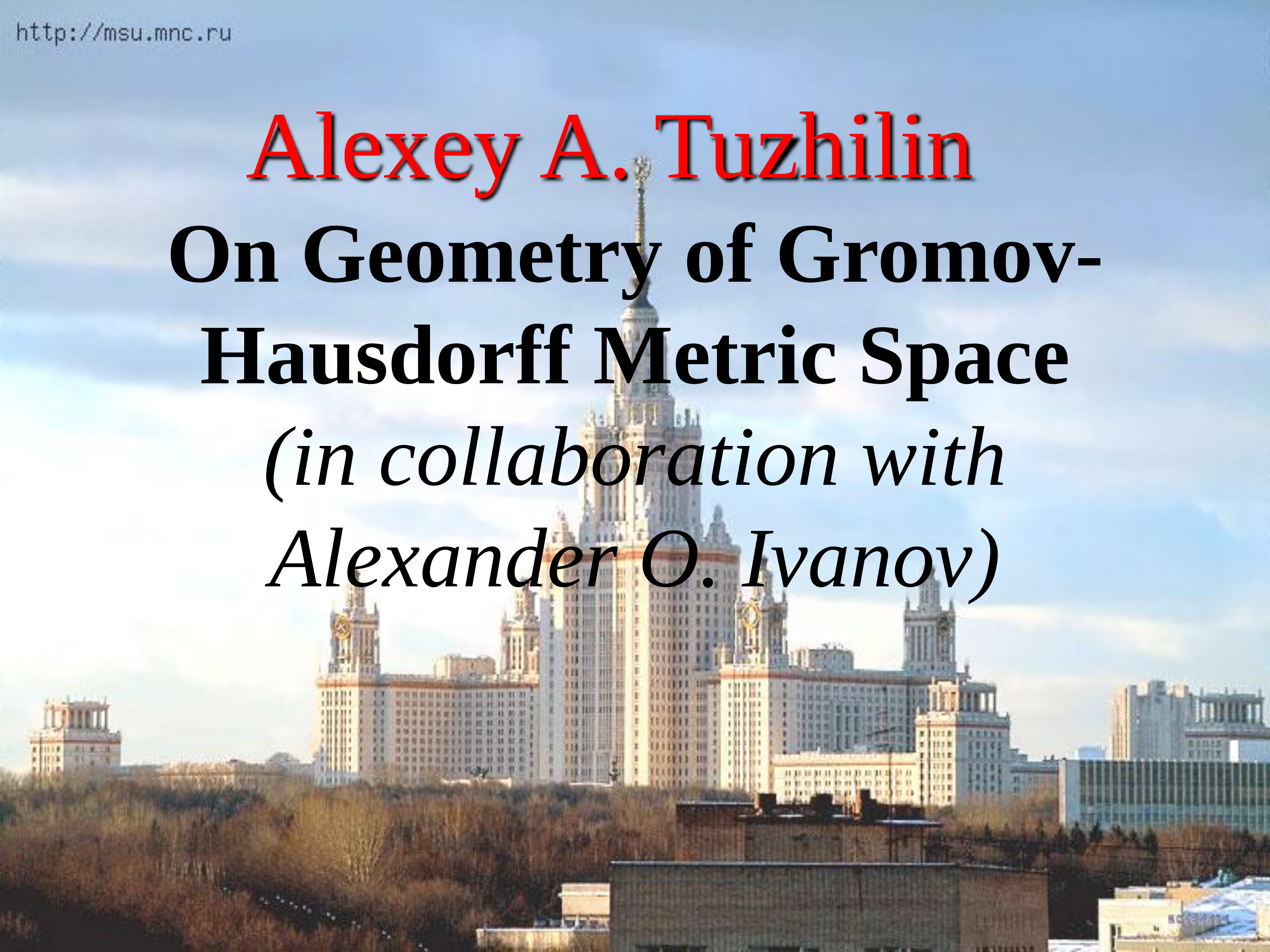


Alexey A. Tuzhilin

On Geometry of Gromov-Hausdorff Metric Space

(in collaboration with Alexander O. Ivanov)

Hausdorff and Gromov-Hausdorff distances.

X is a metric space, $x, y \in X$; $A, B \subset X$ are nonempty,

$|xy|$ is the distance between x and y,

$|xA| = \inf\{|xa| : a \in A\}$,

for $r > 0$ let $U_r(A) = \{ x \in X : |xA| < r \}$ be r-neighborhood of A,

$$d_H(A, B) = \inf \{ r : A \subset U_r(B) \text{ and } B \subset U_r(A) \}.$$

Let A and B be metric spaces.

Realization of (A, B) is a triple (Z, A', B'), where

- Z is a metric space,
- $A', B' \subset Z$,
- A' is isometric to A, B' is isometric to B.

$d_{GH}(A, B) = \inf \{ r : \exists$ a realization (Z, A', B'), $d_H(A', B') < r \}$.

Remark. It suffices to take $Z = A \sqcup B$ and extend the metrics to Z.

Main Properties.

X is a metric space,
$\mathcal{H}(X)$ is the set of all nonempty closed bounded subsets of X.

