Lagrangian submanifolds in the nearly Kähler $S^3 \times S^3$

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Based on joint work Bart Dioos and J Xianfeng Wang (Nankai University)

Zlatibor
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7. Lagrangian submanifolds with constant sectional curvature

Nearly Kähler manifolds

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Definitions

- An almost Hermitian manifold (M, g, J) is a manifold M with metric g and almost complex structure J (endomorphism s.t. $J^2 = -\mathrm{Id}$) satisfying

$$g(JX, JY) = g(X, Y), \qquad X, Y \in TM.$$

- A nearly Kähler manifold is an almost Hermitian manifold (M, g, J) with the extra assumption that ∇J is skew-symmetric:

$$(\nabla_X J)Y + (\nabla_Y J)X = 0, \qquad X, Y \in TM.$$

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First example introduced by Fukami and Ishihara in 1955 and systematically studied by A. Gray in several papers starting from 1970.

Nearly Kähler manifolds

Identities

- For convenience, we will write $G(X,Y) = (\nabla_X J)Y$.
- Some identities:

$$G(X,Y) + G(Y,X) = 0,$$
$$G(X,JY) + JG(X,Y) = 0,$$
$$g(G(X,Y),Z) \text{ and } g(G(X,Y),JZ) \text{ are totally anti symmetric.}$$
$$\overline{\nabla} J = 0.$$

Here $\overline{\nabla}_X Y = \nabla_X Y - \frac{1}{2} JG(X,Y)$ is the canonical Hermitian connection.

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1. by the structure theorems of Nagy, they serve as building blocks for arbitrary nearly Kähler manifolds
2. by a result of Grunewald, provided the nearly Kähler manifold is simply connected, there is a bijective correspondence between nearly Kähler structures and unit real Killing spinors

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All these spaces are compact.

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The nearly Kähler structure

Let

$$Z(p,q) = (pU(p,q), qV(p,q)),$$

be a tangent vector at the point (p,q). Then $U(p,q)$ and $V(p,q)$ are imaginary quaternions.

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- If $\langle \cdot\,,\cdot \rangle$ is the product metric on $S^3 \times S^3$, the metric g is given by

$$\begin{aligned} g(Z,Z') &= \frac{1}{2}\left(\langle Z,Z'\rangle + \langle JZ,JZ'\rangle\right) \\ &= \frac{4}{3}\left(\langle U,U'\rangle + \langle V,V'\rangle\right) \\ &\quad - \frac{2}{3}\left(\langle U,V'\rangle + \langle U',V\rangle\right). \end{aligned}$$

In $S^3 \times S^3$ we have that the tensor G has the following additional properties:

$$
\begin{aligned}
g\big(G(X,Y),G(Z,W)\big) &= \frac{1}{3}\big(g(X,Z)g(Y,W) - g(X,W)g(Y,Z) \\
&\quad + g(JX,Z)g(JW,Y) - g(JX,W)g(JZ,Y)\big), \\
G\big(X,G(Y,Z)\big) &= \frac{1}{3}\big(g(X,Z)Y - g(X,Y)Z \\
&\quad + g(JX,Z)JY - g(JX,Y)JZ\big), \\
(\tilde{\nabla}G)(X,Y,Z) &= \frac{1}{3}\big(g(X,Z)JY - g(X,Y)JZ - g(JY,Z)X\big).
\end{aligned}
$$

The nearly Kähler $S^3 \times S^3$

Isometries

- For unit quaternions a, b and c the map

$$F(p, q) = (apc^{-1}, bqc^{-1})$$

is an isometry.

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$$P^2 = \mathrm{Id},$$
$$PJ = -JP,$$
$$g(PZ, PZ') = g(Z, Z'),$$
$$\text{P is preserved by isometries},$$
$$PG(X, Y) + G(PX, PY)$$

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Note that the usual product structure

$$Q(Z)_{(p,q)} = (-pU, qV).$$

is not compatible with the almost complex metric.

