Lagrangian submanifolds in the nearly Kähler $S^3 \times S^3$

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Based on joint work Bart Dioos and J Xianfeng Wang (Nankai University)

Zlatibor
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Nearly Kähler manifolds
An **almost Hermitian** manifold \((M, g, J)\) is a manifold \(M\) with metric \(g\) and almost complex structure \(J\) (endomorphism s.t. \(J^2 = -\text{Id}\)) satisfying

\[
g(JX, JY) = g(X, Y), \quad X, Y \in TM.
\]

A **nearly Kähler** manifold is an almost Hermitian manifold \((M, g, J)\) with the extra assumption that \(\nabla J\) is skew-symmetric:

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(\nabla_X J)Y + (\nabla_Y J)X = 0, \quad X, Y \in TM.
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Nearly Kähler manifolds
Definitions

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First example introduced by Fukami and Ishihara in 1955 and systematically studied by A. Gray in several papers starting from 1970.
For convenience, we will write $G(X, Y) = (\nabla_X J)Y$.

Some identities:

$G(X, Y) + G(Y, X) = 0,$

$G(X, JY) + JG(X, Y) = 0,$

$g(G(X, Y), Z)$ and $g(G(X, Y), JZ)$ are totally anti symmetric.

$\bar{\nabla}J = 0.$

Here $\bar{\nabla}_X Y = \nabla_X Y - \frac{1}{2} JG(X, Y)$ is the canonical Hermitian connection.
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2. by a result of Grunewald, provided the nearly Kähler manifold is simply connected, there is a bijective correspondence between nearly Kähler structures and unit real Killing spinors
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The nearly Kähler $S^3 \times S^3$
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The nearly Kähler structure

Let

$$Z(p, q) = (pU(p, q), qV(p, q)),$$

be a tangent vector at the point $(p, q)$. Then $U(p, q)$ and $V(p, q)$ are imaginary quaternions.
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- The almost complex structure $J$ on $S^3 \times S^3$ is defined by

$$JZ_{(p, q)} = \frac{1}{\sqrt{3}} (p(2V - U), q(-2U + V)).$$
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- If $\langle \cdot, \cdot \rangle$ is the product metric on $S^3 \times S^3$, the metric $g$ is given by

$$g(Z, Z') = \frac{1}{2} (\langle Z, Z' \rangle + \langle JZ, JZ' \rangle)$$

$$= \frac{4}{3} (\langle U, U' \rangle + \langle V, V' \rangle)$$

$$- \frac{2}{3} (\langle U, V' \rangle + \langle U', V \rangle).$$
In $S^3 \times S^3$ we have that the tensor $G$ has the following additional properties:

\[
g(G(X, Y), G(Z, W)) = \frac{1}{3} \left( g(X, Z)g(Y, W) - g(X, W)g(Y, Z) + g(JX, Z)g(JW, Y) - g(JX, W)g(JZ, Y) \right),
\]

\[
G(X, G(Y, Z)) = \frac{1}{3} \left( g(X, Z)Y - g(X, Y)Z + g(JX, Z)JY - g(JX, Y)JZ \right),
\]

\[
(\nabla G)(X, Y, Z) = \frac{1}{3} \left( g(X, Z)JY - g(X, Y)JZ - g(JY, Z)X \right).
\]
For unit quaternions $a$, $b$ and $c$ the map

$$F(p, q) = (apc^{-1}, bqc^{-1})$$

is an isometry.
The nearly Kähler $S^3 \times S^3$

An almost product structure $P$

We define the almost product structure $P$ as

$$P(Z)_{(p,q)} = (pV, qU).$$
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Properties:

\[ P^2 = \text{Id}, \]
\[ PJ = -JP, \]
\[ g(PZ, PZ') = g(Z, Z'), \]
\[ P \text{ is preserved by isometries}, \]
\[ PG(X, Y) + G(PX, PY) \]

Note that the usual product structure $Q$:

\[ Q(Z)_{(p,q)} = (-pU, qV). \]

is not compatible with the almost complex metric.
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Properties:

