations:

$$\begin{aligned}\frac{dy_1}{dt} &= y_2\\ \frac{dy_2}{dt} &= -(\alpha+\beta y_1^2)y_2 - \big(\omega_0^2+\gamma y_1^2 + f(t,\omega)\big)y_1.\end{aligned}$$

By the representation of the variables $y_1(t)$ and $y_2(t)$ in the standard form

$$\begin{aligned}y_1(t) &= a(t)\cos\phi(t)\\ y_2(t) &= -a(t)\,\omega_0\sin\phi(t)\\ \phi(t) &= \omega_0\,t+\theta(t),\end{aligned}$$

in which $a(t)$ is a solution amplitude for elongation, whereas $\theta(t)$ is a phase and $\phi(t)$ is a phase angle, the preceding system is transformed into the form convenient to the Khas'minskii averiging principle (see [hasm****], [Ibrahim****], [Katica****]), which is based upon the ideas of the Bogoliubov–Mitropolsky ([Bog,Mit****]. Without going into details (for more details se, for example, [Ibrahim****]), as a final results of an application of the averiging principle, one can obtain stochastic differential equations of the Itô type with respect to the averiged amplitude $\overline{a}(t)$ and the averiged phase $\overline{\theta}(t)$. The stochastic differential equation with respect to the averiged amplitude will be of interest to us:

$$\begin{aligned}d\overline{a}_t &= \left[\frac{3}{16}\,\frac{S(2\omega_0)}{\omega_0^2}\,\overline{a}_t\,dt - \frac{1}{2}\left(\alpha+\frac{\beta}{4}\,\overline{a}_t^2\right)\overline{a}_t\right]dt + \sqrt{\frac{S(2\omega_0)}{8\omega_0^2}}\,\overline{a}_t\,dw_t, \quad t>0,\\ \overline{a}_0 &= \eta,\end{aligned} \tag{3.2}$$

in which $w=(w_t, t\geq 0)$ is a normalized Brownian motion, which is the outcome to the effect of the random forces process. Since this equation cannot be effectively solvable (see [Skor.Gih****], [Ja, Katica****], in order to describe its solution different methods are used. For example, since the solution is a homogeneous Markovian process (see [Skor****], [Arnold****]), the corresponding Kolmogorov–Fokker–Planck equation for a conditional probability density is used for determining the stationary probability density of the averiged amplitude (see Ibrahim str 254 1985.****] [Ariaratnam 1980****]. Likewise, a comparison method is presented in [ja,Kat****], [Ja****] to estimate the mean square expectation of this averiged amplitude.

For the sake of simpler writting, we are introducing the following notations:

$$\mu = \frac{3}{16}\,\frac{S(2\omega_0)}{\omega_0^2} - \frac{\alpha}{2}, \qquad \nu = \sqrt{\frac{S(2\omega_0)}{8\omega_0^2}},$$

such that Eq. (3.2) becomes

$$d\overline{a}_t = \left(\mu\overline{a}_t - \frac{\beta}{8}\,\overline{a}_t^3\right)dt + \nu\overline{a}_t\,dw_t, \quad t>0, \quad \overline{a}_0=\eta. \tag{3.3}$$

In connection with the discussion in Section 2, together with the non–linear oscillator motion mathematically described with Eq. (3.1), we consider the corresponding linear one, mathematically described with the following equation

$$\ddot{y} + \alpha\dot{y} + \left(\omega_0^2 + f(t,\omega)\right) y = 0. \tag{3.4}$$

In fact, the quadratic terms in (3.1) could be treated as perturbations with respect to the linear equation (3.4).

By applying the stochastic averiging principle to Eq. (3.4), the corresponding equation with respect to the averiged amplitude $\bar{b}(t)$ is that linear homogeneous stochastic differential equation of the Itô type:

$$d\bar{b}_t = \mu\bar{b}_t\,dt + \nu\bar{b}_t\,dw_t, \quad t > 0, \quad \bar{b}_0 = \eta. \tag{3.5}$$

Thus, Eq. (3.3) could be treated as the perturbed equation with the small perturbation $-\frac{\beta}{8}\,a^3$, while (3.5) is the corresponding unperturbed equation.

