The Moutard transformation of two-dimensional Dirac operators and the Möbius geometry

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The Darboux transformation

$$H = -\frac{d^2}{dx^2} + u(x)$$

— one-dimensional Schrödinger operator.
Let

$$H\omega = 0.$$

The Darboux transformation is defined by a solution ω and maps H into the operator $\widetilde{H}$ and solutions of the equation

$$H\psi = E\psi$$

into solutions $\widetilde{\psi}$ of the equation

$$\widetilde{H}\widetilde{\psi} = E\widetilde{\psi}.$$

Every solution ω of the equation $H\omega = 0$ defines a factorization of H:

$$H = A^\top A, \quad A = -\frac{d}{dx} + v, \quad A^\top = \frac{d}{dx} + v, \quad v = \frac{\omega'}{\omega}.$$

The Darboux transformation of H consists in swapping $A^\top$ and A:

$$H = A^\top A \;\longrightarrow\; \widetilde{H} = AA^\top = -\frac{d^2}{dx^2} + \widetilde{u}(x),$$

where

$$u = v^2 + v' \to \widetilde{u} = v^2 - v'$$

and it acts on eigenfunctions as follows:

$$\psi \longrightarrow \widetilde{\psi} = A\psi.$$

The harmonic oscillator

Let $v = ax, a > 0$, then

$$v' = \mathrm{const} = a$$

and

$$AA^\top = 2H - a, \quad A^\top A = 2H + a,$$

where

$$H = \frac{1}{2}\left(-\frac{d^2}{dx^2} + a^2x^2\right)$$

is the harmonic oscillator operator. It follows from the commutation relation $[A^\top, A] = 2a$ that if

$$H\psi = E\psi,$$

then

$$H(A\psi) = (E + a)(A\psi), \quad H(A^\top\psi) = (E - a)(A^\top\psi).$$

Note that

$$(2E - a)(\psi, \psi) = (AA^{\top}\psi, \psi) = (A^{\top}\psi, A^{\top}\psi) \geq 0,$$

which implies

$$E \geq \frac{a}{2}.$$

The equality is attained on a solution of the equation

$$A^{\top}\psi = \left(\frac{d}{dx} + ax\right)\psi = 0,$$

which is up to a constant multiple equals

$$\psi_1(x) = e^{-\frac{ax^2}{2}}.$$

The basis of eigenfunctions has the form

$$\psi_N = A^{N-1}\psi_1, \quad N = 1, 2, 3, \ldots$$

with eigenvalues

$$\frac{a}{2} + (N-1)a.$$

The Moutard transformation

Let H be a two-dimensional potential Schrödinger operator and ω be a solution of the equation

$$H\omega = (-\Delta + u)\omega = 0,$$

where Δ is the two-dimensional Laplace operator: $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.
The Moutard transformation of H is defined as

$$\widetilde{H} = -\Delta + u - 2\Delta \log \omega = -\Delta - u + 2\frac{\omega_x^2 + \omega_y^2}{\omega^2}.$$

If ψ satisfies $H\psi = 0$, then the function θ, defined via the system

$$(\omega\theta)_x = -\omega^2 \left(\frac{\psi}{\omega}\right)_y, \quad (\omega\theta)_y = \omega^2 \left(\frac{\psi}{\omega}\right)_x,$$

satisfies $\widetilde{H}\theta = 0$.
For $u = u(x)$ and $\omega = f(x)e^{\kappa y}$, the the Moutard transformation reduces to the Darboux transformation defined by f.

The Moutard equation in surface theory

Originally the Moutard transformation was defined for the Moutard equation:

$$\psi_{xy} + U\psi = 0.$$

Let $r(x, y)$ be a surface in $\mathbb{R}^3$ with asymptotic coordinates x, y. This means that $(r_x, n_x) = (r_y, n_y) = 0$ or

$$r_x = \lambda n_x \times n, \quad r_y = \mu n \times n_y.$$

The compatibility condition $r_{xy} = r_{yx}$ implies that $\lambda = \mu$. Put

$$\psi = n\sqrt{\lambda}.$$

Then it is derived by differentiation that $\psi_{xy} \times \psi = 0$, which implies

$$\psi_{xy} + U\psi = 0.$$

Converse: any vector-valued solution to the latter equation defines a negatively-curved surface in $\mathbb{R}^3$ with asymptotic coordinates x, y.