- $P^2 = \text{Id}$,
- $PJ = -JP$,
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- $PG(X, Y) + G(PX, PY)$

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The nearly Kähler $S^3 \times S^3$

Curvature tensor

- The Riemann curvature tensor of $S^3 \times S^3$ is

$$\tilde{R}(U, V)W = \frac{5}{12} (g(V, W)U - g(U, W)V)$$

$$+ \frac{1}{12} (g(JV, W)JU - g(JU, W)JV - 2g(JU, V)JW)$$

$$+ \frac{1}{3} (g(PV, W)PU - g(PU, W)PV)$$

$$+ g(JPV, W)JPU - g(JPU, W)JPV).$$

- $$(\tilde{\nabla}_Z P)Z' = \frac{1}{2} J(G(Z, PZ') + PG(Z, Z'))$$
Relations with the Euclidean induced structure
Remark that the Levi Civita connections $\tilde{\nabla}$ of $g$ and $\hat{\nabla}$ of the usual metric are related by

$$\hat{\nabla}_Z Z' = \tilde{\nabla}_Z Z' + \frac{1}{2}(JG(Z, PZ') + JG(Z', PZ)).$$
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Note that the $\hat{\nabla}$ is given by

$$D_Z Z' = \tilde{\nabla}_Z Z' - \frac{1}{2} \langle Z, QZ' \rangle(-p, q) - \frac{1}{2} \langle Z, Z' \rangle(p, q).$$
Basic properties of Lagrangian submanifolds
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Let $M$ be a Lagrangian submanifold of $S^3 \times S^3$. Then we have

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3. $M$ is minimal
4. $\langle h(X, Y), JZ \rangle$ is totally symmetric

$G$ is related to the canonical volume form $\omega$ by

$$\omega(X, Y, Z) = \sqrt{3}g(JG(X, Y), Z).$$
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1. \( A \) and \( B \) are symmetric operators
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These properties imply that \( A \) and \( B \) can be diagonalised simultaneously and that there exists a basis \( e_1, e_2, e_3 \) of the tangent space at every point such that

\[ Pe_i = \cos 2\theta_i e_i + \sin 2\theta_i Je_i \]
The tensor $T(X, Y) = S_{JX} Y = -Jh(X, Y)$ is minimal and $g(T(X, Y), Z)$ is totally symmetric.
Basic Equations

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2. Gauss equation

$$R(X, Y)Z = \frac{5}{12} (g(Y, Z)X - g(X, Z)Y)$$
$$+ \frac{1}{3} (g(AY, Z)AX - g(AX, Z)AY + g(BY, Z)BX - g(BX, Z)BY)$$
$$+ [S_{JX}, S_{JY}]Z.$$

3. Codazzi equation

$$\nabla h(X, Y, Z) - \nabla h(Y, X, Z) =$$
$$\frac{1}{3} (g(AY, Z)JBX - g(AX, Z)JBY - g(BY, Z)JAX + g(BX, Z)JAY)$$
Covariant derivatives equations for $A$ and $B$:

\[
(\nabla_X A) Y = BS_{JX} Y - Jh(X, BY) + \frac{1}{2}(JG(X, AY) - AJG(X, Y)),
\]

\[
(\nabla_X B) Y = Jh(X, AY) - AS_{JX} Y + \frac{1}{2}(JG(X, BY) - BJG(X, Y)).
\]
Proposition

The sum of the angle functions vanishes modulo $\pi$
Lemma

The derivatives of the angles $\theta_i$ give the components of the second fundamental form

$$E_i(\theta_j) = - h^i_{jj}$$

except $h^3_{12}$. The second fundamental form and covariant derivative are related by

$$h^k_{ij} \cos(\theta_j - \theta_k) = \left( \frac{\sqrt{3}}{6} \varepsilon^k_{ij} - \omega^k_{ij} \right) \sin(\theta_j - \theta_k).$$
Elementary examples of Lagrangian submanifolds
Schäfer-Smoczyk