Proposition.
1) d_H is a metric on $\mathcal{H}(X)$;
2) X is complete $\Leftrightarrow$ $\mathcal{H}(X)$ is complete;
3) X is compact $\Leftrightarrow$ $\mathcal{H}(X)$ is compact.

$\mathcal{M}$ is the set of isometry classes of compact metric spaces.

Proposition.
1) d_{GH} is a metric on $\mathcal{M}$;
2) $\mathcal{M}$ is path-connected, complete, separable, not locally compact.
3) $\mathcal{M}$ is geodesic (A.Ivanov, N.Nikolaeva, A.Tuzhilin, 2015).

$\mathcal{M}$ with d_{GH} is called Gromov-Hausdorff space.

Examples.

If X is a metric space, $\lambda > 0$, then λX is the metric space with the same set X and with the distance $\lambda|xy|$ between x and y.

Let Δ_n be n-points space whose nonzero distances equal 1.

Remark. Δ_1 is one-point space.

For any metric space X we put $\operatorname{diam} X = \sup\{ |xy| : x, y \in X \}$.

Proposition. For any metric spaces X and Y

1) $2d_{GH}(\Delta_1, X) = \operatorname{diam} X$;
2) if $\operatorname{diam} X < \infty$, then $2d_{GH}(X, Y) \geq |\operatorname{diam} X - \operatorname{diam} Y|$;
3) $2d_{GH}(X, Y) \leq \max\{ \operatorname{diam} X, \operatorname{diam} Y \}$;
4) for any $\lambda, \mu > 0$ we have $2d_{GH}(\lambda X, \mu X) = |\lambda - \mu| \operatorname{diam} X$;
5) for any $\lambda > 0$ we have $d_{GH}(\lambda X, \lambda Y) = \lambda\, d_{GH}(X, Y)$, i.e., multiplication $X \to \lambda X$ is a homothety centered at Δ_1.

Steiner Minimal Trees.

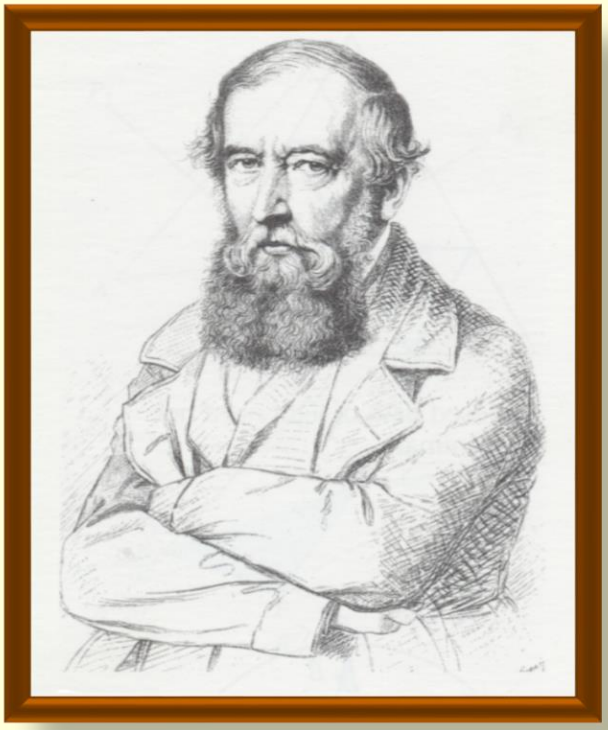
If $G = (V, E)$ is a graph such that $V \subset X$, and $e = vw \in E$, then
$|e| = |vw|$ is the length of e, and $|G| = \sum_{e \in E} |e|$ is the length of G.

Let X be a metric space, and $M \subset X$ be finite, then
$smt(M, X) = \inf \{ |G| : G = (V, E)$ is a tree such that $M \subset V \subset X \}$;

$G = (V, E)$ is a tree such that $M \subset V \subset X$, $|G| = smt(M, X)$, then
G is called a Steiner minimal tree on M.

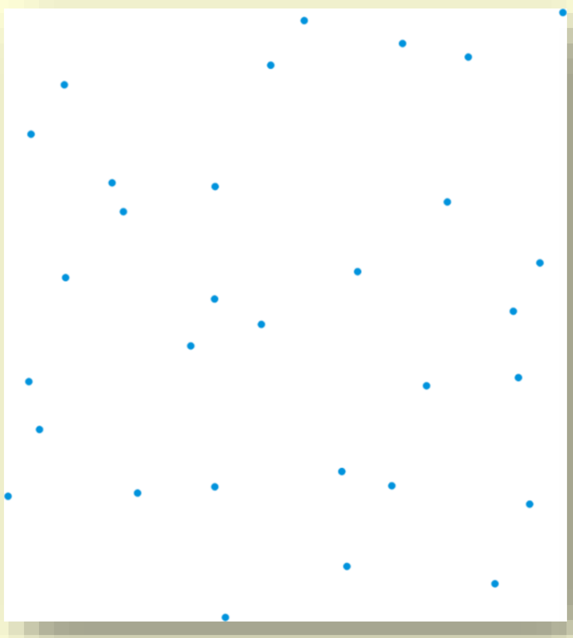
$SMT(M, X) = \{ G : G$ is a Steiner minimal tree on $M \}$.

