INVARIANTS OF GEOMETRIC MAPPINGS

Nenad O. Vesić, Milan Lj. Zlatanović, Mića S. Stanković

University of Niš

Zlatibor, 2016

REMEMBER!!!	ПОМНИТЕ!!!
Basic equations	Основные уравнения
ARE NOT DEFINITIONS!	НЕ ЯВЛЯЮТСЯ ОПРЕДЕЛЕНИЯМИ!
They are	Они являются
JUST EQUIVALENT	ЛИШЬ ЭКВИВАЛЕНТНИЫМИ
to the definitions!	определениями!

Briefly about this presentation

- *Introduction*
- *Generalized Bochner Tensor*
  (Nenad O. Vesić, Milan Lj. Zlatanović)
- *Some Relations Between Generalizations of Weyl Projective Tensor*
  (Nenad O. Vesić, Mića S. Stanković)

Introduction

Definition

A differentiable manifold $\mathcal{M}_N$ endowed with a non-symmetric affine connection $L, (L^i_{jk} \neq L^i_{kj})$ is **the non-symmetric affine connection space** $\mathbb{GA}_N$.

A differentiable manifold $\mathcal{M}_N$ endowed with a non-symmetric metric tensor g_{ij} is **the non-symmetric metric space**. This space, endowed with affine connection coefficients

$$\Gamma_{i.jk} = \frac{1}{2}\big(g_{ji,k} - g_{jk,i} + g_{ik,j}\big) \quad \text{and} \quad \Gamma^i_{jk} = g^{\underline{i\alpha}}\Gamma_{\alpha.jk},$$

for $[g^{\underline{ij}}] = [g_{\underline{ij}}]^{-1}$ and $g_{\underline{ij}} = \frac{1}{2}(g_{ij} + g_{ji})$ is **the generalized Riemannian space** $\mathbb{GR}_N$. It is also $g_{\underset{\vee}{ij}} = \frac{1}{2}(g_{ij} - g_{ji})$.

Introduction

Some basic magnitudes

Because of a difference $L^i_{jk} \neq L^i_{kj}$ between affine connection coefficients L^i_{jk} and L^i_{kj} of the affine connection space $\mathbb{GA}_N$ it is defined:

- A symmetric part $L^i_{\underline{jk}}$ of the affine connection coefficient L^i_{jk} as

$$L^i_{\underline{jk}} = \frac{1}{2}(L^i_{jk} + L^i_{kj}).$$

- An anti-symmetric part $L^i_{\underset{\vee}{jk}}$ of the affine connection coefficient L^i_{jk} as

$$L^i_{\underset{\vee}{jk}} = \frac{1}{2}(L^i_{jk} - L^i_{kj}).$$

Definition

A space $\mathbb{A}_N$ which affine connection coefficients are $L^i_{\underline{jk}}$ is **the associated space of the space** $\mathbb{GA}_N$.

Introduction

Covariant derivatives

- It exists only one kind of covariant differentiation with regard to the affine connection of the associated space $\mathbb{A}_N$ defined as

$$a^i_{j;k} = a^i_{j,k} + L^i_{\underline{\alpha k}} a^\alpha_j - L^\alpha_{\underline{jk}} a^i_\alpha,$$

  for a partial differentiation $\partial/\partial x^i$ denoted by comma.
- It exists four kinds of covariant differentiation with regard to the affine connection of the space $\mathbb{GA}_N$ defined as

$$a^i_{j\underset{1}{|}k} = a^i_{j,k} + L^i_{\alpha k} a^\alpha_j - L^\alpha_{jk} a^i_\alpha, \quad a^i_{j\underset{2}{|}k} = a^i_{j,k} + L^i_{k\alpha} a^\alpha_j - L^\alpha_{kj} a^i_\alpha,$$
$$a^i_{j\underset{3}{|}k} = a^i_{j,k} + L^i_{\alpha k} a^\alpha_j - L^\alpha_{kj} a^i_\alpha, \quad a^i_{j\underset{4}{|}k} = a^i_{j,k} + L^i_{k\alpha} a^\alpha_j - L^\alpha_{jk} a^i_\alpha.$$

Introduction

Curvature tensors

- It exists only one curvature tensor R^i_{jmn} of the associated space $\mathbb{A}_N$ defined as

$$R^i_{jmn} = L^i_{\underline{jm},n} - L^i_{\underline{jn},m} + L^\alpha_{\underline{jm}} L^i_{\underline{\alpha n}} - L^\alpha_{\underline{jn}} L^i_{\underline{\alpha m}}. \tag{1}$$

- It exists twelve curvature tensors of a non-symmetric affine space $\mathbb{GA}_N$. A general curvature tensor K^i_{jmn} of this space is

$$K^i_{jmn} = R^i_{jmn} + uL^i_{\underset{\vee}{jm};n} + u'L^i_{\underset{\vee}{jn};m} + vL^\alpha_{\underset{\vee}{jm}} L^i_{\underset{\vee}{\alpha n}} + v'L^\alpha_{\underset{\vee}{jn}} L^i_{\underset{\vee}{\alpha m}} + wL^\alpha_{\underset{\vee}{mn}} L^i_{\underset{\vee}{\alpha j}},$$

for the corresponding real constants u, u', v, v', w.

Introduction

There are five linearly independent curvature tensor of the space $\mathbb{GA}_N$:

$$\underset{1}{K}{}^i_{jmn} = R^i_{jmn} + L^i_{\underset{\vee}{jm};n} - L^i_{\underset{\vee}{jn};m} + L^\alpha_{\underset{\vee}{jm}} L^i_{\underset{\vee}{\alpha n}} - L^\alpha_{\underset{\vee}{jn}} L^i_{\underset{\vee}{\alpha m}}, \tag{2}$$

$$\underset{2}{K}{}^i_{jmn} = R^i_{jmn} - L^i_{\underset{\vee}{jm};n} + L^i_{\underset{\vee}{jn};m} + L^\alpha_{\underset{\vee}{jm}} L^i_{\underset{\vee}{\alpha n}} - L^\alpha_{\underset{\vee}{jn}} L^i_{\underset{\vee}{\alpha m}}, \tag{3}$$

$$\underset{3}{K}{}^i_{jmn} = R^i_{jmn} + L^i_{\underset{\vee}{jm};n} + L^i_{\underset{\vee}{jn};m} - L^\alpha_{\underset{\vee}{jm}} L^i_{\underset{\vee}{\alpha n}} + L^\alpha_{\underset{\vee}{jn}} L^i_{\underset{\vee}{\alpha m}} - 2L^\alpha_{\underset{\vee}{mn}} L^i_{\underset{\vee}{\alpha j}}, \tag{4}$$

$$\underset{4}{K}{}^i_{jmn} = R^i_{jmn} + L^i_{\underset{\vee}{jm};n} + L^i_{\underset{\vee}{jn};m} - L^\alpha_{\underset{\vee}{jm}} L^i_{\underset{\vee}{\alpha n}} + L^\alpha_{\underset{\vee}{jn}} L^i_{\underset{\vee}{\alpha m}} + 2L^\alpha_{\underset{\vee}{mn}} L^i_{\underset{\vee}{\alpha j}}, \tag{5}$$

$$\underset{5}{K}{}^i_{jmn} = R^i_{jmn} - L^\alpha_{\underset{\vee}{jm}} L^i_{\underset{\vee}{\alpha n}} + L^\alpha_{\underset{\vee}{jn}} L^i_{\underset{\vee}{\alpha m}} \tag{6}$$

for a curvature tensor R^i_{jmn} of the associated space $\mathbb{A}_N$.