Example 1: \( f(g) = (g, 1) \).
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\[ df(E_1(g))) = (gi, 0)_{(g, 1)}, \]
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\[ Pdf(E_1(g))) = (0, i)_{(g,1)}, \]
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\begin{align*}
    Jdf(E_1(g))) &= \frac{1}{\sqrt{3}} (-gi, -2i)_{(g,1)}, \\
    Jdf(E_2(g))) &= \frac{1}{\sqrt{3}} (-gj, -2j)_{(g,1)}, \\
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(2\theta_1, 2\theta_2, 2\theta_3) = \left( \frac{2\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3} \right).\]
Schäfer-Smoczyk

Example 2: \( f(g) = (1, g) \)
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\begin{align*}
    df(E_1(g)) &= (0, gi)(1,g), \\
    df(E_2(g)) &= (0, gj)(1,g), \\
    df(E_3(g)) &= (0, -gk)(1,g).
\end{align*}
\]

\[
\begin{align*}
    P\text{df}(E_1(g)) &= (i, 0)(1,g), \\
    P\text{df}(E_2(g)) &= (j, 0)(1,g), \\
    P\text{df}(E_3(g)) &= (k, 0)(1,g).
\end{align*}
\]

\[
\begin{align*}
    J\text{df}(E_1(g)) &= \frac{1}{\sqrt{3}} (2i, gi)(1,g), \\
    J\text{df}(E_2(g)) &= \frac{1}{\sqrt{3}} (2j, gj)(1,g)), \\
    J\text{df}(E_3(g)) &= \frac{1}{\sqrt{3}} (-2k, gk)(1,g).
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    df(E_1(g)) &= (gi, gi)(g, g) = P \text{df}(E_1(g)), \\
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\[ Jdf(E_1(g)) = \frac{1}{\sqrt{3}} (gi, -gi)_{(g,g)}, \]
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As $P$ is the identity, we see that the angle functions vanish.
Moroianu-Semmelmann

Example 4: \( f(g) = (g, gi) \)
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$Pdf(E_1(g)) = (gi, -g)(g, gi) = dF(E_1(g))$, 
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$(2\theta_1, 2\theta_2, 2\theta_3) = (0, \pi, \pi)$
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Elementary examples of Lagrangian submanifolds

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$$

df(E_1(g)) = (gi, -g)_{(g, gi)}, \\

df(E_2(g)) = (gj, -gk)_{(g, gi)}, \\

df(E_3(g)) = (-gk, -gj)_{(g, gi)}.
$$

$$
Pdf(E_1(g)) = (gi, -g)_{(g, gi)} = dF(E_1(g)), \\
Pdf(E_2(g)) = (gj, -gk)_{(g, gi)} = -dF(E_2(g)), \\
Pdf(E_3(g)) = (-gk, -gj)_{(g, gi)} = -dF(E_3(g)).
$$

$$
Jdf(E_1(g)) = \frac{1}{\sqrt{3}}(gi, g)_{(g, gi)}, \\
Jdf(E_2(g)) = -\sqrt{3}(gj, gk)_{(g, gi)}), \\
Jdf(E_3(g)) = \sqrt{3}(gk, -gj)_{(g, gi)}.
$$

$$(2\theta_1, 2\theta_2, 2\theta_3) = (0, \pi, \pi)$$
Moroianu-Semmelmann

Example 5: \( f(g) = (g^{-1}, \text{gig}^{-1}) \)
Example 5: $f(g) = (g^{-1}, \text{gig}^{-1})$

\[
\begin{align*}
\text{df}(E_1(g)) &= (-ig^{-1}, 0)_{(g^{-1}, \text{gig}^{-1})}, \\
\text{df}(E_2(g)) &= (-jg^{-1}, -2gkg^{-1})_{(g^{-1}, \text{gig}^{-1})}, \\
\text{df}(E_3(g)) &= (kg^{-1}, -2gjg^{-1})_{(g^{-1}, \text{gig}^{-1})}.
\end{align*}
\]