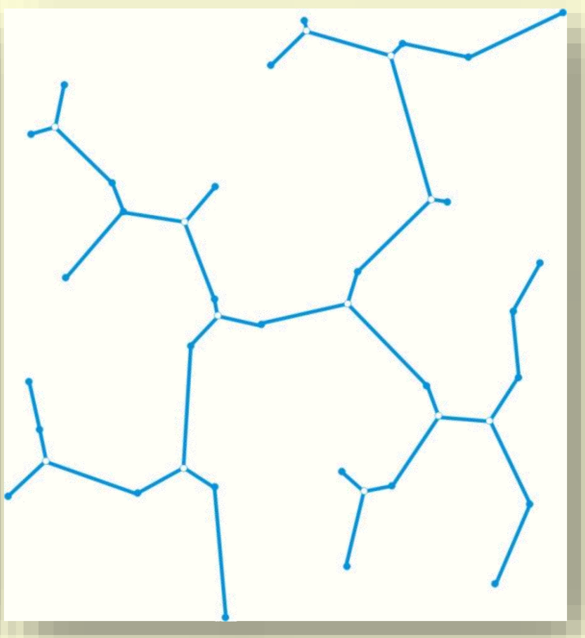
Classical Steiner Problem.



Jacob Steiner
(1796-1863)

Construct a Steiner minimal tree for a given finite subset of the Euclidean plane.





Steiner problem in Gromov-Hausdorff space.

General Steiner problem in a metric space X.

1) Is it true that for any finite $M \subset X$ it holds $SMT(M, X) \neq \emptyset$?

2) Describe the trees from $SMT(M, X)$ for various finite $M \subset X$.

$\mathcal{M}_n = \{X \in \mathcal{M} : \#X \le n\}$.

Theorem [A.Ivanov, N.Nikolaeva, A.Tuzhilin].

For any positive integer n and finite $M \subset \mathcal{M}_n$ we have $SMT(M, \mathcal{M}) \neq \emptyset$. Moreover, there exists an integer $m=m(n) \ge n$ and a Steiner minimal tree $G = (V, E) \in SMT(M, \mathcal{M})$ such that $V \subset \mathcal{M}_m$, i.e., $G \in SMT(M, \mathcal{M}_m)$.

Conjecture. Existence result is valid for any finite $M \subset \mathcal{M}$.

Finite metric spaces, local structure of $\mathcal{M}_n$.

$\mathcal{M}_1$ is one-point metric space.

$\mathcal{M}_2$ is isometric to $\mathbb{R}_+ = \{x \in \mathbb{R}: x \geq 0\}$; the isometry is $X \to \frac{1}{2}\,\mathrm{diam}\,X$.

Theorem [A.Ivanov, A.Tuzhilin].
$\mathcal{M}_3$ is isometric to the cone $C = \{(a, b, c) : 0 \leq a \leq b \leq c \leq a + b\} \subset \mathbb{R}^3$ with the metric generated by max-norm $|(a, b, c)| = \max\{|a|, |b|, |c|\}$.

We say that a finite metric space is in general position if all its nonzero distances are pairwise distinct, and all its triangle inequalities are strict.

Theorem [A.Ivanov, A.Tuzhilin].
Suppose that X is a finite metric space, $\#X = n$, X is in general position. Then there exists a neighborhood $U \subset \mathcal{M}_n$ of X such that U is isometric to an open subset of $\mathbb{R}^{n(n-1)/2}$ with the metric generated by the max-norm.

Finite-spaces universality of Gromov-Hausdorff space.

Denote by $\mathbb{R}^n_\infty$ the space $\mathbb{R}^n$ with the metric generated by the max-norm.
Let $X = \{x_1, \dots x_n\}$ be a finite metric space.
Define Kuratowski mapping $\nu : X \to \mathbb{R}^n_\infty$ as $x_i \to (|x_i x_1|, \dots , |x_i x_n|)$.
It is well-known that ν is an isometric embedding.

Combining the Kuratowski mapping and the previous theorem, we get

Corollary [A.Ivanov, S.Iliadis, A.Tuzhilin].
For any finite metric space M there exists an isometric embedding of M into $\mathcal{M}$.

Remark. It is not true that $\mathcal{M}$ is universal in the following sense: for any finite metric spaces $M \subset N$, any isometric embedding $M \to \mathcal{M}$ can be extended to an isometric embedding $N \to \mathcal{M}$.
Example. $M = \{A, B\}$, $|AB| = 1/2$; $N= \{A, B, C\}$, $|AC|=1/2$, $|BC|=2/3$; $A \to \Delta_1$, $B \to \Delta_2$; this can not be extended to isometric embedding of N.

