Pontryagin algebras of some moment-angle complexes

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Preliminaries

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Simplicial complex is flag if any minimal non-simplex consists of two elements.

Let k be a commutative ring with unit.
The face ring of a simplicial complex $\mathcal{K}$ is the quotient of the graded polynomial ring $k[v_1, \ldots, v_m]$ by the monomial ideal generated by non-simplices

$$k[\mathcal{K}] = k[v_1, \ldots, v_m] / \big(v_{i_1} \ldots v_{i_k} : \{i_1, \ldots, i_k\} \notin \mathcal{K}\big), \quad \deg v_i = 2.$$

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The unit polydisc in the complex space is

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For each $I = \{i_1, \cdots, i_k\} \subset [m]$ define the following subset of the polydisc:

$$B_I = \{(z_1, \ldots, z_m) \in \mathbb{D}^m \colon |z_i|^2 = 1 \quad \text{if } i \notin I\}$$

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The moment-angle complex corresponding to a simplicial complex $\mathcal{K}$ is the space

$$\mathcal{Z}_{\mathcal{K}} = \bigcup_{I \in \mathcal{K}} B_I \subset \mathbb{D}^m,$$

where the union is taken in $\mathbb{D}^m$.

Theorem ([Buchstaber–Panov])

There are isomorphisms of rings:

$$\begin{aligned} H^*(\mathcal{Z}_\mathcal{K}) &\cong \operatorname{Tor}_{k[v_1,\cdots,v_m]}(k[\mathcal{K}], k) \\ &\cong H\big(\Lambda[u_1,\ldots,u_m] \otimes k[\mathcal{K}], d\big), \\ H^p(\mathcal{Z}_\mathcal{K}) &\cong \bigoplus_{I\subset[m]} \widetilde{H}^{p-|I|-1}(\mathcal{K}_I), \end{aligned}$$

where $k[\mathcal{K}]$ *is the face ring,* $du_i = v_i,\ dv_i = 0, \quad \deg u_i = 1,\ \deg v_i = 2,$ $\mathcal{K}_I = \{J \in \mathcal{K}\colon J \subset I\}$ *is the full subcomplex (the restriction of* $\mathcal{K}$ *to* I*).*

Preliminaries

Also, consider the subset:

$$(\mathbb{C}P^\infty)^\mathcal{K} = \bigcup_{I \in \mathcal{K}} BT^I \subset BT^m$$

where

$$BT^m = (\mathbb{C}P^\infty)^m, \quad BT^I = \{(x_1, \dots, x_m) \in BT^m \colon x_i = * \text{ if } i \notin I\}$$

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Theorem ([Buchstaber–Panov])

The moment-angle complex $\mathcal{Z}_\mathcal{K}$ is the homotopy fibre of the inclusion $(\mathbb{C}P^\infty)^\mathcal{K} \to (\mathbb{C}P^\infty)^m$.

Preliminaries

The homotopy fibration $\mathcal{Z}_\mathcal{K} \to (\mathbb{C}P^\infty)^\mathcal{K} \to (\mathbb{C}P^\infty)^m$ gives rise to the exact sequence of Pontryagin algebras:

$$1 \to H_*(\Omega\mathcal{Z}_\mathcal{K}; k) \to H_*\big(\Omega((\mathbb{C}P^\infty)^\mathcal{K}); k\big) \to \Lambda[u_1, \ldots, u_m] \to 0,$$

where $\Lambda[u_1, \ldots, u_m] = H_*(\Omega(\mathbb{C}P^\infty)^m; k) = H_*(T^m; k)$.

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where $\Lambda[u_1, \ldots, u_m] = H_*(\Omega(\mathbb{C}P^\infty)^m; k) = H_*(T^m; k)$.

Therefore, $H_*(\Omega\mathcal{Z}_\mathcal{K})$ is the commutator subalgebra of the non-commutative algebra $H_*\big(\Omega((\mathbb{C}P^\infty)^\mathcal{K}); k\big)$.

Preliminaries

This algebra can be described explicitly when $\mathcal{K}$ is a flag complex:

Theorem (Panov–Ray)

Let $\mathcal{K}$ be a flag complex. Then

$$H_*(\Omega((\mathbb{C}P^\infty)^{\mathcal{K}}); k) \cong T\langle x_1, \dots, x_m\rangle / (x_i^2 = 0, x_i x_j + x_j x_i = 0 \text{ if } \{i,j\} \in \mathcal{K}),$$

where $T\langle x_1, \dots, x_m\rangle$ is a free (tensor) algebra, $\deg x_i = 1$.