\[
\begin{align*}
\text{Pdf}(E_1(g)) &= (0, 1)_{(g^{-1}, \text{gig}^{-1})}, \\
\text{Pdf}(E_2(g)) &= (-2jg^{-1}, -gkg^{-1})_{(g^{-1}, \text{gig}^{-1})}, \\
\text{Pdf}(E_3(g)) &= (2kg^{-1}, -gjg^{-1})_{(g^{-1}, \text{gig}^{-1})}.
\end{align*}
\]

\[
\begin{align*}
\text{Jdf}(E_1(g)) &= \frac{1}{\sqrt{3}} (ig^{-1}, -2)_{(g^{-1}, \text{gig}^{-1})}, \\
\text{Jdf}(E_2(g)) &= \frac{1}{\sqrt{3}} (-jg^{-1}, 0)_{(g^{-1}, \text{gig}^{-1})}, \\
\text{Jdf}(E_3(g)) &= \frac{1}{\sqrt{3}} (3kg^{-1}, 0)_{(g^{-1}, \text{gig}^{-1})}.
\end{align*}
\]
Almost complex surfaces
Elementary examples of Lagrangian submanifolds

Moroianu-Semmelmann

Example 5: \( f(g) = (g^{-1}, gig^{-1}) \)

\[
\begin{align*}
df(E_1(g))) &= (-ig^{-1}, 0)_{(g^{-1}, gig^{-1})}, \\
df(E_2(g))) &= (-jg^{-1}, -2gkg^{-1})_{(g^{-1}, gig^{-1})}, \\
df(E_3(g))) &= (kg^{-1}, -2gjg^{-1})_{(g^{-1}, gig^{-1})}.
\end{align*}
\]

\[
\begin{align*}
Pdf(E_1(g))) &= (0, 1)_{(g^{-1}, gig^{-1})}, \\
Pdf(E_2(g))) &= (-2jg^{-1}, -gkg^{-1})_{(g^{-1}, gig^{-1})}, \\
Pdf(E_3(g))) &= (2kg^{-1}, -gjg^{-1})_{(g^{-1}, gig^{-1})}.
\end{align*}
\]

\[
\begin{align*}
Jdf(E_1(g))) &= \frac{1}{\sqrt{3}}(ig^{-1}, -2)_{(g^{-1}, gig^{-1})}, \\
Jdf(E_2(g))) &= \frac{1}{\sqrt{3}}(-jg^{-1}, 0)_{(g^{-1}, gig^{-1})}, \\
Jdf(E_3(g))) &= \frac{1}{\sqrt{3}}(3kg^{-1}, 0)_{(g^{-1}, gig^{-1})}.
\end{align*}
\]

\[
(2\theta_1, 2\theta_2, 2\theta_3) = \left( \frac{\pi}{3}, \frac{\pi}{3}, \frac{4\pi}{3} \right)
\]
Example 6: \( f(g) = (gig^{-1}, g^{-1}) \)
Almost complex surfaces
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Example 6: \( f(g) = (g_{ig}^{-1}, g^{-1}) \) So this is the previous immersions with both components interchanged. Then

\[
\begin{align*}
df(E_1(g))) &= (0, -ig^{-1})(g_{ig}^{-1}, g^{-1}), \\
df(E_2(g))) &= (-2gkg^{-1}, -jg^{-1})(g_{ig}^{-1}, g^{-1}), \\
df(E_3(g))) &= (-2gjg^{-1}, kg^{-1})(g_{ig}^{-1}, g^{-1}).
\end{align*}
\]

\[
\begin{align*}
Pdf(E_1(g))) &= (1, 0)(g_{ig}^{-1}, g^{-1}), \\
Pdf(E_2(g))) &= (-gkg^{-1}, -2jg^{-1})(g_{ig}^{-1}, g^{-1}), \\
Pdf(E_3(g))) &= (-gjg^{-1}, 2kg^{-1})(g_{ig}^{-1}, g^{-1}).
\end{align*}
\]