One-dimensional Gromov minimal fillings

Let M be a finite metric space,

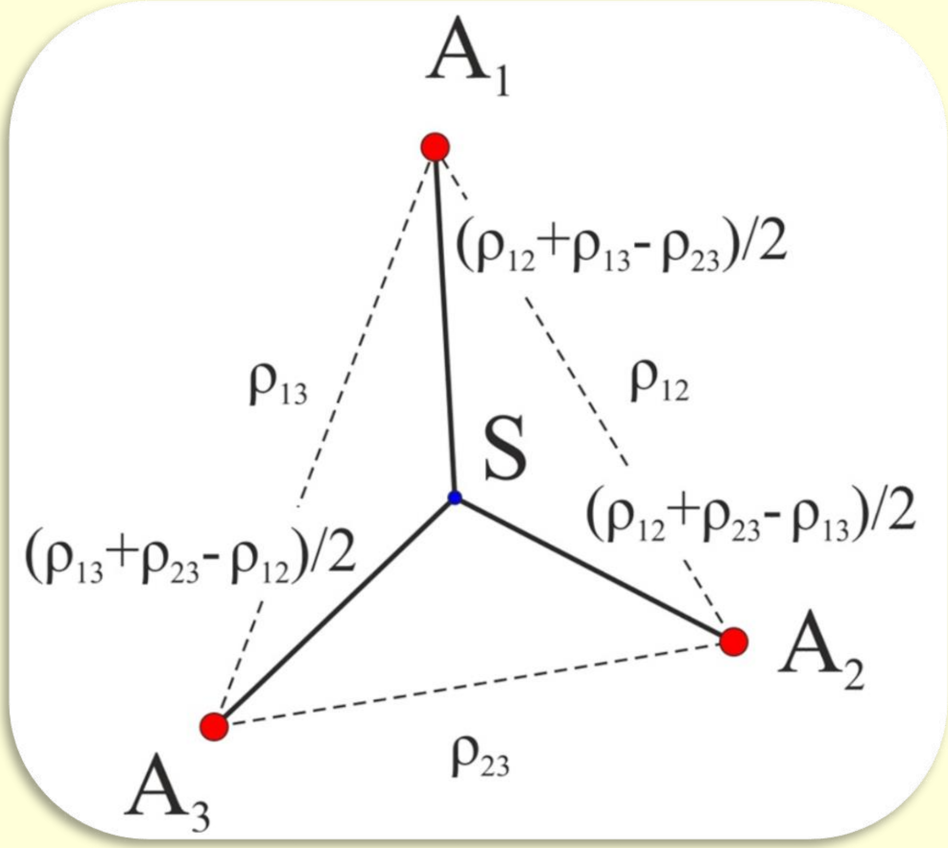
X an arbitrary metric space, $f : M \to X$ an isometric embedding,

$mf(M) = \inf\{ s : \text{there exists } f : M \to X \text{ such that } smt(f(M), X) \le s \}$.

Each $G \in SMT(f(M), X)$ such that $|G| = mf(M)$ is called

(one-dimensional Gromov) minimal filling.

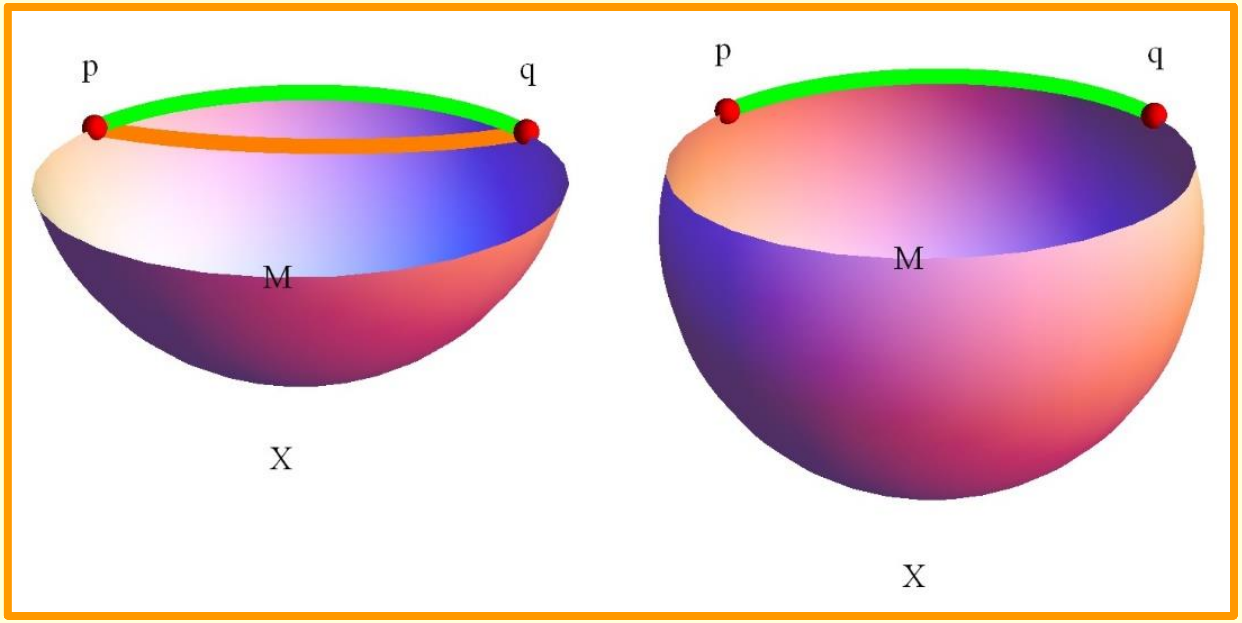
Example: 3-Points Case



General Gromov minimal fillings.