A RECALL: Weyl projective tensor and its generalizations

Let $f : \mathbb{GA}_N \to \mathbb{G}\overline{\mathbb{A}}_N$ be a mapping between non-symmetric affine connection spaces $\mathbb{GA}_N$ and $\mathbb{G}\overline{\mathbb{A}}_N$ and let affine connection coefficients $\underline{L^i_{jk}}$ and $\underline{\overline{L}^i_{jk}}$ of the associated spaces $\mathbb{A}_N$ and $\overline{\mathbb{A}}_N$ satisfy a relation

$$\underline{\overline{L}^i_{jk}} = \underline{L^i_{jk}} + \underline{\overline{\omega}}^i_{jk} - \underline{\omega}^i_{jk}, \qquad (7)$$

for $\underline{\omega}^i_{jk} \in \mathbb{GA}_N$ and $\underline{\overline{\omega}}^i_{jk} \in \mathbb{G}\overline{\mathbb{A}}_N$.

A RECALL: Weyl projective tensor and its generalizations

From the relation (7) we may obtain a magnitude

$$\mathcal{T}^i_{jk} = L^i_{\underline{jk}} - \underline{\omega}^i_{jk},$$

named *generalized Thomas projective parameter* is an invariant of the mapping f but this invariant has no any important rule in processes of generalizations of the Weyl projective tensor until now.

A RECALL: Weyl projective tensor and its generalizations

Using some less or more complicate calculations authors directly from the relation (7) obtain a relation

$$\overline{R}^i_{jmn} = R^i_{jmn} + \underline{\overline{\Omega}}^i_{jmn} - \underline{\Omega}^i_{jmn} \tag{8}$$

between curvature tensors of the associated spaces $\mathbb{A}_N$ and $\overline{\mathbb{A}}_N$ of the spaces $\mathbb{GA}_N$ and $\mathbb{G}\overline{\mathbb{A}}_N$. The corresponding analogue to Weyl projective tensor is

$$\mathcal{W}^i_{jmn} = R^i_{jmn} - \underline{\Omega}^i_{jmn}. \tag{9}$$

A RECALL: Generalized Weyl projective tensor and its generalizations

Let again $f:\mathbb{GA}_N\to\mathbb{G}\overline{\mathbb{A}}_N$ be a mapping between non-symmetric affine connection spaces $\mathbb{GA}_N$ and $\mathbb{G}\overline{\mathbb{A}}_N$. Affine connection coefficients L^i_{jk} and $\overline{L}^i_{jk}$ of these spaces satisfy a relation

$$\overline{L}^i_{jk}=L^i_{jk}+\overline{\underline{\omega}}^i_{jk}-\underline{\omega}^i_{jk}+\xi^i_{jk}, \tag{10}$$

for an anti-symmetric tensor ξ^i_{jk}. Because $\xi^i_{jk}=\overline{L}^i_{\underset{\vee}{jk}}-L^i_{\underset{\vee}{jk}}$ authors presume $\xi^i_{jk}=0$ (equitorsion mappings). Otherwise, the mapping f becomes reduced onto the associated space $\mathbb{A}_N$.

A RECALL: Weyl projective tensor and its generalizations

In attempts to generalize Weyl projective tensor of a mapping $f:\mathbb{GA}_N\to\mathbb{G}\overline{\mathbb{A}}_N$ authors have forgotten the invariant $\mathcal{W}^i_{jmn}$ obtained from the equation (8).

The equation (8) is just a motivation in further research in this subject. Only which authors have used in further researches in this subject is the same method used for obtaining of the rule (8). Nothing more. As we mentioned above, generalized Thomas projective parameter is a completely useless magnitude in this research.

The methodology used in papers [12, 13, 15, 23] motivated a research for eventually universal invariants of geometric mappings determined with an explicit deformation tensor $P^i_{jk}=\overline{L}^i_{jk}-L^i_{jk}$ and maybe simplifications and generalizations of invariants of previously obtained generalizations of Weyl projective tensor.

A RECALL: Weyl projective tensor and its generalizations

Using the necessary guess of $\xi^i_{jk} = 0$, authors transform the equation (10) to

$$\overline{L}^i_{jk} = L^i_{jk} + \overline{\underline{\omega}}^i_{jk} - \underline{\omega}^i_{jk}. \tag{11}$$

From this equation they have obtained a curvature tensor K^i_{jmn} and its image $\overline{K}^i_{jmn}$ satisfy a relation

$$\overline{K}^i_{jmn} = K^i_{jmn} + \overline{\hat{\Omega}}^i_{jmn} - \hat{\Omega}^i_{jmn}.$$

The corresponding generalization of Weyl projective tensor is $\hat{\mathcal{W}}^i_{jmn} = K^i_{jmn} - \hat{\Omega}^i_{jmn}$.

Introduction

Motivation

Is $\xi^i_{jk} = \overline{L}^i_{\underset{\vee}{jk}} - L^i_{\underset{\vee}{jk}}$ the only possible expression of the tensor ξ^i_{jk} from the equation (10) and WHY?

Is it necessary in a process of generalizing of Weyl projective tensor to be forgotten the corresponding invariant (9) and WHY?

Is the corresponding generalized Thomas projective parameter really useless magnitude in a process of generalization of Weyl projective tensor and WHY?

There are so many WHYs without their real BECAUSEs. Only of BECAUSEs is *BECAUSE IT IS THE ONLY USED METHOD*. But, is it the only possible scientifically acceptable method and WHY?

Introduction

Purposes

We will try to involve a generalized Thomas projective parameter in research for general invariants of geometric mappings bellow. We also will not forget a transformation rule of a curvature tensor R^i_{jmn} of the associated space $\mathbb{A}_N$. As to a necessity of equitorsion mappings of non-symmetric affine connection spaces are only mappings which invariants save non-symmetry we will try to go back to definitions of mappings to find a way to try to exclude the guess of $\overline{L}^i_{\underset{\vee}{jm}} = L^i_{\underset{\vee}{jm}}$.

How it works is presented by results obtained in [20]. These results will help us to generalize a Bochner tensor (see [1]). Furthermore, we will find a correlation between generalizations of Weyl projective tensor with regard to some special almost geodesic mappings.

Introduction

A method (Vesić, [19], preprint)

Let $f : \mathbb{GA}_N \to \mathbb{G}\overline{\mathbb{A}}_N$ be a mapping between non-symmetric affine connection spaces $\mathbb{GA}_N$ and $\mathbb{G}\overline{\mathbb{A}}_N$. Symmetric parts of affine connection coefficients of these spaces satisfy the equation

$$\overline{L}^i_{\underline{jk}} = L^i_{\underline{jk}} + P^i_{\underline{jk}},$$

for a deformation tensor $P^i_{\underline{jk}}$.

In the case of $P^i_{\underline{jk}} = \underline{\omega}^i_{jk} - \underline{\overline{\omega}}^i_{jk}, \underline{\omega}^i_{jk} \not\equiv -L^i_{\underline{jk}}$, we obtain a generalized Thomas projective parameter as a magnitude

$$\mathcal{T}^i_{jk} = L^i_{\underline{jk}} + \underline{\omega}^i_{jk},$$

which is invariant under the mapping f.

A method (Vesić, [19], preprint)

Using the fact it is satisfied an equality

$$\mathcal{T}^i_{jm,n} - \mathcal{T}^i_{jn,m} + \mathcal{T}^\alpha_{jm}\mathcal{T}^i_{\alpha n} - \mathcal{T}^\alpha_{jn}\mathcal{T}^i_{\alpha m} = \overline{\mathcal{T}}^i_{jm,n} - \overline{\mathcal{T}}^i_{jn,m} + \overline{\mathcal{T}}^\alpha_{jm}\overline{\mathcal{T}}^i_{\alpha n} - \overline{\mathcal{T}}^\alpha_{jn}\overline{\mathcal{T}}^i_{\alpha m},$$

for the above defined invariant $\mathcal{T}^i_{jk}$ we directly obtain curvature tensors $\overline{R}^i_{jmn}$ and R^i_{jmn} of the associated spaces $\overline{\mathbb{A}}_N$ and $\mathbb{A}_N$ satisfy a relation

$$\overline{R}^i_{jmn} = R^i_{jmn} + \mathcal{D}^i_{jmn} - \overline{\mathcal{D}}^i_{jmn}, \qquad (12)$$

for $\mathcal{D}^i_{jmn} = \underline{\omega}^i_{jm,n} - \underline{\omega}^i_{jn,m} + L^\alpha_{\underline{jm}}\underline{\omega}^i_{\alpha n} + L^i_{\underline{\alpha n}}\underline{\omega}_{\underline{jm}} - L^\alpha_{\underline{jn}}\underline{\omega}^i_{\alpha m} - L^i_{\underline{\alpha m}}\underline{\omega}^\alpha_{jn}$ and the corresponding $\overline{\mathcal{D}}^i_{jmn}$.