\[
\begin{align*}
Jdf(E_1(g))) &= \frac{1}{\sqrt{3}} (2, ig^{-1})(g_{ig}^{-1}, g^{-1}), \\
Jdf(E_2(g))) &= \frac{1}{\sqrt{3}} (0, 3jg^{-1})(g_{ig}^{-1}, g^{-1}), \\
Jdf(E_3(g))) &= \frac{1}{\sqrt{3}} (0, -3kg^{-1})(g_{ig}^{-1}, g^{-1}).
\end{align*}
\]

\[
(2\theta_1, 2\theta_2, 2\theta_3) = \left( \frac{2\pi}{3}, \frac{5\pi}{3}, \frac{5\pi}{3} \right)
\]
Moroianu-Semmelmann

Example 7: $f(g) = (gig^{-1}, gjg^{-1})$

For the tangent map we have

$$df(E_1) = (0, 2gkg^{-1}),$$

$$df(E_2) = (-2gkg^{-1}, 0),$$

$$df(E_3) = 2(-gjg^{-1}, gig^{-1}).$$

Also

$$Jdf(E_1) = 2\sqrt{3}(2gkg^{-1}),$$

$$Jdf(E_2) = 2\sqrt{3}(-gkg^{-1}, -2),$$

$$Jdf(E_3) = -2\sqrt{3}(-gjg^{-1}, gig^{-1}).$$

We also have

$$Pdf(E_1) = (2, 0) = -\frac{1}{2}(df(E_1) - \sqrt{3}Jdf(E_2)),$$

$$Pdf(E_2) = (0, 2) = -\frac{1}{2}(df(E_1) + \sqrt{3}Jdf(E_2)),$$

$$Pdf(E_3) = -2(-gjg^{-1}, -gig^{-1}) = df(E_3).$$

The angles $2\theta_i$ are thus equal to $0$, $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$. 
Example 7: $f(g) = (gig^{-1}, gjg^{-1})$ For the tangent map we have $df(E_1) = (0, 2gkg^{-1})$, $df(E_2) = (-2gkg^{-1}, 0)$, $df(E_3) = 2(-gjg^{-1}, gig^{-1})$. 
Example 7: $f(g) = (g_{ij}^{-1}, g_{jig}^{-1})$ For the tangent map we have $df(E_1) = (0, 2kgg^{-1})$, $df(E_2) = (-2kgg^{-1}, 0)$, $df(E_3) = 2(-g_{jig}^{-1}, g_{jig}^{-1})$. Also $Jdf(E_1) = \frac{2}{\sqrt{3}}(2, kgg^{-1})$, $Jdf(E_2) = \frac{2}{\sqrt{3}}(kgg^{-1}, -2)$ and $Jdf(E_3) = -\frac{2}{\sqrt{3}}(g_{jig}^{-1}, g_{jig}^{-1})$. 
Example 7: $f(g) = (gig^{-1}, gjg^{-1})$ For the tangent map we have $df(E_1) = (0, 2gkg^{-1})$, $df(E_2) = (-2gkg^{-1}, 0)$, $df(E_3) = 2(-gjg^{-1}, gig^{-1})$. Also $Jdf(E_1) = \frac{2}{\sqrt{3}}(2, gkg^{-1})$, $Jdf(E_2) = \frac{2}{\sqrt{3}}(gkg^{-1}, -2)$ and $Jdf(E_3) = -\frac{2}{\sqrt{3}}(gjg^{-1}, gig^{-1})$.

We also have

$$Pdf(E_1) = (2, 0) = -\frac{1}{2}(df(E_1) - \sqrt{3}Jdf(E_2)),$$

$$Pdf(E_2) = (0, 2) = -\frac{1}{2}(df(E_1) + \sqrt{3}Jdf(E_2)),$$

$$Pdf(E_3) = -2(gjg^{-1}, -gig^{-1}) = df(E_3).$$
Example 7: $f(g) = (gig^{-1}, gjg^{-1})$. For the tangent map we have $df(E_1) = (0, 2gkg^{-1})$, $df(E_2) = (-2gkg^{-1}, 0)$, $df(E_3) = 2(-gjg^{-1}, gig^{-1})$.