An n-dimensional manifold X with a metric d is called a filling of an (n-1)-dimensional manifold M with a metric ρ,

if $M = \partial X$, and $\rho(p,q) \le d(p,q)$ for all $p, q \in M$.



Problem (M.Gromov). Given M, find the least possible volume mf(M) of fillings X for M, and describe minimal fillings X, i.e., the ones with

$$mf(M) = volume(X).$$

Minimal fillings, existence, ratios.

Theorem [A.Ivanov, A.Tuzhilin].
For any finite metric space M there exists its minimal filling.

Variational curvature of a metric space.

Define Steiner subratio $ssr(X)$ of a metric space X (Ivanov, Tuzhilin):
for any finite $M \subset X$ we put $ssr(M, X) = mf(M)/smt(M, X)$,
$ssr(X) = \inf \{ ssr(M, X) : M \subset X, 2 \le \#M < \infty \}$.
We call a metric space X variationally flat if $ssr(X) = 1$.

Theorem [Z.Ovsyannikov].
The space $\mathbb{R}^n_\infty$ is variationally flat, i.e., for any finite $M \subset \mathbb{R}^n_\infty$ it holds

$$SMT(M, \mathbb{R}^n_\infty) \neq \varnothing ,$$

and each $G \in SMT(M, \mathbb{R}^n_\infty)$ is a minimal filling for M.

Minimal fillings and Gromov-Hausdorff space.

Recall that $\mathcal{M}_3$ is isometric to the cone

$$C = \{(a, b, c) : 0 \le a \le b \le c \le a + b\} \subset \mathbb{R}^3_\infty .$$

Is it true that $\mathcal{M}_3$ and, probably, $\mathcal{M}_n$ and $\mathcal{M}$ are variationally flat?

Theorem [A.Ivanov, A.Tuzhilin]. For any positive integer N and any n-points metric space X in general position, $n \ge 2$, there exists a neighborhood $U \subset \mathcal{M}_n$ of X such that for any $M \subset U$, $\#M \le N$, each Steiner minimal tree on M is a minimal filling for M.

Example. $M=\{A, B, C\} \subset \mathcal{M}_3$, where the spaces A, B, C are given by their distances vectors $(8, 22, 29.5)$, $(11.5, 18, 29)$, $(12, 21.5, 33)$, respectively. Then $\mathrm{smt}(M, \mathcal{M}) > \mathrm{mf}(M)$. Indeed, by numerical experiment, we have $\mathrm{ssr}(\mathcal{M}) \le \mathrm{ssr}(M) < 0.857$.

Problem. Calculate $\mathrm{ssr}(\mathcal{M})$.

MF-universality of Gromov-Hausdorff space.

Theorem [A.Ivanov, A.Tuzhilin].
Let X be a finite metric space, and $G=(V, E)$ be its minimal filling (in particular, $X \subset V$). Then there exists a positive integer N and an isometric embedding $f : X \to \mathcal{M}_N$, $M = f(X)$, such that

(1) there exists an isometric embedding $F : V \to \mathcal{M}_N$ such that $F|_X = f$;

(2) $F(G) = (F(V), F(E)) \in SMT(M, \mathcal{M})$, i.e., $F(G)$ is a Steiner minimal tree on M.

In other words, any one-dimensional Gromov minimal filling can be realized as a Steiner minimal tree in $\mathcal{M}$. Moreover, one can put this realization into some $\mathcal{M}_N$.

Distances caclulations in Gromov-Hausdorff space.

Recall that for $\lambda > 0$ we defined $\lambda\Delta_k$ as k-points metric space whose nonzero distances equal λ. Let us put $\lambda\Delta_k = \{1, \dots , k\}$.

For a set X and a positive integer k we denote by $\mathcal{D}_k(X)$ the set of all partitions of X into k subsets (k-parts partitions).

For $D = \{X_1, \dots , X_k\} \in \mathcal{D}_k(X)$ we put

$$\operatorname{diam} D = \max_i \operatorname{diam} X_i,$$

$$\alpha(D) = \inf\{ |xy| : x \in X_i,\ y \in X_j,\ i \neq j \},$$

$$\beta(D) = \sup\{ |xy| : x \in X_i,\ y \in X_j,\ i \neq j \}.$$

Theorem [A.Ivanov, A.Tuzhilin].

For any $X \in \mathcal{M}$, a positive integer $k \leq \#X$, and $\lambda > 0$ we have

$$2\, d_{GH}(\lambda\Delta_k , X) = \inf \{\max[\operatorname{diam} D, \lambda - \alpha(D), \beta(D) - \lambda] : D \in \mathcal{D}_k(X)\}.$$

mst-spectrum of finite metric space.

Let M be a finite metric space.
We put

$$mst(M) = \inf\{\, |G| : G = (M, E) \text{ is a tree} \,\},$$
$$MST(M) = \{\, G : G = (M, E) \text{ is a tree}, |G| = mst(M) \,\},$$
$G \in MST(M)$ is called a minimum spanning tree on M.

Evidently, $MST(M) \neq \varnothing$ and $\#MST(M)$ may be more than 1.