The nearly Kähler $S^3 \times S^3$

Curvature tensor

- The Riemann curvature tensor of $S^3 \times S^3$ is

$$\begin{aligned}\tilde{R}(U,V)W = {} & \frac{5}{12}\big(g(V,W)U - g(U,W)V\big) \\ & + \frac{1}{12}\big(g(JV,W)JU - g(JU,W)JV - 2g(JU,V)JW\big) \\ & + \frac{1}{3}\big(g(PV,W)PU - g(PU,W)PV \\ & \qquad + g(JPV,W)JPU - g(JPU,W)JPV\big).\end{aligned}$$

- $(\tilde{\nabla}_Z P)Z' = \frac{1}{2}J(G(Z,PZ') + PG(Z,Z'))$

Relations with the Euclidean induced structure

Remark that the Levi Civita connections $\tilde{\nabla}$ of g and $\hat{\nabla}$ of the usual metric are related by

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Note that the $\hat{\nabla}$ is given by

$$D_Z Z' = \tilde{\nabla}_Z Z' - \tfrac{1}{2}\langle Z, QZ' \rangle(-p, q) - \tfrac{1}{2}\langle Z, Z' \rangle(p, q).$$

Basic properties of Lagrangian submanifolds

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Theorem (Schäfer-Smoczyk)

Let M be a Lagrangian submanifold of $S^3 \times S^3$ Then we have

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3. *M is minimal*
4. *$< h(X,Y), JZ >$ is totally symmetric*

G is related to the canonical volume form ω by

$$\omega(X,Y,Z) = \sqrt{3}g(JG(X,Y),Z).$$

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These properties imply that A and B can be diagonalised simultaneously and that there exists a basis e_1, e_2, e_3 of the tangent space at every point such that

$$Pe_i = \cos 2\theta_i e_i + \sin 2\theta_i Je_i$$

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2. Gauss equation
$$R(X,Y)Z = \frac{5}{12}\big(g(Y,Z)X - g(X,Z)Y\big) + \frac{1}{3}\big(g(AY,Z)AX - g(AX,Z)AY + g(BY,Z)BX - g(BX,Z)BY\big) + [S_{JX}, S_{JY}]Z.$$
3. Codazzi equation
$$\nabla h(X,Y,Z)) - \nabla h(Y,X,Z) = \frac{1}{3}\big(g(AY,Z)JBX - g(AX,Z)JBY - g(BY,Z)JAX + g(BX,Z)JAY\big)$$

Basic Equations

Covariant derivatives equations for A and B:

$$(\nabla_X A)Y = BS_{JX}Y - Jh(X, BY) + \frac{1}{2}\big(JG(X, AY) - AJG(X, Y)\big),$$
$$(\nabla_X B)Y = Jh(X, AY) - AS_{JX}Y + \frac{1}{2}\big(JG(X, BY) - BJG(X, Y)\big).$$

Proposition

The sum of the angle functions vanishes modulo π

Lemma

The derivatives of the angles θ_i give the components of the second fundamental form

$$E_i(\theta_j) = -h^i_{jj}$$

except h^3_{12}. The second fundamental form and covariant derivative are related by

$$h^k_{ij}\cos(\theta_j - \theta_k) = \left(\frac{\sqrt{3}}{6}\varepsilon^k_{ij} - \omega^k_{ij}\right)\sin(\theta_j - \theta_k).$$

Elementary examples of Lagrangian submanifolds

Schäfer-Smoczyk

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$$(2\theta_1, 2\theta_2, 2\theta_3) = \left(\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3}\right)$$

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$$\begin{aligned} Jdf(E_1(g)) &= \frac{1}{\sqrt{3}}(2i, gi)_{(1,g)}, \\ Jdf(E_2(g)) &= \frac{1}{\sqrt{3}}(2j, gj)_{(1,g))}, \\ Jdf(E_3(g)) &= \frac{1}{\sqrt{3}}(-2k, gk)_{(1,g)}. \end{aligned}$$

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As P is the identity, we see that the angle functions vanish.