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where $T\langle x_1, \ldots, x_m \rangle$ is a free (tensor) algebra, $\deg x_i = 1$.

Here $x_i \in H_1(\Omega((\mathbb{C}P^\infty)^{\mathcal{K}}))$ is the canonical generator corresponding to the i-th coordinate mapping

$$S^1 = \Omega(\mathbb{C}P^\infty) \to \Omega(\mathbb{C}P^\infty)^{\mathcal{K}}.$$

From now on we assume that k is a field and do not explicitly indicate it in the notation of homology.

The following result describes multiplicative generators of the commutator algebra $H_*(\Omega\mathcal{Z}_\mathcal{K})$.

Theorem ([Grbić–Panov–Theriault–Wu])

Let $\mathcal{K}$ be a flag complex. Then the subalgebra $H_*(\Omega\mathcal{Z}_\mathcal{K}) \subset H_*\big(\Omega((\mathbb{C}P^\infty)^\mathcal{K})\big)$ is multiplicatively generated by iterated commutators of the form:

$$[x_i, x_j], \quad [x_{k_1}, [x_j, x_i]], \quad \ldots, \quad [x_{k_1}, [x_{k_2}, [x_{k_3}, \ldots, [x_{k_{m-2}}, [x_j, x_i]] \ldots]]],$$

where $k_1 < k_2 < \cdots < k_p < j > i$, $k_s \neq i$ for any s, and i is the smallest vertex in a connected component not containing j of the full subcomplex $\mathcal{K}_{\{k_1,\ldots,k_p,j,i\}} \subset \mathcal{K}$.

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where $k_1 < k_2 < \cdots < k_p < j > i$, $k_s \neq i$ for any s, and i is the smallest vertex in a connected component not containing j of the full subcomplex $\mathcal{K}_{\{k_1,\ldots,k_p,j,i\}} \subset \mathcal{K}$.
Furthemore, this multiplicative generating set is minimal, that is, the commutators above form a basis in the submodule of indecomposables in $H_*(\Omega\mathcal{Z}_\mathcal{K})$.

The case of the pentagon

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The nontrivial cohomology groups are

$$H^0(\mathcal{Z}_{\mathcal{K}}) \cong \widetilde{H}^{-1}(\mathcal{K}_{\varnothing}) \cong k,$$

$$H^3(\mathcal{Z}_{\mathcal{K}}) \cong \bigoplus_{|I|=2} \widetilde{H}^0(\mathcal{K}_I) \cong k^5,$$

$$H^4(\mathcal{Z}_{\mathcal{K}}) \cong \bigoplus_{|I|=3} \widetilde{H}^0(\mathcal{K}_I) \cong k^5,$$

$$H^7(\mathcal{Z}_{\mathcal{K}}) \cong \widetilde{H}^1(\mathcal{K}) \cong k.$$

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A generator of $H^7(\mathcal{Z}_\mathcal{K})$ is represented by any monomial $u_i u_{i+1} u_{i+2} v_{i+3} v_{i+4} \in \Lambda[u_1, \dots, u_5] \otimes k[\mathcal{K}]$, where the indices are considered modulo 5.

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We denote $t = [u_1 u_2 u_3 v_4 v_5] \in H^7(\mathcal{Z}_\mathcal{K})$ and calculate the product in the cohomology ring.

The case of the pentagon

We choose bases for $H^3(\mathcal{Z}_\mathcal{K})$ and $H^4(\mathcal{Z}_\mathcal{K})$ as shown in the Table. For any given basis element of $H^3(\mathcal{Z}_\mathcal{K})$ there is a unique basis element of $H^4(\mathcal{Z}_\mathcal{K})$ such that the product of these two elements is t. The product of any other two elements of $H^3(\mathcal{Z}_\mathcal{K})$ and $H^4(\mathcal{Z}_\mathcal{K})$ is zero. For example,

$$[u_1 v_3] \cdot [u_4 u_5 v_2] = [u_1 u_4 u_5 v_2 v_3] = [u_1 u_2 u_3 v_4 v_5] = t.$$

H^3	H^4	Product
$[u_1 v_3]$	$[u_4 u_5 v_2]$	t
$[u_2 v_4]$	$-[u_1 u_5 v_3]$	t
$[u_3 v_5]$	$[u_1 u_2 v_4]$	t
$[u_4 v_1]$	$[u_2 u_3 v_5]$	t
$[u_5 v_2]$	$[u_3 u_4 v_1]$	t

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To describe the homology of $\mathcal{Z}_\mathcal{K}$ we use the cell decomposition.
Each factor $\mathbb{D}$ is decomposed into three cells:
0-cell: $1 \in \mathbb{D}$
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By taking a product we obtain a cellular decomposition of $\mathbb{D}^m$ whose cells have the form

$$\prod_{i \in I} D_i \times \prod_{j \in J} S_j,$$

where $I, J \subset [m]$ and $I \cap J = \varnothing$.