Let $\#M = n$. For each $G \in MST(M)$ we define $s(G) = (s_1, \ldots, s_{n-1})$ to be the vector obtained from the lengths of edges of G by ordering them descending.

Proposition. For any finite metric space M and any $G_1, G_2 \in MST(M)$ it holds $s(G_1) = s(G_2)$.

The vector from the above proposition is called the mst-spectrum of M and is denoted by $s(M)$.

Theorem [A.Tuzhilin]. For a finite metric space M, $s(M) = (s_1 , \ldots , s_{n-1})$, a positive integer $k \le n - 1$, and $\lambda \ge 2\,\mathrm{diam}\, M$ it holds

$$s_k = \lambda - 2\, d_{GH}(\lambda\Delta_{k+1} , M).$$

Recall that for an arbitrary metric space X and any its k-parts partition $D = \{ X_1, \ldots , X_k \} \in \mathcal{D}_k(X)$ we have defined

$$\alpha(D) = \inf\{ |xy| : x \in X_i, y \in X_j, i \ne j \}.$$

Now we put

$$\sigma_k(X) = \sup\{ \alpha(D) : D \in \mathcal{D}_{k+1}(X) \}.$$

Theorem [A.Ivanov, A.Tuzhilin (variational def. of mst edges lengths)]. If M is a metric space, $\#M = n$, $s(M) = (s_1 , \ldots , s_{n-1})$, then $s_k = \sigma_k(M)$.

For an arbitrary metric space X we define

$$\varepsilon(X) = \inf\{ |xy| : x, y \in X, x \ne y \}.$$

Remark. If $\#M = n$, $s(M) = (s_1 , \ldots , s_{n-1})$, then $s_{n-1} = \sigma_{n-1}(M) = \varepsilon(M)$.

Now, let

$$d_k(X) = \inf\{ \operatorname{diam} D : D \in \mathcal{D}_k(X) \}.$$

Remark. If $\#M = n$, then $d_n(M) = 0$, $d_{n-1}(M) = \varepsilon(M)$, $d_1(M) = \operatorname{diam} M$.

Theorem [A.Ivanov, A.Tuzhilin].

For any $X \in \mathcal{M}$, a positive integer $n \ge 2$, and $\lambda > 0$ we have

$2\, d_{GH}(\lambda\Delta_k , X) =$

- $\max\{\lambda, \operatorname{diam} X - \lambda\}$ for $\#X < k$,
- $\max\{\lambda - \sigma_{k-1}(X), \operatorname{diam} X - \lambda\}$ for $\#X = k$,
- $\max\{\sigma_{k-1}(X), \lambda - \sigma_{k-2}(X), \operatorname{diam} X - \lambda\}$ for $\#X = k + 1$,
- $\lambda - \sigma_{k-1}(X)$ for $\#X \ge k$ and $\lambda \ge \operatorname{diam} X + \sigma_{k-1}(X)$,
- $\operatorname{diam} X$ for $\#X \ge k$, $\lambda < \operatorname{diam} X + \sigma_{k-1}(X)$, and $d_k(X) = \operatorname{diam} X$,
- $\max\{d_k(X), \operatorname{diam} X - \lambda\}$ for $\#X \ge k$ and $\lambda \le (\operatorname{diam} X)/2$.

Remark. Indeed, it remained a partial lacuna for

$\#X \ge k$ and $(\operatorname{diam} X)/2 < \lambda < \operatorname{diam} X + \sigma_{k-1}(X)$.

Nontriviality of the distances caclulations.