Moroianu-Semmelmann

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Example 5: $f(g) = (g^{-1}, gig^{-1})$

$$df(E_1(g))) = (-ig^{-1}, 0)_{(g^{-1},gig^{-1})},$$
$$df(E_2(g))) = (-jg^{-1}, -2gkg^{-1})_{(g^{-1},gig^{-1})},$$
$$df(E_3(g))) = (kg^{-1}, -2gjg^{-1})_{(g^{-1},gig^{-1})}.$$

$$Pdf(E_1(g))) = (0, 1)_{(g^{-1},gig^{-1})},$$
$$Pdf(E_2(g))) = (-2jg^{-1}, -gkg^{-1})_{(g^{-1},gig^{-1})},$$
$$Pdf(E_3(g))) = (2kg^{-1}, -gjg^{-1})_{(g^{-1},gig^{-1})}.$$

$$Jdf(E_1(g))) = \frac{1}{\sqrt{3}}(ig^{-1}, -2)_{(g^{-1},gig^{-1})},$$
$$Jdf(E_2(g))) = \frac{1}{\sqrt{3}}(-jg^{-1}, 0)_{(g^{-1},gig^{-1})},$$
$$Jdf(E_3(g))) = \frac{1}{\sqrt{3}}(3kg^{-1}, 0)_{(g^{-1},gig^{-1})}.$$

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$$\begin{aligned}
df(E_1(g))) &= (-ig^{-1}, 0)_{(g^{-1}, gig^{-1})},\\
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df(E_3(g))) &= (kg^{-1}, -2gjg^{-1})_{(g^{-1}, gig^{-1})}.
\end{aligned}$$

$$\begin{aligned}
Pdf(E_1(g))) &= (0, 1)_{(g^{-1}, gig^{-1})},\\
Pdf(E_2(g))) &= (-2jg^{-1}, -gkg^{-1})_{(g^{-1}, gig^{-1})},\\
Pdf(E_3(g))) &= (2kg^{-1}, -gjg^{-1})_{(g^{-1}, gig^{-1})}.
\end{aligned}$$

$$\begin{aligned}
Jdf(E_1(g))) &= \tfrac{1}{\sqrt{3}}(ig^{-1}, -2)_{(g^{-1}, gig^{-1})},\\
Jdf(E_2(g))) &= \tfrac{1}{\sqrt{3}}(-jg^{-1}, 0)_{(g^{-1}, gig^{-1})},\\
Jdf(E_3(g))) &= \tfrac{1}{\sqrt{3}}(3kg^{-1}, 0)_{(g^{-1}, gig^{-1})}.
\end{aligned}$$

$$(2\theta_1, 2\theta_2, 2\theta_3) = \left(\tfrac{\pi}{3}, \tfrac{\pi}{3}, \tfrac{4\pi}{3}\right)$$

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Example 6: $f(g) = (gig^{-1}, g^{-1})$

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Example 6: $f(g) = (gig^{-1}, g^{-1})$ So this is the previous immersions with both components interchanged. Then

$$df(E_1(g))) = (0, -ig^{-1})_{(gig^{-1},g^{-1})},$$
$$df(E_2(g))) = (-2gkg^{-1}, -jg^{-1})_{(gig^{-1},g^{-1})},$$
$$df(E_3(g))) = (-2gjg^{-1}, kg^{-1})_{(gig^{-1},g^{-1})}.$$

$$Pdf(E_1(g))) = (1, 0)_{(gig^{-1},g^{-1})},$$
$$Pdf(E_2(g))) = (-gkg^{-1}, -2jg^{-1})_{(gig^{-1},g^{-1})},$$
$$Pdf(E_3(g))) = (-gjg^{-1}, 2kg^{-1})_{(gig^{-1},g^{-1})}.$$

$$Jdf(E_1(g))) = \frac{1}{\sqrt{3}}(2, ig^{-1})_{(gig^{-1},g^{-1})},$$
$$Jdf(E_2(g))) = \frac{1}{\sqrt{3}}(0, 3jg^{-1})_{(gig^{-1},g^{-1})},$$
$$Jdf(E_3(g))) = \frac{1}{\sqrt{3}}(0, -3kg^{-1})_{(gig^{-1},g^{-1})}.$$

$$(2\theta_1, 2\theta_2, 2\theta_3) = (\tfrac{2\pi}{3}, \tfrac{5\pi}{3}, \tfrac{5\pi}{3})$$

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Example 7: $f(g) = (gig^{-1}, gjg^{-1})$

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Example 7: $f(g) = (gig^{-1}, gjg^{-1})$ For the tangent map we have $df(E_1) = (0, 2gkg^{-1})$, $df(E_2) = (-2gkg^{-1}, 0)$, $df(E_3) = 2(-gjg^{-1}, gig^{-1})$.