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The cells of $\mathcal{Z}_\mathcal{K} \subset \mathbb{D}^m$ are specified by the condition $I \in \mathcal{K}$.

The case of the pentagon

We now describe cellular cycles dual to the cohomology classes in the previous table. These are given in the table below.
For example, the cochain $u_1 v_3$ takes value 1 on the chain $S_1 D_3 + D_1 S_3$ and vanishes on all other chains given.

Table: Cellular cycles representing basis homology classes.

H_3	H_4
$S_1 D_3 + D_1 S_3$	$-D_2 S_4 S_5 + S_2 S_4 D_5$
$S_2 D_4 + D_2 S_4$	$S_1 S_3 D_5 + S_1 D_3 S_5$
$S_3 D_5 + D_3 S_5$	$D_1 S_2 S_4 - S_1 S_2 D_4$
$S_4 D_1 + D_4 S_1$	$D_2 S_3 S_5 - S_2 S_3 D_5$
$S_5 D_2 + D_5 S_2$	$-D_1 S_3 S_4 + S_1 S_3 D_4$

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The commutator $[x_i, x_j] = x_i x_j + x_j x_i$ in $H_*(\Omega(\mathbb{C}P^\infty)^{\mathcal{K}})$ is adjoint to the Whitehead product $[\widehat{\mu}_i, \widehat{\mu}_j]\colon S^3 \to (\mathbb{C}P^\infty)^{\mathcal{K}}$.

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It lifts to $\mathcal{Z}_\mathcal{K}$ via the homotopy fibration $\mathcal{Z}_\mathcal{K} \to (\mathbb{C}P^\infty)^\mathcal{K} \to (\mathbb{C}P^\infty)^m$, and therefore we can view $[x_i, x_j]$ as an element of $H_*(\Omega\mathcal{Z}_\mathcal{K})$.

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We consider iterated Whitehead products $[\widehat{\mu}_{i_1}, [\widehat{\mu}_{i_2}, \cdots [\widehat{\mu}_{i_{k-1}}, \widehat{\mu}_{i_k}] \cdots]]\colon S^{k+1} \to \mathcal{Z}_\mathcal{K}$ and iterated commutators $[x_{i_1}, [x_{i_2}, \cdots [x_{i_{k-1}}, x_{i_k}] \cdots]] \in H_k(\Omega\mathcal{Z}_\mathcal{K})$ similarly.

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Let $h\colon \pi_k(\mathcal{Z}_\mathcal{K}) \to H_k(\mathcal{Z}_\mathcal{K})$ denote the Hurewicz homomorphism. From the description above it is easy to deduce the following relation between iterated commutators and cellular chains:

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The Hurewicz image $h([\widehat{\mu}_{i_1},[\widehat{\mu}_{i_2},\cdots[\widehat{\mu}_{i_{k-1}},\widehat{\mu}_{i_k}]\cdots]]) \in H_k(\mathcal{Z}_\mathcal{K})$ is represented by the cellular chain $S_{i_1}\cdots D_{i_{k-1}}S_{i_k} + S_{i_1}\cdots S_{i_{k-1}}D_{i_k}$.

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This gives rise to the correspondence between cellular chains of $\mathcal{Z}_\mathcal{K}$ and commutators in $H_*(\Omega \mathcal{Z}_\mathcal{K})$.

$$S_i D_j + D_i S_j \leftrightarrow [x_i, x_j]$$

$$S_i S_j D_k + S_i D_j S_k \leftrightarrow [x_i, [x_j, x_k]]$$

The case of the pentagon

We obtain the following elements in $H_*(\Omega\mathcal{Z}_{\mathcal{K}})$:

Table: Commutators.

$\deg = 2$	$\deg = 3$
$[x_3, x_1]$	$-[x_4, [x_5, x_2]]$
$[x_4, x_2]$	$[x_1, [x_5, x_3]]$
$[x_5, x_3]$	$[x_2, [x_4, x_1]]$
$[x_4, x_1]$	$[x_3, [x_5, x_2]]$
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Theorem

Let $\mathcal{K}$ be the boundary of a pentagon.