Also $Jdf(E_1) = \frac{2}{\sqrt{3}}(2, gkg^{-1})$, $Jdf(E_2) = \frac{2}{\sqrt{3}}(gkg^{-1}, -2)$ and $Jdf(E_3) = -\frac{2}{\sqrt{3}}(gjg^{-1}, gig^{-1})$.

We also have

$$Pdf(E_1) = (2, 0) = -\frac{1}{2}(df(E_1) - \sqrt{3}Jdf(E_2)),$$

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$$Pdf(E_3) = -2(gjg^{-1}, -gig^{-1}) = df(E_3).$$

The angles $2\theta_i$ are thus equal to $0$, $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$. 
Example 8. Consider the immersion \( f : \mathbb{R}^3 \to S^3 \times S^3 : (\tilde{u}, \tilde{v}, \tilde{w}) \mapsto (p(\tilde{u}, \tilde{w}), q(\tilde{u}, \tilde{v})) \) where \( p \) and \( q \) are constant mean curvature torii in \( S^3 \) given by:

\[
p(\tilde{u}, \tilde{w}) = \left( \cos(\tilde{u}) \cos(\tilde{w}), \cos(\tilde{u}) \sin(\tilde{w}), \sin(\tilde{u}) \cos(\tilde{w}), \varepsilon_1 \sin(\tilde{u}) \sin(\tilde{w}) \right),
\]

\[
q(\tilde{u}, \tilde{v}) = \left( \cos(\tilde{u}) \cos(\tilde{v}), \cos(\tilde{u}) \sin(\tilde{v}), \sin(\tilde{u}) \cos(\tilde{v}), \varepsilon_2 \sin(\tilde{u}) \sin(\tilde{v}) \right).
\]

Straightforward computations give that \( u, v, w \) determined by \( \tilde{u} = \sqrt{\frac{3}{2}}u \), \( \tilde{v} = \sqrt{\frac{3}{2}}v \) and \( \tilde{w} = \sqrt{\frac{3}{2}}w \) are the standard flat coordinates. So it is an immersion of a torus.

The angles \( 2\theta_i \) are again equal to 0, \( \frac{2\pi}{3} \) and \( \frac{4\pi}{3} \).
Example 8. Consider the immersion $f : \mathbb{R}^3 \to S^3 \times S^3 : (\tilde{u}, \tilde{v}, \tilde{w}) \mapsto (p(\tilde{u}, \tilde{w}), q(\tilde{u}, \tilde{v}))$ where $p$ and $q$ are constant mean curvature torii in $S^3$ given by

$$p = (\cos(\tilde{u}) \cos(\tilde{w}), \cos(\tilde{u}) \sin(\tilde{w}), \sin(\tilde{u}) \cos(\tilde{w}), \varepsilon_1 \sin(\tilde{u}) \sin(\tilde{w})),
$$

$$q = (\cos(\tilde{u}) \cos(\tilde{v}), \cos(\tilde{u}) \sin(\tilde{v}), \sin(\tilde{u}) \cos(\tilde{v}), \varepsilon_2 \sin(\tilde{u}) \sin(\tilde{w}))d,$$

where $d$ is the unit quaternion given by
Example 8. Consider the immersion $f : \mathbb{R}^3 \to S^3 \times S^3 : (\tilde{u}, \tilde{v}, \tilde{w}) \mapsto (p(\tilde{u}, \tilde{w}), q(\tilde{u}, \tilde{v}))$ where $p$ and $q$ are constant mean curvature torii in $S^3$ given by