A method (Vesić, [19], preprint)

In this way, it is obtained a magnitude

$$\widetilde{\mathcal{W}}^i_{jmn} = R^i_{jmn} + \mathcal{D}^i_{jmn},$$

is an invariant of the mapping $f : \mathbb{GA}_N \to \mathbb{G}\overline{\mathbb{A}}_N$ obtained from the change of the curvature tensor R^i_{jmn}. Furthermore, if $\overline{L}^i_{\underset{\vee}{jm}\bar{;}n} - L^i_{\underset{\vee}{jm};n} = \overline{\zeta}^i_{jmn} - \zeta^i_{jmn}$ for $\zeta^i_{jmn} \not\equiv L^i_{\underset{\vee}{jm};n}$ and $\overline{L}^\alpha_{\underset{\vee}{jm}}\overline{L}^i_{\underset{\vee}{\alpha n}} - L^\alpha_{\underset{\vee}{jm}}L^i_{\underset{\vee}{\alpha n}} = \hat{\overline{\zeta}}^i_{jmn} - \hat{\zeta}^i_{jmn}$ for $\hat{\zeta}^i_{jmn} \not\equiv L^\alpha_{\underset{\vee}{jm}}L^i_{\underset{\vee}{\alpha n}}$, we obtain it holds an equation

$$\overline{K}^i_{jmn} = K^i_{jmn} + \hat{\mathcal{D}}^i_{jmn} - \hat{\overline{\mathcal{D}}}^i_{jmn},$$

for $\hat{\mathcal{D}}^i_{jmn} = \mathcal{D}^i_{jmn} - u\zeta^i_{jmn} - u'\zeta^i_{jnm} - v\hat{\zeta}^i_{jmn} - v'\hat{\zeta}^i_{jnm} - w\hat{\zeta}^i_{mnj}$.

A method (Vesić, [19], preprint)

It is proved a magnitude

$$\hat{\widetilde{\mathcal{W}}}{}^i_{jmn} = K^i_{jmn} + \hat{\mathcal{D}}^i_{jmn},$$

for the above defined magnitude $\hat{\mathcal{D}}^i_{jmn}$ is an invariant of the mapping $f : \mathbb{GA}_N \to \mathbb{G}\overline{\mathbb{A}}_N$.

Theorem (Vesić, [19], preprint)

Let $f : \mathbb{GA}_N \to \mathbb{G}\overline{\mathbb{A}}_N$ *be a mapping between non-symmetric affine connection spaces* $\mathbb{GA}_N$.

- *The above obtained invariants* $\widetilde{\mathcal{W}}^i_{jmn} = R^i_{jmn} + \mathcal{D}^i_{jmn}$ *and* $\hat{\widetilde{\mathcal{W}}}^i_{jmn} = K^i_{jmn} + \hat{\mathcal{D}}^i_{jmn}$ *satisfy an equation*

$$\hat{\widetilde{\mathcal{W}}}^i_{jmn} = \widetilde{\mathcal{W}}^i_{jmn} + uL^i_{\underset{\vee}{jm};n} + u'L^i_{\underset{\vee}{jn};m} + vL^\alpha_{\underset{\vee}{jm}}L^i_{\underset{\vee}{\alpha n}} + v'L^\alpha_{\underset{\vee}{jn}}L^i_{\underset{\vee}{\alpha m}} + wL^\alpha_{\underset{\vee}{mn}}L^i_{\underset{\vee}{\alpha j}}$$
$$- u\zeta^i_{jmn} - u'\zeta^i_{jnm} - v\hat{\zeta}^i_{jmn} - v'\hat{\zeta}^i_{jnm} - w\hat{\zeta}^i_{mnj}.$$

- *The invariant* $\hat{\widetilde{\mathcal{W}}}^i_{jmn}$ *and an invariant* $\hat{\hat{\widetilde{\mathcal{W}}}}^i_{jmn} = K^i_{jmn} + \hat{\hat{\mathcal{D}}}^i_{jmn}$ *of the mapping* f *satisfy an equation*

$$\hat{\hat{\widetilde{\mathcal{W}}}}^i_{jmn} = \hat{\widetilde{\mathcal{W}}}^i_{jmn} + \hat{\hat{\mathcal{D}}}^i_{jmn} - \hat{\mathcal{D}}^i_{jmn}.$$

Corollary (Vesić, [19], preprint)

Let $f : \mathbb{GA}_N \to \mathbb{G}\overline{\mathbb{A}}_N$ be an equitorsion mapping.

- *Curvature tensors K^i_{jmn} and $\overline{K}^i_{jmn}$ satisfy a relation $\overline{K}^i_{jmn} = K^i_{jmn} + \hat{\tilde{\mathcal{D}}}^i_{jmn} - \hat{\tilde{\tilde{\mathcal{D}}}}^i_{jmn}$, for*

$$\hat{\tilde{\mathcal{D}}}^i_{jmn} = \mathcal{D}^i_{jmn} - u\big(\underline{\omega}^i_{\alpha n} L^\alpha_{\underset{\vee}{jm}} - \underline{\omega}^\alpha_{jn} L^i_{\underset{\vee}{\alpha m}} - \underline{\omega}^\alpha_{mn} L^i_{\underset{\vee}{j\alpha}}\big) - u'\big(\underline{\omega}^i_{\alpha m} L^\alpha_{\underset{\vee}{jm}} - \underline{\omega}^\alpha_{jm} L^i_{\underset{\vee}{\alpha n}} - \underline{\omega}^\alpha_{nm} L^i_{\underset{\vee}{j\alpha}}\big),$$

for the above mentioned magnitude $\underline{\omega}^i_{jk}$ so a magnitude

$$\hat{\tilde{\tilde{\mathcal{W}}}}^i_{jmn} = K^i_{jmn} + \hat{\tilde{\mathcal{D}}}^i_{jmn}$$

is an invariant of this mapping.

- *The invariants $\hat{\tilde{\tilde{\mathcal{W}}}}^i_{jmn}$ and $\widetilde{\mathcal{W}}^i_{jmn}$ of the mapping f satisfy an equation*

$$\hat{\tilde{\tilde{\mathcal{W}}}}^i_{jmn} = \widetilde{\mathcal{W}}^i_{jmn} + uL^i_{\underset{\vee}{jm},n} + u'L^i_{\underset{\vee}{jn},m} + vL^\alpha_{\underset{\vee}{jm}} L^i_{\underset{\vee}{\alpha n}} + v'L^\alpha_{\underset{\vee}{jn}} L^i_{\underset{\vee}{\alpha m}} + wL^\alpha_{\underset{\vee}{mn}} L^i_{\underset{\vee}{\alpha j}}.$$

Generalized Bochner Tensor

Nenad O. Vesić, Milan Lj. Zlatanović

Generalized Bochner tensor

A RECALL: Conformal mappings of a space $\mathbb{R}_N$

A mapping $f : \mathbb{R}_N \to \overline{\mathbb{R}}_N$ defined as $g_{ij} \xrightarrow{f} \overline{g}_{ij} = e^{2\psi} g_{ij}$, for a metric tensor g symmetric by indices i and j and a scalar function ψ is said to be **the conformal mapping of the space** $\mathbb{R}_N$.
A basic equation of this mapping is

$$\overline{\Gamma}_{i.jk} = \Gamma_{i.jk} + g_{ij}\psi_k + g_{ik}\psi_j - g_{jk}\psi_i,$$

for $\psi_i = \partial\psi/\partial x^i$ and a Christofell symbol $\Gamma_{i.jk}$ of the space $\mathbb{R}_N$.
Weyl conformal curvature tensor (see Bochner, 1949, [1])

$$W_{ijmn} = R_{ijmn} - \frac{1}{N-2}\big(g_{jm}R_{in} - g_{jn}R_{im} + g_{in}R_{jm} - g_{im}R_{jn}\big) - \frac{R}{(N-1)(N-2)}\big(g_{im}g_{jn} - g_{in}g_{jm}\big)$$

is an invariant of this mapping obtained from the change of the curvature tensor R_{ijmn} under this mapping.