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Example 7: $f(g) = (gig^{-1}, gjg^{-1})$ For the tangent map we have $df(E_1) = (0, 2gkg^{-1})$, $df(E_2) = (-2gkg^{-1}, 0)$, $df(E_3) = 2(-gjg^{-1}, gig^{-1})$.
Also $Jdf(E_1) = \frac{2}{\sqrt{3}}(2, gkg^{-1})$, $Jdf(E_2) = \frac{2}{\sqrt{3}}(gkg^{-1}, -2)$ and $Jdf(E_3) = -\frac{2}{\sqrt{3}}(gjg^{-1}, gig^{-1})$.

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We also have

$$Pdf(E_1) = (2, 0) = -\frac{1}{2}(df(E_1) - \sqrt{3}Jdf(E_2)),$$
$$Pdf(E_2) = (0, 2) = -\frac{1}{2}(df(E_1) + \sqrt{3}Jdf(E_2)),$$
$$Pdf(E_3) = -2(gjg^{-1}, -gig^{-1}) = df(E_3).$$

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We also have

$$\begin{aligned} Pdf(E_1) &= (2,0) = -\tfrac{1}{2}(df(E_1) - \sqrt{3}Jdf(E_2)), \\ Pdf(E_2) &= (0,2) = -\tfrac{1}{2}(df(E_1) + \sqrt{3}Jdf(E_2)), \\ Pdf(E_3) &= -2(gjg^{-1}, -gig^{-1}) = df(E_3). \end{aligned}$$

The angles $2\theta_i$ are thus equal to 0, $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$.

New

Example 8. Consider the immersion $f:\mathbb{R}^3 \to S^3 \times S^3 : (\tilde{u}, \tilde{v}, \tilde{w}) \mapsto (p(\tilde{u}, \tilde{w}), q(\tilde{u}, \tilde{v}))$ where p and q are constant mean curvature torii in S^3 given by

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$$p = (\cos(\tilde{u})\cos(\tilde{w}), \cos(\tilde{u})\sin(\tilde{w}), \sin(\tilde{u})\cos(\tilde{w}), \varepsilon_1 \sin(\tilde{u})\sin(\tilde{w})),$$
$$q = (\cos(\tilde{u})\cos(\tilde{v}), \cos(\tilde{u})\sin(\tilde{v}), \sin(\tilde{u})\cos(\tilde{v}), \varepsilon_2 \sin(\tilde{u})\sin(\tilde{w}))d,$$

where d is the unit quaternion given by

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$$(\tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{2}}, 0, 0)$$

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where d is the unit quaternion given by

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Straightforward computations give that u, v, w determined by $\tilde{u} = \frac{\sqrt{3}}{2}u$, $\tilde{v} = \frac{\sqrt{3}}{2}v$ and $\tilde{w} = \frac{\sqrt{3}}{2}w$ are the standard flat coordinates. So it is an immersion of a torus.

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Totally geodesic Lagrangian submanifolds

Theorem

Let M^3 be a totally geodesic Lagrangian immersion in $S^3 \times S^3$ then M is locally congruent with either

1. $g \mapsto (g, 1)$
2. $g \mapsto (1, g)$
3. $g \mapsto (g, g)$
4. $g \mapsto (g, gi)$
5. $g \mapsto (g^{-1}, gig^{-1})$
6. $g \mapsto (gig^{-1}, g^{-1})$

Lagrangian submanifolds with constant sectional curvature

Theorem

Let M^3 be a totally geodesic Lagrangian immersion in $S^3 \times S^3$ then M is totally geodesic or locally congruent with either

1. $g \mapsto (gig^{-1}, gjg^{-1})$
2. *the flat torus described earlier.*