(a) *There is a homotopy equivalence $\mathcal{Z}_{\mathcal{K}} \simeq (S^3 \times S^4)^{\#5}$, where $(S^3 \times S^4)^{\#5}$ is a connected sum of 5 copies $S^3 \times S^4$.*

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(b) *$H_*(\Omega \mathcal{Z}_{\mathcal{K}})$ is an algebra with 10 generators:*

$$a_1 = [x_3, x_1], a_2 = [x_4, x_1], a_3 = [x_4, x_2], a_4 = [x_5, x_2], a_5 = [x_5, x_3],$$

$$b_1 = [x_4, [x_5, x_2]], b_2 = [x_3, [x_5, x_2]], b_3 = [x_1, [x_5, x_3]],$$
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$$b_4 = [x_3, [x_4, x_1]], b_5 = [x_2, [x_4, x_1]],$$

that satisfy a single relation:

$$-[a_1, b_1] + [a_2, b_2] + [a_3, b_3] - [a_4, b_4] + [a_5, b_5] = 0, \qquad (1)$$

where $\deg x_i = 1, \deg a_i = 2, \deg b_i = 3$,
$[a, b] = ab - (-1)^{deg\ a \cdot deg\ b} ba$.

Proof of the theorem

The generator $a_1 \in H_2(\Omega\mathcal{Z}_\mathcal{K})$ is given by the map $S^2 \to \Omega\mathcal{Z}_\mathcal{K}$ which is adjoint to the lift $S^3 \to \mathcal{Z}_\mathcal{K}$ of the Whitehead product $[\widehat{\mu}_3, \widehat{\mu}_1]$, as described in the diagram

$$\begin{array}{ccccc} & & S^3 & & \\ & \swarrow & \downarrow{\scriptstyle [\widehat{\mu}_3,\widehat{\mu}_1]} & & \\ \mathcal{Z}_\mathcal{K} & \longrightarrow & (\mathbb{C}P^\infty)^\mathcal{K} & \longrightarrow & (\mathbb{C}P^\infty)^m \end{array}$$

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$$\begin{array}{ccccc} & & S^4 & & \\ & \swarrow & \downarrow{\scriptstyle [\widehat{\mu}_4,[\widehat{\mu}_5,\widehat{\mu}_2]]} & & \\ \mathcal{Z}_\mathcal{K} & \longrightarrow & (\mathbb{C}P^\infty)^\mathcal{K} & \longrightarrow & (\mathbb{C}P^\infty)^m \end{array}$$

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Using the commutation relations $x_1x_2 = -x_2x_1$, $x_2x_3 = -x_3x_2$, $x_3x_4 = -x_4x_3$, $x_4x_5 = -x_5x_4$, $x_1x_5 = -x_5x_1$ in $H_*(\Omega((\mathbb{C}P^\infty)^\mathcal{K}))$ we reduce each expanded commutator to the following canonical form: take x_j with the minimal index j (in our case x_1) and move it to the left as far as possible using the commutation relations. Next, take x_2 and move it to the left as far as possible without using the commutation relations with x_1. Proceed in this fashion until we reach x_5.

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E.g., the canonical form of the monomial $x_4x_2x_3x_5x_1$ is $x_3x_4x_1x_2x_5$.

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Summing up with the appropriate signs we obtain the required relation.

Proof of the theorem

Consider the map $f\colon (S^3 \vee S^4)^{\vee 5} \to \mathcal{Z}_{\mathcal{K}}$ defined as the wedge of the maps corresponding to the generators $a_1, \ldots, a_5$, $b_1, \ldots, b_5$.

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Because of the relation (1), the map f extends to a map from the connected sum:

$$\begin{array}{ccc} (S^3 \vee S^4)^{\vee 5} & \xrightarrow{\;f\;} & \mathcal{Z}_{\mathcal{K}} \\ \big\downarrow & \overset{\widehat{f}}{\nearrow} & \\ (S^3 \times S^4)^{\# 5} & & \end{array}$$

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The map $\widehat{f}$ induces an isomorphism in homology, so it is a homotopy equivalence, because all spaces are simply connected.

Proof of the theorem

To finish the proof we need to show that there are no other relations on the generators $a_1, \ldots, a_5$, $b_1, \ldots, b_5$.

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The space X is obtained from a wedge of spheres $X = (S^3 \vee S^4)^{\vee 5}$ by attaching a 7-dimensional cell along a sum of 5 Whitehead products.