Generalized Bochner tensor

A RECALL: Conformal mappings of a space $\mathbb{GR}_N$

A mapping $f : \mathbb{GR}_N \to \mathbb{G}\overline{\mathbb{R}}_N$ defined as $g_{ij} \xrightarrow{f} \overline{g}_{ij} = e^{2\psi} g_{ij}$, for a metric tensor g non-symmetric by indices i and j and a scalar function ψ is said to be **the conformal mapping of the space** $\mathbb{GR}_N$.
A basic equation of this mapping is

$$\overline{\Gamma}_{i.jk} = \Gamma_{i.jk} + g_{ij}\psi_k + g_{ik}\psi_j - g_{jk}\psi_i + \xi_{i.jk}, \qquad (13)$$

i.e.

$$\overline{\Gamma}^i_{jk} = \Gamma^i_{jk} + \delta^i_j\psi_k + \delta^i_k\psi_j - g_{jk}\psi_\alpha g^{\underline{i\alpha}} + \xi^i_{jk}, \qquad (14)$$

for $\psi_i = \partial\psi/\partial x^i$, Christofell symbols $\Gamma_{i.jk}$ and Γ^i_{jk} of the space $\mathbb{R}_N$, $\xi_{i.jk} = \overline{\Gamma}_{i.\underset{\vee}{jk}} - \Gamma_{i.\underset{\vee}{jk}}$ and $\xi^i_{jk} = \overline{\Gamma}^i_{\underset{\vee}{jk}} - \Gamma^i_{\underset{\vee}{jk}}$.

A RECALL: Conformal mappings of a space $\mathbb{GR}_N$

We see into the basic equations (13, 14) are some forms of the above mentioned anti-symmetric tensor ξ. The recall from the introduction betokens there should be presumed $\xi = 0$, but WHY?!
Weyl tensor $W^i_{jmn} = g^{\underline{i\alpha}} W_{\alpha jmn}$ is generalized in (Stanković et all., 2009, [12]) as invariants of equitorsion conformal mappings. But again, is it the most possible general result which may be obtained in this part of research?!

Generalized Bochner tensor

(Resulsts from (Vesić,[20], submitted))

It holds $\Gamma_{i.\underset{\vee}{jk}} = \frac{1}{2}\left(g_{\underset{\vee}{ji},k} - g_{\underset{\vee}{jk},i} + g_{\underset{\vee}{ik},j}\right)$ and $\Gamma^i_{\underset{\vee}{jk}} = \frac{1}{2} g^{\underline{i\alpha}}\left(g_{\underset{\vee}{j\alpha},k} - g_{\underset{\vee}{jk},\alpha} + g_{\underset{\vee}{\alpha k},j}\right)$.

It is obtained (Ivanov, Zlatanović, 2016, [4]) it holds $g_{\underset{\vee}{ij}} = A^\alpha_i g_{\underline{\alpha j}}$ for a tensor A^i_j.

It directly holds the tensor A^i_j has a form $A^i_j = g^{\underline{i\alpha}} g_{\underset{\vee}{j\alpha}}$.

Based on the definition of the conformal mappings we conclude it holds equalities

$$\boxed{\overline{A}^i_j =} \overline{g}^{\underline{i\alpha}} \overline{g}_{\underset{\vee}{j\alpha}} = e^{-2\psi} g^{\underline{i\alpha}} e^{2\psi} g_{\underset{\vee}{j\alpha}} = g^{\underline{i\alpha}} g_{\underset{\vee}{j\alpha}} \boxed{= A^i_j},$$

which proves the magnitude A^i_j is an invariant of the conformal mapping f.

Generalized Bochner tensor

Remark

The magnitude A^i_j is boxed onto the previous slide because this magnitude is not necessary to be obtained following results. This magnitude just gave a more intensive motivation for this research but its expression $g^{\underline{i\alpha}}g_{j\underset{\vee}{\alpha}}$ is very enough for our further study. However, this magnitude will be used because of a simplification of formulas.

Generalized Bochner tensor

It is satisfied $\Gamma^i_{\underset{\vee}{jk}} = \frac{1}{2}\left(A^i_{j;k} - A^i_{k;j} - g^{\underline{i\alpha}} g_{\underset{\vee}{jk};\alpha}\right)$.
We may easily obtain it holds an equation

$$\overline{A}^i_{j\bar{;}k} = A^i_{j;k} - \overline{\Gamma}^i_{\underline{\alpha k}} \overline{g}^{\underline{\alpha\beta}} \overline{g}_{\underset{\vee}{j\beta}} + \overline{\Gamma}^\alpha_{\underline{jk}} \overline{g}^{\underline{i\beta}} \overline{g}_{\underset{\vee}{\alpha\beta}} + \Gamma^i_{\underline{\alpha k}} g^{\underline{\alpha\beta}} g_{\underset{\vee}{j\beta}} - \Gamma^\alpha_{\underline{jk}} g^{\underline{i\beta}} g_{\underset{\vee}{\alpha\beta}}.$$

It also holds an equation

$$\overline{g}^{\underline{i\alpha}} \overline{g}_{\underset{\vee}{jk}\bar{;}\alpha} = g^{\underline{i\alpha}} g_{\underset{\vee}{jk};\alpha} - \overline{\Gamma}^\beta_{\underline{j\alpha}} \overline{g}^{\underline{i\alpha}} \overline{g}_{\underset{\vee}{\beta k}} - \overline{\Gamma}^\beta_{\underline{k\alpha}} \overline{g}^{\underline{i\alpha}} \overline{g}_{\underset{\vee}{j\beta}} + \Gamma^\beta_{\underline{j\alpha}} g^{\underline{i\alpha}} g_{\underset{\vee}{\beta k}} + \Gamma^\beta_{\underline{k\alpha}} g^{\underline{i\alpha}} g_{\underset{\vee}{j\beta}},$$

which proves it holds a relation

$$\begin{aligned}\overline{\Gamma}^i_{\underset{\vee}{jk}} = \Gamma^i_{\underset{\vee}{jk}} &- \frac{1}{2}\left(\overline{\Gamma}^i_{\underline{\alpha k}} \overline{g}^{\underline{\alpha\beta}} \overline{g}_{\underset{\vee}{j\beta}} - \overline{\Gamma}^\alpha_{\underline{jk}} \overline{g}^{\underline{i\beta}} \overline{g}_{\underset{\vee}{\alpha\beta}} + \overline{\Gamma}^\beta_{\underline{j\alpha}} \overline{g}^{\underline{i\alpha}} \overline{g}_{\underset{\vee}{\beta k}} + \overline{\Gamma}^\beta_{\underline{k\alpha}} \overline{g}^{\underline{i\alpha}} \overline{g}_{\underset{\vee}{j\beta}}\right) \\ &+ \frac{1}{2}\left(\Gamma^i_{\underline{\alpha k}} g^{\underline{\alpha\beta}} g_{\underset{\vee}{j\beta}} - \Gamma^\alpha_{\underline{jk}} g^{\underline{i\beta}} g_{\underset{\vee}{\alpha\beta}} + \Gamma^\beta_{\underline{j\alpha}} g^{\underline{i\alpha}} g_{\underset{\vee}{\beta k}} + \Gamma^\beta_{\underline{k\alpha}} g^{\underline{i\alpha}} g_{\underset{\vee}{j\beta}}\right).\end{aligned} \tag{15}$$

Generalized Bochner tensor

It is proved a magnitude

$$\underset{(1)}{\mathcal{C}}{}^{i}_{jk} = \Gamma^{i}_{\underset{\vee}{jk}} + \frac{1}{2}\left(\Gamma^{i}_{\underline{\alpha k}} g^{\underline{\alpha\beta}} g_{\underset{\vee}{j\beta}} - \Gamma^{\alpha}_{\underline{jk}} g^{\underline{i\beta}} g_{\underset{\vee}{\alpha\beta}} + \Gamma^{\beta}_{\underline{j\alpha}} g^{\underline{i\alpha}} g_{\underset{\vee}{\beta k}} + \Gamma^{\beta}_{\underline{k\alpha}} g^{\underline{i\alpha}} g_{\underset{\vee}{j\beta}}\right)$$

is an invariant of the conformal mapping f in this way.