Blowing up solutions of the Novikov–Veselov equation (T.–Tsarev, 2007)

The Novikov–Veselov equation:

$$U_t = \partial^3 U + \bar{\partial}^3 U + 3\partial(UV) + 3\bar{\partial}(\bar{V}U) = 0,$$

$$\bar{\partial} V = \partial U.$$

The one-dimensional reduction

$$U = U(x), \quad U = V = \bar{V}$$

leads to the Korteweg–de Vries equation

$$U_t = \frac{1}{4} U_{xxx} + 6UU_x.$$

The Novikov–Veselov equation is the compatibility condition for the system

$$\begin{aligned} H\psi = (\partial\bar{\partial} + U)\psi = 0, \\ \partial_t\psi = -A\psi = (\partial^3 + \bar{\partial}^3 + 3V\partial + 3\bar{V}\bar{\partial})\psi \end{aligned} \tag{1}$$

and is represented by a "Manakov triple" of the form

$$H_t = [H, A] + BH.$$

Equations represented by such triples preserve the "spectrum on the zero energy level" deforming "eigenfunctions" via

$$(\partial_t + A)\psi = 0.$$

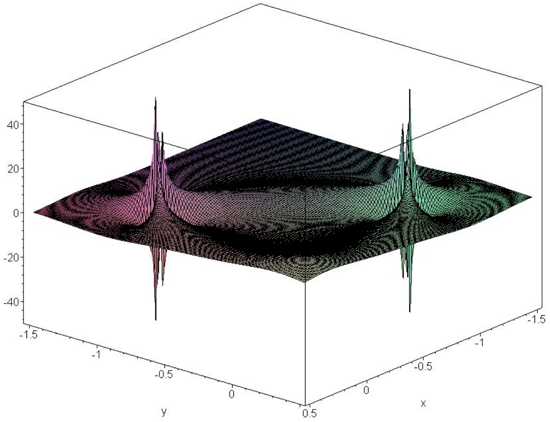
The Moutard transformation is extended to a transformation of solutions of the NV equation.

By applying this construction we construct and a solution

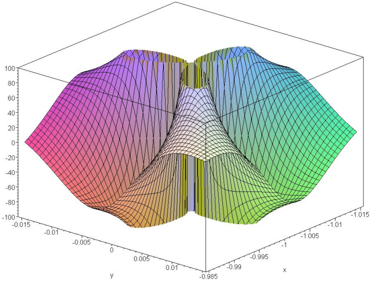
$$U = 2\partial\bar{\partial}\log\Phi, \quad V = 2\partial^2\log\Phi,$$

$$\Phi(x, y, t) = 3(x^2 + y^2) + 4(x^3 + y^3) + 30 - 12t.$$

It decays as r^{-3}, is nonsingular for $0 \leq t < T_* = \frac{29}{12}$ and is singular for $t \geq T_* = \frac{29}{12}$.



Pic. 1. The potential U as $t \to \frac{29}{12}$.



Pic. 2. The potential U at $t = \frac{29}{12}$ near the point $(-1, 0)$.

The two-dimensional Dirac operator

$$\mathcal{D} = \begin{pmatrix} 0 & \partial \\ -\bar{\partial} & 0 \end{pmatrix} + \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix},$$

$\partial = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$, $\bar{\partial} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$, and $U = \bar{U}$.
To every solution

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

of the Dirac equation

$$\mathcal{D}\psi = 0$$

we correspond a matrix-valued (or quaternion-valued) solution

$$\Psi = \begin{pmatrix} \psi_1 & -\bar{\psi}_2 \\ \psi_2 & \bar{\psi}_1 \end{pmatrix}$$

The Moutard type transformation (Yu–Liu–Wang, 2001)

For every pair Ψ and Φ of H-valued functions correspond a 1-form ω

$$\omega(\Phi, \Psi) = \Phi^\top \Psi dy - i\Phi^\top \sigma_3 \Psi dx =$$

$$-\frac{i}{2}\left(\Phi^\top \sigma_3 \Psi + \Phi^\top \Psi\right) dz - \frac{i}{2}\left(\Phi^\top \sigma_3 \Psi - \Phi^\top \Psi\right) d\bar{z}$$

and a matrix-valued function

$$S(\Phi, \Psi)(z, \bar{z}, t) = \Gamma \int_0^z \omega(\Phi, \Psi), \quad \Gamma = i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

which is defined up to constant matrices $C \in su(2)$ formed by integration constants.

Given a solution Ψ_0 of the Dirac equation,

$$K(\Psi_0) = \Psi_0 S^{-1}(\Psi_0, \Psi_0)\Gamma\Psi_0^\top \Gamma^{-1} = \begin{pmatrix} iW & a \\ -\bar{a} & -iW \end{pmatrix}, \quad W = \bar{W},$$

and for every solution Ψ of the Dirac equation the function

$$\widetilde{\Psi} = \Psi - \Psi_0 S^{-1}(\Psi_0, \Psi_0) S(\Psi_0, \Psi)$$

satisfies the equation

$$\widetilde{\mathcal{D}}\widetilde{\Psi} = 0$$

for the Dirac operator $\widetilde{\mathcal{D}}$ with potential

$$\widetilde{U} = U + W$$

The modified Novikov–Veselov (mNV) equation

This transformation is extended to transformations of solutions of the mNV equation

$$U_t = \left(U_{zzz} + 3U_z V + \frac{3}{2} U V_z\right) + \left(U_{\bar{z}\bar{z}\bar{z}} + 3U_{\bar{z}} \bar{V} + \frac{3}{2} U \bar{V}_{\bar{z}}\right),$$

where

$$V_{\bar{z}} = (U^2)_z.$$

This equation takes the form of Manakov triple:

$$\mathcal{D}_t + [\mathcal{D}, A] - B\mathcal{D} = 0$$

and for $U = U(x), V = U^2$ reduces to the mKdV equation.