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By passing to the Pontryagin algebra, we obtain that $H_*(\Omega X)$ is the quotient of a free algebra on 10 generators by a single relation, i.e. has exactly the form decribed above.

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The homotopy equivalence $X \to \mathcal{Z}_\mathcal{K}$ implies an isomorphism of Pontryagin algebras, so there are no other relations except the described above.

The case of the hexagon

Let $\mathcal{K}$ be the boundary of a hexagon. We describe the cohomology, the homology and the Pontryagin algebra of $\mathcal{Z}_{\mathcal{K}}$.

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The nontrivial cohomology groups are

$$H^0(\mathcal{Z}_{\mathcal{K}}) \cong \widetilde{H}^{-1}(\mathcal{K}_{\varnothing}) \cong k,$$
$$H^3(\mathcal{Z}_{\mathcal{K}}) \cong \bigoplus_{|I|=2} \widetilde{H}^0(\mathcal{K}_I) \cong k^9,$$
$$H^4(\mathcal{Z}_{\mathcal{K}}) \cong \bigoplus_{|I|=3} \widetilde{H}^0(\mathcal{K}_I) \cong k^{16},$$
$$H^5(\mathcal{Z}_{\mathcal{K}}) \cong \bigoplus_{|I|=4} \widetilde{H}^0(\mathcal{K}_I) \cong k^9,$$
$$H^8(\mathcal{Z}_{\mathcal{K}}) \cong \widetilde{H}^1(\mathcal{K}) \cong k.$$

The case of the hexagon

To describe the cohomology ring $H^*(\mathcal{Z}_\mathcal{K})$ it is convenient to work with the Koszul algebra $\Lambda[u_1, \dots, u_6] \otimes k[\mathcal{K}]$.

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A generator of $H^8(\mathcal{Z}_{\mathcal{K}})$ is represented by any monomial $u_i u_{i+1} u_{i+2} u_{i+3} v_{i+4} v_{i+5} \in \Lambda[u_1, \dots, u_6] \otimes k[\mathcal{K}]$, where the indices are considered modulo 6.

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We denote $t = [u_1 u_2 u_3 u_4 v_5 v_6] \in H^8(\mathcal{Z}_\mathcal{K})$ and calculate the product in the cohomology ring.

The case of the hexagon

We choose bases for $H^3(\mathcal{Z}_\mathcal{K})$ and $H^5(\mathcal{Z}_\mathcal{K})$ as shown in the table. For any given basis element of $H^3(\mathcal{Z}_\mathcal{K})$ there is a unique basis element of $H^5(\mathcal{Z}_\mathcal{K})$ such that the product of these two elements is t. The product of any other two elements of $H^3(\mathcal{Z}_\mathcal{K})$ and $H^5(\mathcal{Z}_\mathcal{K})$ is zero.

Table: Cohomology classes and their product.

H^3	H^5	Product
$[u_1 v_3]$	$[u_4 u_5 u_6 v_2]$	t
$[u_1 v_4]$	$-[u_2 u_3 u_5 v_6] + [u_2 u_3 u_6 v_5]$	t
$[u_1 v_5]$	$[u_2 u_3 u_4 v_6]$	t
$[u_2 v_4]$	$-[u_1 u_5 u_6 v_3]$	t
$[u_2 v_5]$	$-[u_1 u_3 u_6 v_4] + [u_1 u_4 u_6 v_3]$	t
$[u_2 v_6]$	$-[u_3 u_4 u_5 v_1]$	t
$[u_3 v_5]$	$[u_1 u_2 u_6 v_4]$	t
$[u_3 v_6]$	$[u_1 u_2 u_4 v_5] - [u_1 u_2 u_5 v_4]$	t
$[u_4 v_6]$	$-[u_1 u_2 u_3 v_5]$	t

The case of the hexagon

We choose bases for $H^4(\mathcal{Z}_\mathcal{K})$ as shown in the table. For any given basis element of $H^4(\mathcal{Z}_\mathcal{K})$ there is a unique basis element of $H^4(\mathcal{Z}_\mathcal{K})$ such that the product of these two elements is t. The product of any other two elements of $H^4(\mathcal{Z}_\mathcal{K})$ and $H^4(\mathcal{Z}_\mathcal{K})$ is zero.

Table: Cohomology classes and their product.