Let us note the following important fact: By applying the comparison method exposed in the papers [Ja, kat****] and [Ja****], we find that $\overline{a}_t^2$ can be compared with the solution z_t of the linear stochastic differential equation

$$dz_t = (2\mu + \nu^2)\,z_t\,dt + 2\nu z_t\,dw_t, \quad t > 0, \quad z_0 = \eta^2,$$

in the sense that

$$\overline{a}_t^2 \le z_t, \quad t \ge 0 \quad \text{with probability one.}$$

Since the solution of this equation may be written in the form (see [Arnold****], [Ibrahim****], [Skor****], [Lipc****])

$$z_t = \eta^2 e^{(2\mu-\nu^2)t+2\nu\,w_t}, \quad t \ge 0,$$

and according to the law of iterated logarithm, from which it follows that the sample function of the Brownian motion approaches the limiting value $\sqrt{2t\log(\log t)}$ as $t \to \infty$ with probability one, it follows that $2\mu - \nu^2 < 0$, or in terms of the original damping ratio,

$$\alpha > \frac{S(2\omega_0)}{4\omega_0^2}\,, \tag{3.6}$$

is a necessary and sufficient condition for the equilibrium solution to be asymptotically stable with probability one. Therefore, it is reasonable to think, under the condition (3.6), that the averiged amplitude $\overline{a}(t)$ is bounded with some constant, with probability one. This conclusion will be very important in the sequel.

Remember that, strictly mathematically, the usual procedure for determining the stationary probability density for the averiged amplitude from the corresponding Kolmogorov–Fokker–Planck equation, requires that the drift and difussion coefficients of Eq. (3.3) be bounded, which is a atrong assumption. In practice there is an abundance of examples (see [Ariaratnam****], [Roberts****], [Caugi****], [Ibrahim****]) to show that this boundedness condition is not essential. However, the situation remains not completely clear and the previous commentary could be an acceptable argument.

Since it is reasonable to think that the averiged amplitude is bounded with a constant $q > 0$ with probability one, determined in advance with the physical characteristics of the oscillator, then $\frac{\beta}{8} \,|\, a \,|^3 \leq \frac{\beta}{8}\, q^3$, and, therefore,

$$\phi(\beta) = \frac{\beta}{8}\, q^3.$$

Now, we could estimate the $(2k)$–th moment closeness of $\overline{a}_t$ and $\overline{b}_t$ by using (2.10), but because of the linearity of Eq. (3.5) we shall obtain one better estimation.

We need the following Itô's differential formula, so–called the Itô's diferentiation rule (see [Skor****], [Arnold****], Lip****], [Ja-Inst****], [Ikeda****], Wong****]): If the stochastic process $(x(t), t \in [0,T])$ has the stochastic differential $dx_t = a_t\, dt + b_t\, dw_t$ and if the non–random function $f(t,x)$ is continuous together with its derivatives f'_t, f'_x, f''_{xx}, then the process $f(t, x_t)$ has the stochastic differential

$$\begin{aligned} df(t,x_t) = &\left(f'_t(t,x_t) + a_t\, f'_x(t,x_t) + \frac{1}{2}\, b_t^2\, f''_{xx}(t,x_t) \right) dt \\ &+ b_t\, f'_x(t,x_t)\, dw_t, \quad t \in [0,T]. \end{aligned}$$

Let us subtract the equations (3.3) and (3.5) in integral form,

$$\overline{a}_t - \overline{b}_t = \int_0^t \left(\mu(\overline{a}_s - \overline{b}_s) - \frac{\beta}{8}\, \overline{a}_s^3 \right) ds + \int_0^t \nu(\overline{a}_s - \overline{b}_s)\, dw_s, \quad t \geq 0,$$

and then apply the Itô's differentiation rule to $f(x) = x^{2k}$:

$$\begin{aligned} (\overline{a}_t - \overline{b}_t)^{2k} = &2k \int_0^t \left(\mu(\overline{a}_s - \overline{b}_s) - \frac{\beta}{8}\, \overline{a}_s^3 \right) (\overline{a}_s - \overline{b}_s)^{2k-1} ds \\ &+ k(2k-1)\nu^2 \int_0^t (\overline{a}_s - \overline{b}_s)^{2k} ds + 2k\nu \int_0^t (\overline{a}_s - \overline{b}_s)^{2k} dw_s, \quad t \geq 0. \end{aligned}$$

From the basic property of the Itô's integral it follows that $E \int_0^t (\overline{a}_s - \overline{b}_s)^{2k} dw_s = 0$, and thus

$$\begin{aligned} E|\overline{a}_t - \overline{b}_t|^{2k} \leq &\left[\, 2k|\mu| + k(2k-1)\nu^2 \right] \int_0^t E|\overline{a}_s - \overline{b}_s|^{2k} ds \\ &+ \frac{\beta}{4}\, q^3 k \int_0^t E|\overline{a}_s - \overline{b}_s|^{2k-1} ds, \quad t \geq 0. \end{aligned}$$