Example. Let $M = \{1, 2, 3, 4\}$ and let metric be given by the matrix

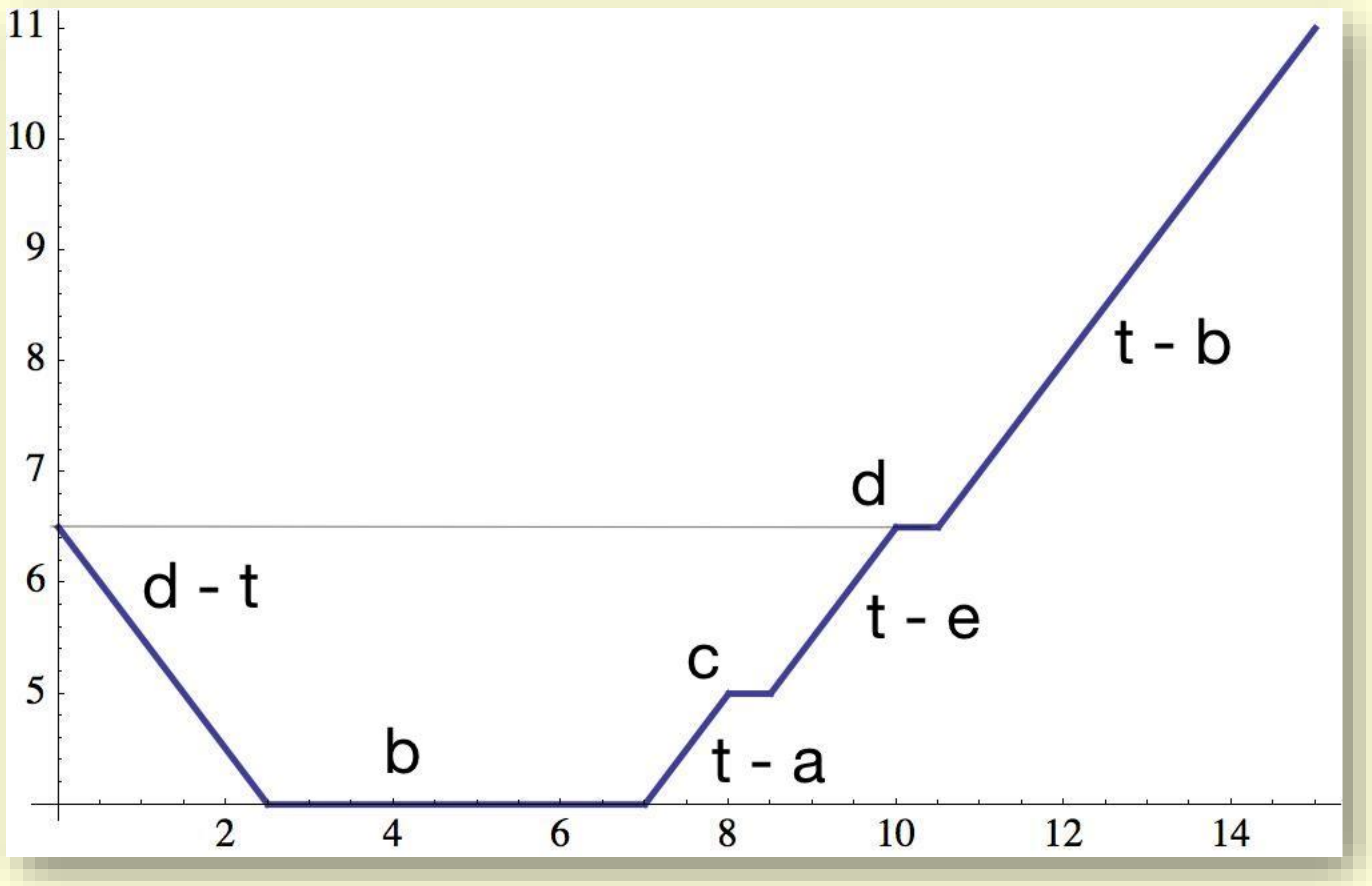
$$\begin{pmatrix} 0 & a & b & d \\ a & 0 & c & e \\ b & c & 0 & f \\ d & e & f & 0 \end{pmatrix}.$$

Let $a < e < b < c < f < d$ and $t > 0$, then

$$s(M) = (b, e, a),\ \operatorname{diam} M = d,\ d_2(M) = b,$$

$$2d_{GH}(t\,\Delta_2, M) = \min\{\max[d, t-b], \max[c, t-e], \max[b, t-a, d-t]\}.$$

Notice that the distance depends on c which does not belong to mst-spectrum, and is distinct from $\operatorname{diam} M$ and $d_2(M)$.



Remark. This c can be obtained in a dual object to minimum spanning trees.

Maximum Spanning Trees and Steiner Minimal Trees.

Let M be a finite metric space, then

$$xst(M) = \max\{\,|G| : G = (M, E) \text{ is a tree}\,\}.$$

$$XST(M) = \{\,G : G = (M, E) \text{ is a tree}, |G| = xst(M)\,\},$$

$G \in XST(M)$ is called a maximum spanning tree on M.

xst-spectrum of finite metric space.

Let $\#M = n$. For each $G \in XST(M)$ we define $S(G) = (S_1, \ldots, S_{n-1})$ to be the vector obtained from the lengths of edges of G by ordering them ascending.

Proposition. For any finite metric space M and any $G_1, G_2 \in XST(M)$ it holds $S(G_1) = S(G_2)$.

The vector from the above proposition is called the xst-spectrum of M and is denoted by $S(M)$.

Recall that for an arbitrary metric space X and any its k-parts partition $D = \{ X_1, \dots , X_k \} \in \mathcal{D}_k(X)$ we have defined

$$\beta(D) = \sup\{ |xy| : x \in X_i, y \in X_j, i \neq j \}.$$

Now we put

$$\Sigma_k(X) = \inf\{ \beta(D) : D \in \mathcal{D}_{k+1}(X) \}.$$

Compare with

$$\alpha(D) = \inf\{ |xy| : x \in X_i, y \in X_j, i \neq j \},$$
$$\sigma_k(X) = \sup\{ \alpha(D) : D \in \mathcal{D}_{k+1}(X) \}.$$