H^4	H^4	Product
$[u_1u_5v_3]$	$-[u_2u_6v_4]$	t
$[u_3u_5v_1]$	$-[u_4u_6v_2]+[u_2u_6v_4]$	t
$[u_2u_3v_6]$	$-[u_4u_5v_1]$	t
$[u_5u_6v_2]$	$[u_3u_4v_1]$	t
$[u_1u_6v_3]$	$[u_4u_5v_2]$	t
$[u_3u_4v_6]$	$-[u_2u_5v_1]+[u_1u_5v_2]$	t
$[u_5u_6v_3]$	$-[u_2u_4v_1]+[u_1u_4v_2]$	t
$[u_1u_6v_4]$	$-[u_3u_5v_2]+[u_2u_5v_3]$	t

The case of the hexagon

We now describe cellular cycles dual to the cohomology classes in the previous tables. These are given in the tables below.

Table: Basis of homology.

H_3	H_5
$S_1D_3 + D_1S_3$	$-D_2S_4S_5S_6 - S_2S_4S_5D_6$
$S_1D_4 + D_1S_4$	$S_2S_3S_5D_6 + D_2S_3S_5S_6$
$S_1D_5 + D_1S_5$	$-D_2S_3S_4S_6 - S_2S_3S_4D_6$
$S_2D_4 + D_2S_4$	$-S_1S_3S_5D_6 + S_1D_3S_5S_6$
$S_2D_5 + D_2S_5$	$-S_1D_3S_4S_6 + S_1S_3S_4D_6$
$S_2D_6 + D_2S_6$	$D_1S_3S_4S_5 + S_1S_3S_4D_5$
$S_3D_5 + D_3S_5$	$-S_1S_2D_4S_6 - S_1S_2S_4D_6$
$S_3D_6 + D_3S_6$	$-D_1S_2S_4S_5 - S_1S_2S_4D_5$
$S_4D_6 + D_4S_6$	$S_1S_2S_3D_5 + D_1S_2S_3S_5$

The case of the hexagon

Table: Basis of homology.

H_4	H_4
$-S_1S_3D_5 - S_1D_3S_5$	$D_2S_4S_6 + S_2D_4S_6$
$S_1S_3D_5 - D_1S_3S_5$	$-S_2S_4D_6 + D_2S_4S_6$
$-S_2S_3D_6 + D_2S_3S_6$	$-S_1S_4D_5 + D_1S_4S_5$
$-D_2S_5S_6 + S_2S_5D_6$	$S_1S_3D_4 - D_1S_3S_4$
$-S_1D_3S_6 - S_1S_3D_6$	$S_2S_4D_5 - D_2S_4S_5$
$-S_3S_4D_6 + D_3S_4S_6$	$-S_1S_2D_5 + D_1S_2S_5$
$S_3S_5D_6 - D_3S_5S_6$	$-D_1S_2S_4 + S_1S_2D_4$
$-S_1S_4D_6 - S_1D_4S_6$	$-D_2S_3S_5 + S_2S_3D_5$

The case of the hexagon

We obtain the following elements in $H_*(\Omega\mathcal{Z}_{\mathcal{K}})$:

Table: Commutators.

1	2	Additional commutators
$[x_3, x_1]$	$-[x_4, [x_5, [x_6, x_2]]]$	$k_1 \cdot [[x_2, x_5], [x_4, x_6]]$
$[x_4, x_1]$	$[x_3, [x_5, [x_6, x_2]]]$	$k_2 \cdot [[x_5, x_3], [x_6, x_2]] +$ $k_3 \cdot [[x_6, x_3], [x_5, x_2]]$
$[x_5, x_1]$	$-[x_3, [x_4, [x_6, x_2]]]$	$k_4 \cdot [[x_4, x_2], [x_6, x_3]]$
$[x_4, x_2]$	$[x_1, [x_5, [x_6, x_3]]]$	$k_5 \cdot [[x_6, x_3], [x_5, x_1]]$
$[x_5, x_2]$	$-[x_1, [x_4, [x_6, x_3]]]$	$k_6 \cdot [[x_4, x_1], [x_6, x_3]] +$ $k_7 \cdot [[x_6, x_4], [x_3, x_1]]$
$[x_6, x_2]$	$[x_3, [x_4, [x_5, x_1]]]$	$k_8 \cdot [[x_5, x_3], [x_4, x_1]]$
$[x_5, x_3]$	$-[x_1, [x_2, [x_6, x_4]]]$	$k_9 \cdot [[x_4, x_1], [x_6, x_2]]$
$[x_6, x_3]$	$-[x_2, [x_4, [x_5, x_1]]]$	$k_{10} \cdot [[x_4, x_2], [x_5, x_1]] +$ $k_{11} \cdot [[x_4, x_1], [x_5, x_2]]$
$[x_6, x_4]$	$[x_2, [x_3, [x_5, x_1]]]$	$k_{12} \cdot [[x_5, x_2], [x_3, x_1]]$

The case of the hexagon

Table: Commutators.