$$p = (\cos(\tilde{u}) \cos(\tilde{w}), \cos(\tilde{u}) \sin(\tilde{w}), \sin(\tilde{u}) \cos(\tilde{w}), \varepsilon_1 \sin(\tilde{u}) \sin(\tilde{w})), $$
$$q = (\cos(\tilde{u}) \cos(\tilde{v}), \cos(\tilde{u}) \sin(\tilde{v}), \sin(\tilde{u}) \cos(\tilde{v}), \varepsilon_2 \sin(\tilde{u}) \sin(\tilde{v}))d,$$

where $d$ is the unit quaternion given by

$$\left( \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0, 0 \right)$$
Example 8. Consider the immersion $f : \mathbb{R}^3 \to S^3 \times S^3 : (\tilde{u}, \tilde{v}, \tilde{w}) \mapsto (p(\tilde{u}, \tilde{w}), q(\tilde{u}, \tilde{v}))$ where $p$ and $q$ are constant mean curvature torii in $S^3$ given by

$$p = (\cos(\tilde{u}) \cos(\tilde{w}), \cos(\tilde{u}) \sin(\tilde{w}), \sin(\tilde{u}) \cos(\tilde{w}), \varepsilon_1 \sin(\tilde{u}) \sin(\tilde{w})),$$

$$q = (\cos(\tilde{u}) \cos(\tilde{v}), \cos(\tilde{u}) \sin(\tilde{v}), \sin(\tilde{u}) \cos(\tilde{v}), \varepsilon_2 \sin(\tilde{u}) \sin(\tilde{v}))d,$$

where $d$ is the unit quaternion given by

$$\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0, 0\right)$$

Straightforward computations give that $u, v, w$ determined by $\tilde{u} = \frac{\sqrt{3}}{2} u$, $\tilde{v} = \frac{\sqrt{3}}{2} v$ and $\tilde{w} = \frac{\sqrt{3}}{2} w$ are the standard flat coordinates. So it is an immersion of a torus.
Example 8. Consider the immersion $f : \mathbb{R}^3 \to S^3 \times S^3 : (\tilde{u}, \tilde{v}, \tilde{w}) \mapsto (p(\tilde{u}, \tilde{w}), q(\tilde{u}, \tilde{v}))$ where $p$ and $q$ are constant mean curvature torii in $S^3$ given by

$$p = (\cos(\tilde{u}) \cos(\tilde{w}), \cos(\tilde{u}) \sin(\tilde{w}), \sin(\tilde{u}) \cos(\tilde{w}), \varepsilon_1 \sin(\tilde{u}) \sin(\tilde{w}))),$$

$$q = (\cos(\tilde{u}) \cos(\tilde{v}), \cos(\tilde{u}) \sin(\tilde{v}), \sin(\tilde{u}) \cos(\tilde{v}), \varepsilon_2 \sin(\tilde{u}) \sin(\tilde{v})))d,$$

where $d$ is the unit quaternion given by

$$(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0, 0)$$

Straightforward computations give that $u, v, w$ determined by $\tilde{u} = \frac{\sqrt{3}}{2}u$, $\tilde{v} = \frac{\sqrt{3}}{2}v$ and $\tilde{w} = \frac{\sqrt{3}}{2}w$ are the standard flat coordinates. So it is an immersion of a torus. The angles $2\theta_i$ are again equal to $0$, $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$. 
**Theorem**

Let $M^3$ be a totally geodesic Lagrangian immersion in $S^3 \times S^3$ then $M$ is locally congruent with either

1. $g \mapsto (g, 1)$
2. $g \mapsto (1, g)$
3. $g \mapsto (g, g)$
4. $g \mapsto (g, gi)$
5. $g \mapsto (g^{-1}, gig^{-1})$
6. $g \mapsto (gig^{-1}, g^{-1})$
Lagrangian submanifolds with constant sectional curvature

Theorem

Let $M^3$ be a totally geodesic Lagrangian immersion in $S^3 \times S^3$ then $M$ is totally geodesic or locally congruent with either

1. $g \mapsto (g_i g^{-1}, g_j g^{-1})$
2. the flat torus described earlier.