The Weierstrass representation of surfaces.I

Let

$$r : \mathcal{U} \to \mathbb{R}^3, \quad \mathcal{U} \subset \mathbb{C},$$

be a conformal immersion of a domain $\mathcal{U}$ into $\mathbb{R}^3$,

$$ds^2 = e^{2\alpha(z,\bar{z})} dz\, d\bar{z}.$$

The conformality condition reads

$$\left(\frac{\partial x^1}{\partial z}\right)^2 + \left(\frac{\partial x^2}{\partial z}\right)^2 + \left(\frac{\partial x^3}{\partial z}\right)^2 = 0,$$

where $(x^1, x^2, x^3) \in \mathbb{R}^3$. The points of this quadric in $\mathbb{C}^3$ are parameterized by pairs $(\psi_1, \bar{\psi}_2) \in \mathbb{C}^2$ as follows:

$$\frac{\partial x^1}{\partial z} = \frac{i}{2}(\psi_1^2 + \bar{\psi}_2^2), \quad \frac{\partial x^2}{\partial z} = \frac{1}{2}(\bar{\psi}_2^2 - \psi_1^2), \quad \frac{\partial x^3}{\partial z} = \psi_1 \bar{\psi}_2.$$

The function $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ defines a surface via these formulas if and only if it satisfies the Dirac equation with a real-valued potential U.

The Weierstrass representation of surfaces.II

Given any solution ψ of the Dirac equation we construct a surface

$$x^1(P) = \frac{i}{2}\int_{P_0}^{P} \left((\psi_1^2 + \bar{\psi}_2^2)dz - (\bar{\psi}_1^2 + \psi_2^2)d\bar{z}\right) + x^1(P_0),$$

$$x^2(P) = \frac{1}{2}\int_{P_0}^{P} \left((\bar{\psi}_2^2 - \psi_1^2)dz + (\psi_2^2 - \bar{\psi}_1^2)d\bar{z}\right) + x^2(P_0),$$

$$x^3(P) = \int_{P_0}^{P} \left(\psi_1\bar{\psi}_2 dz + \bar{\psi}_1\psi_2\bar{z}\right) + x^3(P_0).$$

$$e^\alpha = |\psi_1|^2 + |\psi_2|^2, \quad U = \frac{e^\alpha H}{2},$$

where H is the mean curvature of the surface, and the Willmore functional is

$$\mathcal{W} = 4\int U^2\, dx \wedge dy$$

The Moutard transformation and the Möbius inversion (T., 2014)

Let Ψ_0 and $C = r(P_0) \in su(2)$ define a surface with potential U via the Weierstrass representation. Then

$$S(\Psi_0, \Psi_0)(P) = \int_{P_0}^{P} d \begin{pmatrix} ix^3 & -x^1 - ix^2 \\ x^1 - ix^2 & -ix^3 \end{pmatrix} + C,$$

i.e., $S(\Psi_0, \Psi_0)$ is this surface realized in $\mathbb{R}^3 = su(2)$.
Let $\widetilde{S}$ be a surface obtained from S by the inversion

$$S \to S^{-1}.$$

Then $\widetilde{S}$ is defined by the spinor

$$\widetilde{\Psi}_0 = \Psi_0 S^{-1}$$

via the Weierstrass representation and $\widetilde{\Psi}_0$ meets the Dirac equation with the potential $\widetilde{U} = U + W$.

The blowing up solution of the mNV equation (T., 2014)

Let us consider a stationary solution $U = V = 0$ and construct a new solution by using the spinor $\psi_1 = z, \psi_2 = 1$ (the Enneper surface). We obtain a solution

$$\widetilde{U}(x,y,t) = -\frac{3((x^2+y^2+3)(x^2-y^2)-6x(C-t))}{Q(x,y,t)},$$

$$Q(x,y,t) = (x^2+y^2)^3 + 3(x^4+y^4) + 18x^2y^2 + 9(x^2+y^2) + \quad (2)$$
$$+9(C-t)^2 + (6x^3 - 18xy^2 - 18x)(C-t),$$

such that

- *it is really-analytical everywhere outside a single point* $x = y = 0, t = C = \mathrm{const}$ *at which it has different finite limits along the rays* $x/y = \mathrm{const}, t = C$*;*
- *in all planes* $t = \mathrm{const}$ *it decays as* $O(1/r^2)$*, and has finite* L_2*-norms;*
- *the conservation law* $\int_{\mathbb{R}^2} \widetilde{U}^2 dx\, dy$ *is equal to* 3π *for* $t \neq C$ *and jumps to* 2π *for* $t = C$*.*