1	2
$-[x_1,[x_5,x_3]]$	$-[x_6,[x_4,x_2]]$
$-[x_3,[x_5,x_1]]$	$[x_4,[x_6,x_2]]$
$[x_3,[x_6,x_2]]$	$[x_4,[x_5,x_1]]$
$-[x_5,[x_6,x_2]]$	$-[x_3,[x_4,x_1]]$
$-[x_1,[x_6,x_3]]$	$-[x_4,[x_5,x_2]]$
$[x_4,[x_6,x_3]]$	$[x_2,[x_5,x_1]]$
$-[x_5,[x_6,x_3]]$	$[x_2,[x_4,x_1]]$
$-[x_1,[x_6,x_4]]$	$[x_3,[x_5,x_2]]$

The case of the hexagon

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$-[x_1, [x_5, x_3]]$	$-[x_6, [x_4, x_2]]$
$-[x_3, [x_5, x_1]]$	$[x_4, [x_6, x_2]]$
$[x_3, [x_6, x_2]]$	$[x_4, [x_5, x_1]]$
$-[x_5, [x_6, x_2]]$	$-[x_3, [x_4, x_1]]$
$-[x_1, [x_6, x_3]]$	$-[x_4, [x_5, x_2]]$
$[x_4, [x_6, x_3]]$	$[x_2, [x_5, x_1]]$
$-[x_5, [x_6, x_3]]$	$[x_2, [x_4, x_1]]$
$-[x_1, [x_6, x_4]]$	$[x_3, [x_5, x_2]]$

The coefficients of additional commutators can be calculated as follows

$$k_1 = 1, k_2 = 1, k_3 = -1, k_4 = 1, k_5 = -1, k_6 = 1, k_7 = -1, k_8 = -1,$$

$$k_9 = 0, k_{10} = 0, k_{11} = 0, k_{12} = 0.$$

The case of the hexagon

The relation for $H_*(\Omega\mathcal{Z}_{\mathcal{K}})$:

$$
\begin{gathered}
-[[x_3,x_1],[x_4,[x_5,[x_6,x_2]]]]+[[x_3,x_1],[[x_2,x_5],[x_4,x_6]]]+\\
[[x_4,x_1],[x_3,[x_5,[x_6,x_2]]]]+[[x_4,x_1],[[x_5,x_3],[x_6,x_2]]]-\\
[[x_4,x_1],[[x_6,x_3],[x_5,x_2]]]-[[x_5,x_1],[x_3,[x_4,[x_6,x_2]]]]+\\
[[x_5,x_1],[[x_4,x_2],[x_6,x_3]]]+[[x_4,x_2],[x_1,[x_5,[x_6,x_3]]]]-\\
[[x_4,x_2],[[x_6,x_3],[x_5,x_1]]]-[[x_5,x_2],[x_1,[x_4,[x_6,x_3]]]]+\\
[[x_5,x_2],[[x_4,x_1],[x_6,x_3]]]-[[x_5,x_2],[[x_6,x_4],[x_3,x_1]]]+\\
[[x_6,x_2],[x_3,[x_4,[x_5,x_1]]]]-[[x_6,x_2],[[x_5,x_3],[x_4,x_1]]]-\\
[[x_5,x_3],[x_1,[x_2,[x_6,x_4]]]]-[[x_6,x_3],[x_2,[x_4,[x_5,x_1]]]]+\\
[[x_6,x_4],[x_2,[x_3,[x_5,x_1]]]]+[[x_1,[x_5,x_3]],[x_6,[x_4,x_2]]]-\\
[[x_3,[x_5,x_1]],[x_4,[x_6,x_2]]]+[[x_3,[x_6,x_2]],[x_4,[x_5,x_1]]]+\\
[[x_5,[x_6,x_2]],[x_3,[x_4,x_1]]]+[[x_1,[x_6,x_3]],[x_4,[x_5,x_2]]]+\\
[[x_4,[x_6,x_3]],[x_2,[x_5,x_1]]]-[[x_5,[x_6,x_3]],[x_2,[x_4,x_1]]]-\\
[[x_1,[x_6,x_4]],[x_3,[x_5,x_2]]]=0.
\end{gathered}
$$

The case of the hexagon

Theorem

Let $\mathcal{K}$ be the boundary of a hexagon.