Based on this invariance it holds an equality

$$\underset{(1)}{\overline{\mathcal{C}}}{}^{\alpha}_{jm}\underset{(1)}{\overline{\mathcal{C}}}{}^{i}_{\alpha n} = \underset{(1)}{\mathcal{C}}{}^{\alpha}_{jm}\underset{(1)}{\mathcal{C}}{}^{i}_{\alpha n}$$

which proves it holds an expression

$$\overline{\Gamma}^{\alpha}_{\underset{\vee}{jm}}\overline{\Gamma}^{i}_{\underset{\vee}{\alpha n}} = \Gamma^{\alpha}_{\underset{\vee}{jm}}\Gamma^{i}_{\underset{\vee}{\alpha n}} - \overline{\mathcal{X}}^{i}_{jmn} + \mathcal{X}^{i}_{jmn}, \tag{16}$$

for

$$\begin{aligned}
\mathcal{X}^{i}_{jmn} = {} & \frac{1}{2}\Gamma^{\alpha}_{\underset{\vee}{jm}}\big(\Gamma^{i}_{\underline{\beta n}}g^{\underline{\beta\gamma}}g_{\underset{\vee}{\alpha\gamma}} - \Gamma^{\beta}_{\underline{\alpha n}}g^{\underline{i\gamma}}g_{\underset{\vee}{\beta\gamma}} + \Gamma^{\gamma}_{\underline{\alpha\beta}}g^{\underline{i\beta}}g_{\underset{\vee}{\gamma n}} + \Gamma^{\gamma}_{\underline{n\beta}}g^{\underline{i\beta}}g_{\underset{\vee}{\alpha\gamma}}\big)\\
& + \frac{1}{2}\Gamma^{i}_{\underset{\vee}{\alpha n}}\big(\Gamma^{\alpha}_{\underline{\beta m}}g^{\underline{\beta\gamma}}g_{\underset{\vee}{j\gamma}} - \Gamma^{\beta}_{\underline{jm}}g^{\underline{\alpha\gamma}}g_{\underset{\vee}{\beta\gamma}} + \Gamma^{\gamma}_{\underline{j\beta}}g^{\underline{\alpha\beta}}g_{\underset{\vee}{\gamma m}} + \Gamma^{\gamma}_{\underline{m\beta}}g^{\underline{\alpha\beta}}g_{\underset{\vee}{j\gamma}}\big)\\
& + \frac{1}{4}\big(\Gamma^{\alpha}_{\underline{\beta m}}g^{\underline{\beta\gamma}}g_{\underset{\vee}{j\gamma}} - \Gamma^{\beta}_{\underline{jm}}g^{\underline{\alpha\gamma}}g_{\underset{\vee}{\beta\gamma}} + \Gamma^{\gamma}_{\underline{j\beta}}g^{\underline{\alpha\beta}}g_{\underset{\vee}{\gamma m}} + \Gamma^{\gamma}_{\underline{m\beta}}g^{\underline{\alpha\beta}}g_{\underset{\vee}{j\gamma}}\big)\\
& \cdot\big(\Gamma^{i}_{\underline{\delta n}}g^{\underline{\delta\epsilon}}g_{\underset{\vee}{\alpha\epsilon}} - \Gamma^{\delta}_{\underline{\alpha n}}g^{\underline{i\epsilon}}g_{\underset{\vee}{\delta\epsilon}} + \Gamma^{\epsilon}_{\underline{\alpha\delta}}g^{\underline{i\delta}}g_{\underset{\vee}{\epsilon n}} + \Gamma^{\epsilon}_{\underline{n\delta}}g^{\underline{i\delta}}g_{\underset{\vee}{\alpha\epsilon}}\big).
\end{aligned}$$

Generalized Bochner tensor

Using the invariance of the magnitude $\underset{(1)}{\mathcal{C}}{}^i_{jk}$ we obtain it holds the following expression

$$\underset{(1)}{\overline{\mathcal{C}}}{}^i_{jm\bar{;}n} = \underset{(1)}{\mathcal{C}}{}^i_{jm;n} + \overline{\Gamma}^i_{\underline{\alpha n}}\underset{(1)}{\overline{\mathcal{C}}}{}^\alpha_{jm} - \overline{\Gamma}^\alpha_{\underline{jn}}\underset{(1)}{\overline{\mathcal{C}}}{}^i_{\alpha m} - \overline{\Gamma}^\alpha_{\underline{mn}}\underset{(1)}{\overline{\mathcal{C}}}{}^i_{j\alpha} - \Gamma^i_{\underline{\alpha n}}\underset{(1)}{\mathcal{C}}{}^\alpha_{jm} + \Gamma^\alpha_{\underline{jn}}\underset{(1)}{\mathcal{C}}{}^i_{\alpha m} + \Gamma^\alpha_{\underline{mn}}\underset{(1)}{\mathcal{C}}{}^i_{j\alpha},$$

or equivalently

$$\overline{\Gamma}^i_{\underset{\vee}{jm}\bar{;}n} = \Gamma^i_{\underset{\vee}{jm};n} - \widetilde{\widetilde{\mathcal{X}}}{}^i_{jmn} + \widetilde{\mathcal{X}}^i_{jmn}, \tag{17}$$

for

$$\begin{aligned}\widetilde{\mathcal{X}}^i_{jmn} = &-\Gamma^i_{\underline{\alpha n}}\Big(\Gamma^\alpha_{\underset{\vee}{jm}} + \frac{1}{2}\big(\Gamma^\alpha_{\underline{\beta m}}g^{\underline{\beta\gamma}}g_{\underset{\vee}{j\gamma}} - \Gamma^\beta_{\underline{jm}}g^{\underline{\alpha\gamma}}g_{\underset{\vee}{\beta\gamma}} + \Gamma^\gamma_{\underline{j\beta}}g^{\underline{\alpha\beta}}g_{\underset{\vee}{\gamma m}} + \Gamma^\gamma_{\underline{m\beta}}g^{\underline{\alpha\beta}}g_{\underset{\vee}{j\gamma}}\big)\Big)\\
&+\Gamma^\alpha_{\underline{jn}}\Big(\Gamma^i_{\underset{\vee}{\alpha m}} + \frac{1}{2}\big(\Gamma^i_{\underline{\beta m}}g^{\underline{\beta\gamma}}g_{\underset{\vee}{\alpha\gamma}} - \Gamma^\beta_{\underline{\alpha m}}g^{\underline{i\gamma}}g_{\underset{\vee}{\beta\gamma}} + \Gamma^\gamma_{\underline{\alpha\beta}}g^{\underline{i\beta}}g_{\underset{\vee}{\gamma m}} + \Gamma^\gamma_{\underline{m\beta}}g^{\underline{i\beta}}g_{\underset{\vee}{\alpha\gamma}}\big)\Big)\\
&+\Gamma^\alpha_{\underline{mn}}\Big(\Gamma^i_{\underset{\vee}{j\alpha}} + \frac{1}{2}\big(\Gamma^i_{\underline{\beta\alpha}}g^{\underline{\beta\gamma}}g_{\underset{\vee}{j\gamma}} - \Gamma^\beta_{\underline{j\alpha}}g^{\underline{i\gamma}}g_{\underset{\vee}{\beta\gamma}} + \Gamma^\gamma_{\underline{j\beta}}g^{\underline{i\beta}}g_{\underset{\vee}{\gamma\alpha}} + \Gamma^\gamma_{\underline{\alpha\beta}}g^{\underline{i\beta}}g_{\underset{\vee}{j\gamma}}\big)\Big).\end{aligned}$$