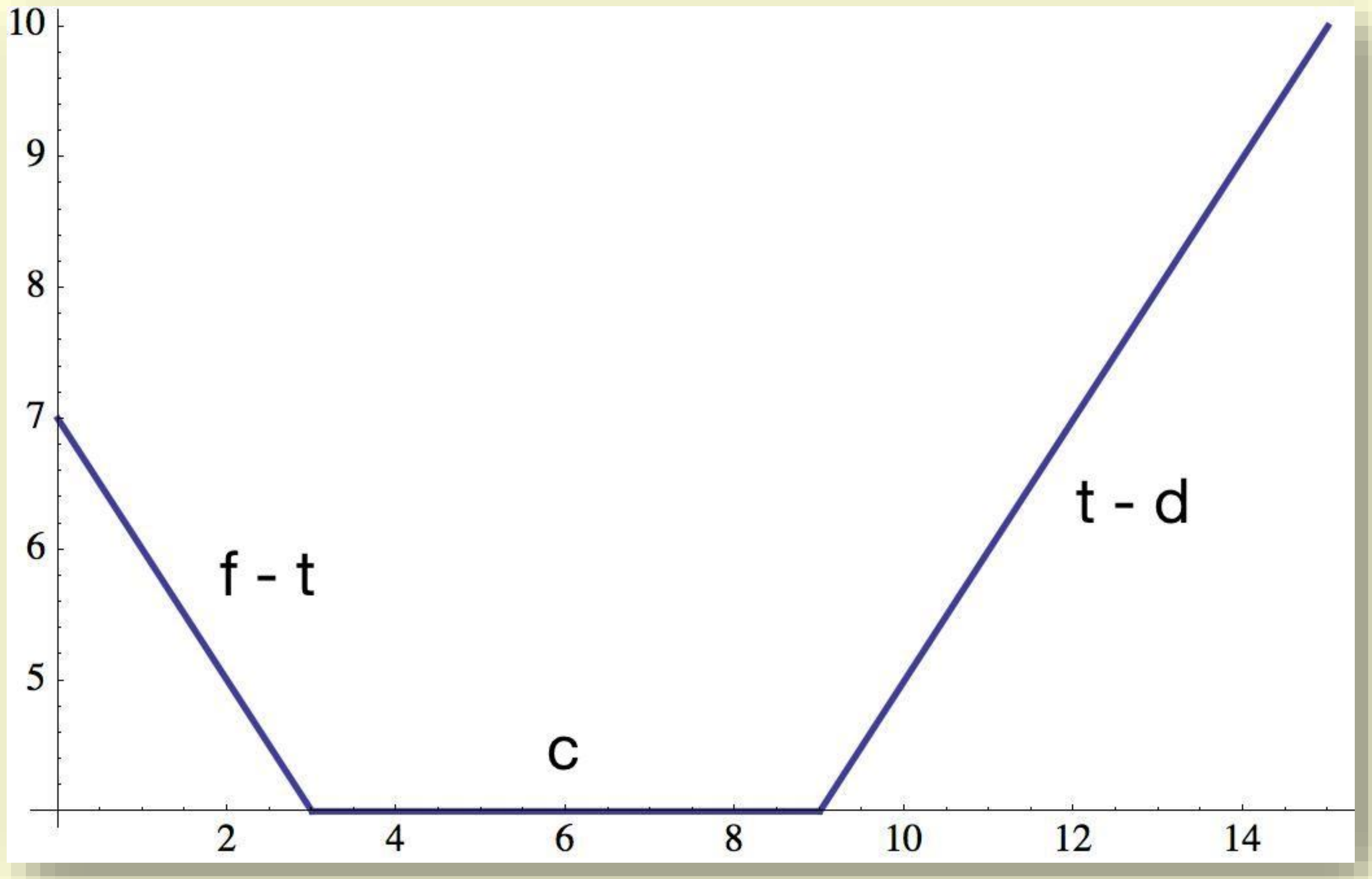
Theorem [A.Ivanov, A.Tuzhilin (variational def. of xst edges lengths)]. If M is a metric space, $\#M = n$, $S(M) = (S_1 , \dots , S_{n-1})$, then $S_k = \Sigma_k(M)$.

Remark. If $\#M = n$, $S(M) = (S_1 , \dots , S_{n-1})$, then $S_{n-1} = S_{n-1}(M) = \operatorname{diam} M$.

In the example above, $S(M) = (c, f, d)$, thus, in this example the formula for $2d_{GH}(t\,\Delta_2, M)$ uses only mst-spectrum and xst-spectrum.

Example. The same M and the distance matrix, but now $a < b < c < d < e < f$, then $s(M) = (d, b, a)$, $S(M) = (d, e, f)$,

$$2d_{GH}(t\,\Delta_2, M) = \max[c, t - d, f - t].$$



Notice that the distance depends on c, which does not belong to spectra. However, $d_2(M) = c$.

Some problems concerning Gromov-Hausdorff space.

1) Is it true that the spheres in $\mathcal{M}$ are linear connected?
2) Is it true that the balls in $\mathcal{M}$ are convex?
3) Describe topological type of $\{M \in \mathcal{M} : \mathrm{diam}\, M = \mathrm{const}\}$.
4) Describe maximal shortest curves.
5) Describe all compact metric spaces which have the same distances to all simplexes.
6) Is it possible to isometrically embed any countable (compact, Polish) metric space into $\mathcal{M}$?
7) Describe various kinds of isometries in $\mathcal{M}$.
8) It is true that any finite set of $\mathcal{M}$ can be joined by a Steiner minimal tree?
9) Describe those boundaries which Steiner minimal trees are minimal fillings.
10) Calculate Steiner subratio of $\mathcal{M}$.
11) Investigate the properties of the mapping $f : \mathcal{M} \to \mathcal{M}$, $X \to \mathcal{H}(X)$.

Thank You!