Since mathematical expectation satisfies the well–known Hölder's inequality $E|X| \le \big(E|X|^p\big)^{1/p}$ for any real number $p > 1$, by taking $p = \frac{2k}{2k-1}$ we obtain $E|\overline{a}_s - \overline{b}_s|^{2k} \le \big(E|\overline{a}_s - \overline{b}_s|^{2k}\big)^{(2k-1)/(2k)}$, such that

$$E|\overline{a}_t - \overline{b}_t|^{2k} \le \big[\, 2k|\mu| + k(2k-1)\nu^2\big] \int_0^t E|\overline{a}_s - \overline{b}_s|^{2k} ds + \frac{\beta}{4}\, q^3 k \int_0^t \big(E|\overline{a}_s - \overline{b}_s|^{2k}\big)^{(2k-1)/(2k)} ds, \quad t \ge 0.$$

To estimate $E|\overline{a}_t - \overline{b}_t|^{2k}$ from this integral inequality, we shall apply the following version of the well–known Gronwall–Bellman's lemma [Bainov****, p. 39]: Let $u(t)$, $a(t)$ and $b(t)$ be nonnegative continuous functions in $[0, T]$ and let $c > 0$, $0 \le \gamma < 1$ be constants. If

$$u(t) \le c + \int_0^t a(s) u(s)\, ds + \int_0^t b(s)\, u^\gamma(s)\, ds, \quad t \in [0, T],$$

then

$$u(t) \le \left(c^{1-\gamma}\, e^{(1-\gamma)\int_0^t a(s)\, ds} + (1-\gamma) \int_0^t b(s)\, e^{(1-\gamma)\int_s^t a(r)\, dr}\, ds \right)^{\frac{1}{1-\gamma}}, \quad t \in [0, T].$$

Because $\overline{a}_t$ and $\overline{b}_t$ are continuous with probability one, the expectation $E|\overline{a}_t - \overline{b}_t|^{2k}$ is also continuous. By taking $u(t) = E|\overline{a}_t - \overline{b}_t|^{2k}$, $\gamma = (2k-1)/(2k)$ and by applying the cited Gronwall–Bellman's lemma, we find

$$E|\overline{a}_t - \overline{b}_t|^{2k} \le C \cdot \beta^{2k} e^{2k[\,|\mu| + (2k-1)/2 \cdot \nu^2] t}, \quad t \ge 0,$$

where $C = \left(\frac{q^3 k}{8[\,|\mu| + (2k-1)/2 \cdot \nu^2]} \right)^{2k}$. In accordance with the considerations in Section 2, for any number $r \in (0, 1)$ let us determine $T(\beta)$ from the relation

$$[\,|\mu| + (2k-1)/2 \cdot \nu^2]\, T(\beta) = -r \ln \beta.$$

Therefore,

$$\sup_{t \in [0, T(\beta)]} E|\overline{a}_t - \overline{b}_t|^{2k} \le C \cdot \beta^{2k(1-r)} \to 0 \quad \text{as} \quad \beta \to 0 \tag{3.7}$$

and $T(\beta) \to \infty$ as $\beta \to 0$.

Remember that the $(2k)$–th moment of the solution $\overline{b}_t$ of the linear stochastic differential equation (3.5) may be written in the form (see [Arnold****], [Gihm****], [Ibrahim****], [Jank****])

$$E|\overline{b}(t)|^{2k} = \eta^{2k} e^{2k[\,\mu + (2k-1)/2 \cdot \nu^2] t}, \quad t > 0.$$

Therefore, this averiged amplitude is exponentially stable in the $(2k)$–th moment sense if and only if $\mu + (2k-1)/2 \cdot \nu^2 < 0$, or in terms of the original damping ratio,

$$\alpha > (k+1)\,\frac{S(2\omega_0)}{4\omega_0^2}. \tag{3.8}$$

Note that then the condition (3.6) is also satisfied. Thus, for an arbitrary small value $\theta > 0$, from (3.7) we find

$$\beta < (\theta/C)^{1/(2k(1-r))}. \tag{3.9}$$

However, if the damping factors of Eq. (3.1) satisfy (3.8) and (3.9) for a given small θ, then $\overline{a}_t$ is different from $\overline{b}_t$, in the $(2k)$–th moment sense, for at most θ on the time–interval $[0, T(\beta)]$. In accordance with this fact, it is reasonable to believe that the behavior of this non–linear system is stable, approximately, in the $(2k)$–th moment sense, analogously to the linear one.

Let us give some concluding remarks: The method exposed in this paper for the description of the behavior of the non–linear oscillator amplitude is illustrated on a simple example, comparable with the linear one. Likewise, it could be applied on some other situations, when mostly classical estimations did not give suitable results. Of course, it could be necessary to know conditions under which the corresponding unperturbed equations are asymptotically stable in the $(2k)$–th moment sense. Moreover, this pethod could be applied to describe the behavior of nonlinear dynamic systems subjected to more independent random excitations of a Gaussian white noise type, what will be a subject of our forthcoming work.