Generalized Bochner tensor

Theorem

Let $f : \mathbb{GR}_N \to \mathbb{G}\overline{\mathbb{R}}_N$ be a conformal mapping of a generalized Riemannian space $\mathbb{GR}_N$. A magnitude

$$\underset{\vee}{\mathcal{C}}{}^i_{jmn} = W^i_{jmn} + u\Gamma^i_{\underset{\vee}{jm};n} + u'\Gamma^i_{\underset{\vee}{jn};m} + v\Gamma^\alpha_{\underset{\vee}{jm}}\Gamma^i_{\underset{\vee}{\alpha n}} + v'\Gamma^\alpha_{\underset{\vee}{jn}}\Gamma^i_{\underset{\vee}{\alpha m}} + w\Gamma^\alpha_{\underset{\vee}{mn}}\Gamma^i_{\underset{\vee}{\alpha j}}$$
$$+ u\widetilde{\mathcal{X}}^i_{jmn} + u'\widetilde{\mathcal{X}}^i_{jnm} + v\mathcal{X}^i_{jmn} + v'\mathcal{X}^i_{jnm} + w\mathcal{X}^i_{mnj},$$

for Weyl conformal curvature tensor W^i_{jmn} is an invariant of the mapping f.

Corollary

A magnitude

$$\underset{\vee}{\mathcal{C}}_{ijmn} = W_{ijmn} + u\Gamma_{i.\underset{\vee}{jm};n} + u'\Gamma_{i.\underset{\vee}{jn};m} + v\Gamma^{\alpha}_{\underset{\vee}{jm}}\Gamma_{i.\underset{\vee}{\alpha n}} + v'\Gamma^{\alpha}_{\underset{\vee}{jn}}\Gamma_{i.\underset{\vee}{\alpha m}} + w\Gamma^{\alpha}_{\underset{\vee}{mn}}\Gamma_{i.\underset{\vee}{\alpha j}}$$
$$+ u\widetilde{\mathcal{X}}_{ijmn} + u'\widetilde{\mathcal{X}}_{ijnm} + v\mathcal{X}_{ijmn} + v'\mathcal{X}_{ijnm} + w\mathcal{X}_{imnj},$$

for Weyl conformal curvature tensor W_{ijmn}, $\widetilde{\mathcal{X}}_{ijmn} = g_{\underline{i\alpha}}\widetilde{\mathcal{X}}^{\alpha}_{jmn}$ *and* $\mathcal{X}_{ijmn} = g_{\underline{i\alpha}}\mathcal{X}^{\alpha}_{jmn}$ *is an invariant of the conformal mapping* f.

Remark

It is confirmed above as in [20] guesses $\xi^i_{jk} = 0$ and $\xi_{i.jk} = 0$ are redundant ones in the case of research for invariants of conformal mappings of a generalized Riemannian space $\mathbb{GR}_N$.

Generalized Bochner tensor

A non-symmetric affine connection space $\mathbb{GA}_N$ is a semi-symmetric one if its torsion tensor $T^i_{jk} = \frac{1}{2}(L^i_{jk} - L^i_{kj}) = L^i_{\underset{\vee}{jk}}$ satisfies the equation

$$T^i_{jk} = \delta^i_j p_k - \delta^i_k p_j,$$

for some covariant vector p. The space $\mathbb{GA}_N$ which torsion tensor T^i_{jk} may be expressed in the form

$$T^i_{jk} = \pi_k \varphi^i_j - \pi_j \varphi^i_k$$

for a covariant vector π and an (1,1)-tensor φ is said to be a quarter-symmetric space.

Generalized Bochner tensor

If $\mathbb{GR}_N$ is a semi-symmetric space it holds $p_i = 0$ which is contradiction so a semi-symmetric generalized Riemannian space does not exist.

Proposition

Let $f : \mathbb{GR}_N \to \mathbb{G}\overline{\mathbb{R}}_N$ be a conformal mapping between quarter-symmetric spaces $\mathbb{GR}_N$ and $\mathbb{G}\overline{\mathbb{R}}_N$ with torsions

$$T^i_{jk} = \pi_k \varphi^i_j - \pi_j \varphi^i_k \quad \textit{and} \quad \overline{T}^i_{jk} = \overline{\pi}_k \overline{\varphi}^i_j - \overline{\pi}_j \overline{\varphi}^i_k .$$

A magnitude

$$\varrho^i_{jk} = \pi_k \varphi^i_j - \pi_j \varphi^i_k$$

is an invariant of the mapping f.

Generalized Bochner tensor

Proposition

Let $f : \mathbb{GR}_N \to \mathbb{G}\overline{\mathbb{R}}_N$ *be a conformal mapping between quarter-symmetric spaces* $\mathbb{GR}_N$ *and* $\mathbb{G}\overline{\mathbb{R}}_N$. *This mapping is an equitorsion one.*

Generalized Bochner tensor

S. Bochner (see Bochner, 1949, [1]) had involved "a formal analogy" with the Weyl conformal tensor W_{ijmn} given above. This magnitude in a non-symmetric $2K$-dimensional affine connection space with a metric

$$ds^2 = 2g_{\alpha\beta^*}dz_\alpha d\overline{z}_\beta, \tag{18}$$

which satisfies Kaehler's assumption

$$\frac{\partial g_{\alpha\gamma^*}}{\partial z_\beta} = \frac{\partial g_{\beta\gamma^*}}{\partial z_\alpha} \tag{19}$$

for

$$h_{\alpha\beta^*} = g_{\alpha\beta^*} = -h_{\beta^*\alpha} \quad \text{and} \quad h_{\alpha\beta} = h_{\alpha^*\beta^*} = g_{\alpha\beta} = g_{\alpha^*\beta^*} = 0, \tag{20}$$

is

$$\begin{aligned}\kappa_{\alpha\beta^*\gamma\delta^*} = R_{\alpha\beta^*\gamma\delta^*} &- \frac{1}{K+2}\big(g_{\alpha\beta^*}R_{\gamma\delta^*} + g_{\alpha\delta^*}R_{\gamma\beta^*} + g_{\gamma\beta^*}R_{\alpha\delta^*} + g_{\gamma\delta^*}R_{\alpha\beta^*}\big)\\ &+ \frac{R}{2(K+2)(K+1)}\big(g_{\alpha\beta^*}g_{\gamma\delta^*} + g_{\alpha\delta^*}g_{\beta^*\gamma}\big),\end{aligned} \tag{21}$$

named *Bochner tensor*.

Remark

S. Bochner defined the tensor $\kappa_{\alpha\beta^*\gamma\delta^*}$ *as a formal analogy to Weyl covariant conformal curvature tensor. For this reason we need to find the corresponding analogies and to further generalize above defined Bochner tensor as sums of formal analogies.*

Generalized Bochner tensor

A formal analogy to the anti-symmetric part $g_{\underset{\vee}{ij}}$ is $g_{\alpha\beta^*}$.

It holds $\Gamma_{\alpha.\beta^*\gamma} = \frac{1}{2}\big(g_{\beta^*\alpha,\gamma} - g_{\beta^*\gamma,\alpha} + g_{\alpha\gamma,\beta^*}\big) = 0$.
It holds $\Gamma_{\alpha.\beta^*\gamma^*} = \frac{1}{2}\big(g_{\alpha\gamma^*,\beta^*} - g_{\alpha\beta^*,\gamma^*}\big) \neq 0$.
A formal analogy to $\Gamma^i_{\underset{\vee}{jk}}$ is $g^{k\alpha}\Gamma_{k.\beta^*\gamma^*}$.