(a) *There is a homotopy equivalence $\mathcal{Z}_{\mathcal{K}} \simeq (S^3 \times S^5)^{\# 9} \# (S^4 \times S^4)^{\# 8}$.*

The case of the hexagon

Theorem

Let $\mathcal{K}$ be the boundary of a hexagon.

(a) *There is a homotopy equivalence* $\mathcal{Z}_{\mathcal{K}} \simeq (S^3 \times S^5)^{\#9} \# (S^4 \times S^4)^{\#8}$.

(b) $H_*(\Omega \mathcal{Z}_{\mathcal{K}})$ *is an algebra with* 34 *generators:*

$$a_1 = [x_3, x_1], a_2 = [x_4, x_1], a_3 = [x_5, x_1], a_4 = [x_4, x_5], a_5 = [x_5, x_2],$$
$$a_6 = [x_6, x_2], a_7 = [x_5, x_3], a_8 = [x_6, x_3], a_9 = [x_6, x_4],$$
$$b_1 = [x_1, [x_5, x_3]], b_2 = [x_3, [x_5, x_1]], b_3 = [x_3, [x_6, x_2]],$$
$$b_4 = [x_5, [x_6, x_2]], b_5 = [x_1, [x_6, x_3]], b_6 = [x_4, [x_6, x_3]],$$
$$b_7 = [x_5, [x_6, x_3]], b_8 = [x_1, [x_6, x_4]],$$
$$d_1 = [x_6, [x_4, x_2]], d_2 = [x_4, [x_6, x_2]], d_3 = [x_4, [x_5, x_1]],$$
$$d_4 = [x_3, [x_4, x_1]], d_5 = [x_4, [x_5, x_2]], d_6 = [x_2, [x_5, x_1]],$$
$$d_7 = [x_2, [x_4, x_1]], d_8 = [x_3, [x_5, x_2]],$$

The case of the hexagon

$$c_1 = [x_4, [x_5, [x_6, x_2]]], c_2 = [x_3, [x_5, [x_6, x_2]]], c_3 = [x_3, [x_4, [x_6, x_2]]],$$
$$c_4 = [x_1, [x_5, [x_6, x_3]]], c_5 = [x_1, [x_4, [x_6, x_3]]], c_6 = [x_3, [x_4, [x_5, x_1]]],$$
$$c_7 = [x_1, [x_2, [x_6, x_4]]], c_8 = [x_2, [x_4, [x_5, x_1]]], c_9 = [x_2, [x_3, [x_5, x_1]]],$$

The case of the hexagon

$$c_1 = [x_4, [x_5, [x_6, x_2]]], c_2 = [x_3, [x_5, [x_6, x_2]]], c_3 = [x_3, [x_4, [x_6, x_2]]],$$
$$c_4 = [x_1, [x_5, [x_6, x_3]]], c_5 = [x_1, [x_4, [x_6, x_3]]], c_6 = [x_3, [x_4, [x_5, x_1]]],$$
$$c_7 = [x_1, [x_2, [x_6, x_4]]], c_8 = [x_2, [x_4, [x_5, x_1]]], c_9 = [x_2, [x_3, [x_5, x_1]]],$$

that satisfy a single relation:

$$\sum_{i=1}^{9} [a_i, c_i'] + \sum_{j=1}^{8} \sigma_j \cdot [b_j, d_j] = 0,$$

where $\deg x_i = 1, \deg a_i = 2, \deg b_i = \deg d_i = 3, \deg c_i = 4,$
$c_1' = -c_1 + [a_5, a_9], c_2' = c_2 + [a_7, a_6] - [a_8, a_5], c_3' = -c_3 + [a_4, a_8],$
$c_4' = c_4 - [a_8, a_3], c_5' = -c_5 + [a_2, a_8] - [a_9, a_1], c_6' = c_6 - [a_7, a_2],$
$c_7' = -c_7 + 0 \cdot [a_2, a_6], c_8' = -c_8 + 0 \cdot [a_4, a_3] + 0 \cdot [a_2, a_5], c_9' = c_9 + 0 \cdot [a_5, a_1],$

$$\sigma_j = \begin{cases} -1, & j \in \{2, 7, 8\} \\ 1, & j \in \{1, 3, 4, 5, 6\} \end{cases}.$$

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