Generalized Bochner tensor

Theorem

Let $\mathbb{GR}_N$ be an $N = 2K$-dimensional affine connection space endowed with a metric tensor g which satisfies the equation (18) and the Kaehler's assumption (19). Generalized Bochner tensors of this space obtained from curvature tensors (2-6) as analogies to generalized Weyl conformal tensors are:

$$\underset{1}{\kappa}_{\alpha\beta^*\gamma\delta^*} = \kappa_{\alpha\beta^*\gamma\delta^*} - \Gamma_{\alpha.\beta^*\delta^*;\gamma}, \tag{22}$$

$$\underset{2}{\kappa}_{\alpha\beta^*\gamma\delta^*} = \kappa_{\alpha\beta^*\gamma\delta^*} + \Gamma_{\alpha.\beta^*\delta^*;\gamma}, \tag{23}$$

$$\underset{3}{\kappa}_{\alpha\beta^*\gamma\delta^*} = \kappa_{\alpha\beta^*\gamma\delta^*} + \Gamma_{\alpha.\beta^*\delta^*;\gamma}, \tag{24}$$

$$\underset{4}{\kappa}_{\alpha\beta^*\gamma\delta^*} = \kappa_{\alpha\beta^*\gamma\delta^*} + \Gamma_{\alpha.\beta^*\delta^*;\gamma}, \tag{25}$$

$$\underset{5}{\kappa}_{\alpha\beta^*\gamma\delta^*} = \kappa_{\alpha\beta^*\gamma\beta^*}, \tag{26}$$

Some relations between generalizations of Weyl projective tensor

Nenad O. Vesić, Mića S. Stanković

Definition

A curve $\ell = \ell(t)$ from a non-symmetric affine connection space $\mathbb{G}\overline{\mathbb{A}}_N$ which tangential vector $\lambda^i = d\ell/dx^i \neq 0$ satisfies a system of differential equations

$$\underset{\theta}{\overline{\lambda}}{}^h{}_{(2)} = \underset{\theta}{\overline{a}}(t)\lambda^h + \underset{\theta}{\overline{b}}(t)\underset{\theta}{\overline{\lambda}}{}^h{}_{(1)}, \quad \underset{\theta}{\overline{\lambda}}{}^h{}_{(1)} = \lambda^h{}_{\underset{\theta}{\|}\alpha}\lambda^\alpha, \quad \underset{\theta}{\overline{\lambda}}{}^h{}_{(2)} = \underset{\theta}{\overline{\lambda}}{}^h{}_{(1)\underset{\theta}{\|}\alpha}\lambda^\alpha, \tag{27}$$

for functions $\underset{\theta}{a}$ and $\underset{\theta}{b}$ of a parameter t, $\theta = 1, 2$ is said to be **the almost geodesic line** of the θ-th kind.
A mapping $f : \mathbb{G}\mathbb{A}_N \to \mathbb{G}\overline{\mathbb{A}}_N$ which any geodesic line of the space $\mathbb{G}\mathbb{A}_N$ turns into a θ-th kind, $\theta = 1, 2$, almost geodesic line of the space $\mathbb{G}\overline{\mathbb{A}}_N$ is **the almost geodesic line** of the θ-th kind.

Some relations between generalizations of Weyl projective tensor

Let $f : \mathbb{GA}_N \to \mathbb{G}\overline{\mathbb{A}}_N$ be a third type almost geodesic mapping which satisfies the property of reciprocity. A magnitude (M. S. Stanković [11])

$$\begin{aligned} \underset{3}{W}{}^i_{jmn} &= R^i_{jmn} - q_\alpha R^\alpha_{jmn}\varphi^i - \omega_{jm}(\delta^i_n - eq_n\varphi^i) + \omega_{jn}(\delta^i_m - eq_m\varphi^i) \\ &+\omega_{[mn]}(\delta^i_j - eq_j\varphi^i) - s_{jm}(eL_n\varphi^i - L^i_{\underset{\vee}{\alpha n}}\varphi^\alpha) + s_{jn}(eL_m\varphi^i - L^i_{\underset{\vee}{\alpha m}}\varphi^\alpha), \end{aligned} \tag{28}$$

for q_i such that $q_\alpha\varphi^\alpha = e, e = \pm 1$, $L_j = L^\beta_{\underset{\vee}{\alpha j}}\varphi^\alpha q_\beta$ and $\sigma_{ij} = \bar{s}_{ij} - s_{ij}$ for

$$s_{ij} = -eq_{i\underset{1}{|}j} - \frac{1}{N}q_j\Big[L^\alpha_{\alpha i} + \frac{e}{N-1}q_i\big(\varphi^\beta L^\alpha_{\alpha\beta} + e\varphi^\alpha\varphi^\beta q_{\alpha\underset{1}{|}\beta}\big) + e\varphi^\alpha q_{\alpha\underset{1}{|}i}\Big]$$

Some relations between generalizations of Weyl projective tensor

and $\omega_{ij} = p_{ij} + q_{ij}$ for

$$
\begin{aligned}
p_{ij} &= R^{\alpha}_{ij\alpha} - eq_{\alpha}\varphi^{\beta}R^{\alpha}_{ij\beta} \\
&\quad + \frac{1}{N}\Big\{ - R^{\alpha}_{\alpha ji} + eq_{\alpha}\varphi^{\beta}R^{\alpha}_{\beta ji} - e\varphi^{\beta}R^{\alpha}_{i\beta\alpha}q_j \\
&\quad + eq_i\big(\varphi^{\beta}R^{\alpha}_{\beta j\alpha} - \varphi^{\beta}R^{\alpha}_{j\beta\alpha} - eq_{\alpha}\varphi^{\beta}\varphi^{\gamma}R^{\alpha}_{\gamma j\beta} - \frac{e}{N-1}q_j q^{\beta}\varphi^{\delta}R^{\alpha}_{\beta\delta\alpha}\big)\Big\}, \\
q_{ij} &= s_{\alpha i}\varphi^{\alpha}L_j - \frac{e}{N}\big(s_{\alpha j}\varphi^{\alpha}L_i - s_{\alpha i}\varphi^{\alpha}L_j - s_{\alpha\beta}\varphi^{\alpha}\varphi^{\beta}\big),
\end{aligned}
$$

is an invariant of this mapping.

Some relations between generalizations of Weyl projective tensor

Remark

We do not have an invariant in a part of an anti-symmetric part $L^i_{\underset{\vee}{jk}}$ *of an affine connection coefficient* L^i_{jk} *of the space* $\mathbb{GA}_N$ *such as it was the case when we generalized Weyl conformal projective tensor. For this reason we will restrict our research onto invariants of equitorsion almost geodesic mappings of the first kind.*

Remark

Invariants of almost geodesic mappings mentioned into the previous remark are found in [21] but we are going to obtain a correlation between the invariant (28) obtained from a change of curvature tensor R^i_{jmn} *and an invariant of this mapping obtained from a change of a general curvature tensor* K^i_{jmn}.

Some relations between generalizations of Weyl projective tensor

Theorem

Let $f : \mathbb{GA}_N \to \mathbb{G}\overline{\mathbb{A}}_N$ be a third type almost geodesic mapping of the first kind which satisfies the property of reciprocity. Magnitudes:

$$\widehat{\widetilde{\mathcal{W}}}{}^i_{\underset{3}{}jmn} = \underset{3}{W}{}^i_{jmn} + uL^i_{\underset{\vee}{jm},n} + u'L^i_{\underset{\vee}{jn},m} + vL^\alpha_{\underset{\vee}{jm}}L^i_{\underset{\vee}{\alpha n}} + v'L^\alpha_{\underset{\vee}{jn}}L^i_{\underset{\vee}{\alpha m}} + wL^\alpha_{\underset{\vee}{mn}}L^i_{\underset{\vee}{\alpha j}},$$

are invariants of the mapping f obtained from the corresponding curvature tensor

$$K^i_{jmn} = R^i_{jmn} + uL^i_{\underset{\vee}{jm};n} + u'L^i_{\underset{\vee}{jn};m} + vL^\alpha_{\underset{\vee}{jm}}L^i_{\underset{\vee}{\alpha n}} + v'L^\alpha_{\underset{\vee}{jn}}L^i_{\underset{\vee}{\alpha m}} + wL^\alpha_{\underset{\vee}{mn}}L^i_{\underset{\vee}{\alpha j}}.$$

□

Some relations between generalizations of Weyl projective tensor

The corresponding invariants obtained from the linearly independent curvature tensors $\underset{1}{K}{}^i_{jmn}, \ldots, \underset{5}{K}{}^i_{jmn}$ from the equations (2-6) are:

$$\underset{\substack{3\\1}}{\hat{\widetilde{\mathcal{W}}}}{}^i_{jmn} = \underset{3}{W}{}^i_{jmn} + L^i_{\underset{\vee}{jm},n} - L^i_{\underset{\vee}{jn},m} + L^\alpha_{\underset{\vee}{jm}} L^i_{\underset{\vee}{\alpha n}} - L^\alpha_{\underset{\vee}{jn}} L^i_{\underset{\vee}{\alpha m}}, \tag{29}$$

$$\underset{\substack{3\\2}}{\hat{\widetilde{\mathcal{W}}}}{}^i_{jmn} = \underset{3}{W}{}^i_{jmn} - L^i_{\underset{\vee}{jm},n} + L^i_{\underset{\vee}{jn},m} + L^\alpha_{\underset{\vee}{jm}} L^i_{\underset{\vee}{\alpha n}} - L^\alpha_{\underset{\vee}{jn}} L^i_{\underset{\vee}{\alpha m}}, \tag{30}$$

$$\underset{\substack{3\\3}}{\hat{\widetilde{\mathcal{W}}}}{}^i_{jmn} = \underset{3}{W}{}^i_{jmn} + L^i_{\underset{\vee}{jm},n} + L^i_{\underset{\vee}{jn},m} - L^\alpha_{\underset{\vee}{jm}} L^i_{\underset{\vee}{\alpha n}} + L^\alpha_{\underset{\vee}{jn}} L^i_{\underset{\vee}{\alpha m}} - 2L^\alpha_{\underset{\vee}{mn}} L^i_{\underset{\vee}{\alpha j}}, \tag{31}$$

$$\underset{\substack{3\\4}}{\hat{\widetilde{\mathcal{W}}}}{}^i_{jmn} = \underset{3}{W}{}^i_{jmn} + L^i_{\underset{\vee}{jm},n} + L^i_{\underset{\vee}{jn},m} - L^\alpha_{\underset{\vee}{jm}} L^i_{\underset{\vee}{\alpha n}} + L^\alpha_{\underset{\vee}{jn}} L^i_{\underset{\vee}{\alpha m}} + 2L^\alpha_{\underset{\vee}{mn}} L^i_{\underset{\vee}{\alpha j}}, \tag{32}$$

$$\underset{\substack{3\\5}}{\hat{\widetilde{\mathcal{W}}}}{}^i_{jmn} = \underset{3}{W}{}^i_{jmn} - L^\alpha_{\underset{\vee}{jm}} L^i_{\underset{\vee}{\alpha n}} + L^\alpha_{\underset{\vee}{jn}} L^i_{\underset{\vee}{\alpha m}}. \tag{33}$$

References I

[1] **S. Bochner**, *Curvature and Betti Numbers II*, Annals of Mathematics, Second Series, Vol. 50, No. 1 (Jan., 1949), pp. 77-93.

[2] **C. García-Meca, A. Martínez, U. Leonhardt**, *Engineering antenna radiation patterns via quasi-conformal mappings*, Optics Express, (2010), Vol. 19, Issue 24, 23743-23750.

[3] **M. Gürlek, G. Civi**, *Conformal Mappings Preserving Einstein Tensor of Weyl Manifolds*, Bull. Iranian Math. Soc. Vol. 41 (2015), No. 2, 463-475.

[4] **S. Ivanov, M. Zlatanović**, *Connections on Non-symmetric (Generalized) Riemannian Manifold and Gravity*, Class. Quantum Grav. 33 (2016) 075016 (23pp).

[5] **J. Mikeš, V. Kiosak, A. Vanžurová**, *Geodesic Mappings of Manifolds with Affine Connection*, Palacký University, Olomouc, 2008.

References II

[6] **J. Mikeš, A. Vanžurová, I. Hinterleitner**, *Geodesic Mappings and Some Generalizations*, Palacky University Olomouc, 2009.

[7] **S. M. Minčić**, *On curvature tensors and pseudotensors of the spaces with non-symmetric affine connection*, Mathematica Balkanica, 4, 76, (1974), 427-430.

[8] **S. M. Minčić**, *Generalized Riemannian Spaces*, Ph. D. Thesis, Faculty of Sciences, Novi Sad, 1975, (in Serbian).

[9] **S. M. Minčić**, *Independent curvature tensors and pseudotensors of spaces with non-symmetric affine connexion*, Colloquia Mathematica Societatis Janos Bolayai, 31, Differential Geometry, Budapest (Hungary), (1979), 445-460.

[10] **N. S. Sinyukov**, *Geodesic mappings of Riemannian spaces*, "Nauka", Moscow, 1979 (in Russian).

References III

[11] **M. S. Stanković**, *Special equitorsion almost geodesic mappings of the third type of non-symmetric affine connection spaces*, Applied Mathematics and Computation 244 (2014) 695–701.

[12] **Mića S. Stanković, S. M. Minčić, Lj. S. Velimirović, M. Lj. Zlatanović**, *Equitorsion conformal Mappings of Generalized Riemannian Spaces*, Matematički Vesnik 61 (2009), 119-129.

[13] **M. S. Stanković, S. M. Minčić, Lj. S. Velimirović, M. Lj. Zlatanović**, *On Equitorsion Geodesic Mappings of General Affine Connection Spaces*, Rend. Sem. Mat. Univ. Padova, Vol. 124 (2010), 77–90.

[14] **M. S. Stanković, N. O. Vesić**, *Some Relations in Non-symmetric Affine Connection Spaces with Regard to a Special Almost Geodesic Mappings of the Third Type*, FILOMAT (NIŠ), (2015), vol. 29 br. 9, 1941-1951.

References IV

[15] **M. S. Stanković, M. Lj. Zlatanović, Lj. Zlatanović, Lj. S. Velimirović**, *Equitorsion holomorphically projective mappings of generalized Kählerian space of the first kind*, Czechoslovak Mathematical Journal, 60 (135) (2010), 635–653.

[16] **M. S. Stanković, M. Lj. Zlatanović, N. O. Vesić**, *Basic Equations of G-almost Geodesic Mappings of the Second Type which have the Property of Reciprocity*, Czech. Math. Journal, vol. 65 br. 3, 787-799.

[17] **J. P. Turpin, A. T. Massoud, Z. H. Jiang, P. L. Werner, D. H. Werner**, *Conformal mappings to achieve simple material parameters for transformation optics devices*, Optics Express, (2010), Vol. 18, Issue 1, 244-252.

[18] **N. O. Vesić**, *Curvature Tensors and the Third Type Almost Geodesic Mappings*, FACTA UNIVERSITATIS (NIŠ), Ser. Math. Inform. Vol. 29, No 4 (2014), 445-460.

[19] **N. O. Vesić**, *A Method for Computation of Invariants of Geometric Mappings*, preprint.

[20] **N. O. Vesić**, *Generalized Weyl Conformal Curvature Tensor of a Generalized Riemannian Space*, submitted.

[21] **N. O. Vesić, Lj. S. Velimirović, M. S. Stanković**, *New Invariants of Equitorsion Third Type Almost Geodesic Mappings*, accepted for publication, DOI 10.1007/s00009-016-0763-z.

[22] **J. W. York**, *Conformatlly invariant orthogonal decomposition of symmetric tensors on Riemannian manifolds and the initial value problem of general relativity*, J.Math.Phys. 14 (1973), 456-464.

[23] **M. Lj. Zlatanović**, *New projective tensors for equitorsion geodesic mappings*, Applied Mathematics Letters, 25, (2012), 890-897.