Upgrading axiomatic system to *n* - dimensional space

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Historical introduction (dimension 3)

- Mathematic, or more precisely geometry was the first science where was developed formal approach.
- It is well known Euclid's axiomatic system, form 5th century B.C.
- More details about developing axiomatic system is given in:
- https://en.wikipedia.org/wiki/Euclidean_geometry
- https://en.wikipedia.org/wiki/Axiomatic_system
- Z. Lučić: Ogledi iz istorije antičke geometrije, Službeni glasnik, 2009.

Historical introduction (dimension *n*)

- The idea of the higher-dimensional space was expressed by I. Kant (1746), while J. d'Alembert (1764) wrote on attaching to space to time as a fourth coordinate.

(form: Encyclopaedia of Mathematics, Volume 4, Kluwer Academic Publishers, 1989.)

- The first paper dealing explicitely with geometry of *n* dimensions was one by A. Cayley in 1843. Importance of the subject was recognised
- by *British* matematicians W.K. Clifford and J.J. Sylvester.
- In *Germany*: H.Grassmann (1844.), V. Schlegel
- L. Schläfli – Swiss mathematician (1852.)

References

1. L. Schläfli: Theorie der vielfahen Kontinuität, 1901.
2. D. Hilbert: Osnove geometrije, Matematički Institut, Beograd, 1957. (Grundlagen der Geometrie, 1899.)
3. D.M.Y. Somerville: An introduction to the geometry of *n* dimensions, London, 1929.
4. H.S.M. Coxeter: Regular Polytopes (3rd edition), Dover, New York, 1973.
5. T.F. Banchoff: Beyond the third dimensions, The Scientific American Library, New York, 1990.
6. D. Lopandić: Geometrija za III razred usmerenog obrazovanja, Naučna knjiga, 1979.
7. Z. Lučić: Euklidska i hiperbolička geometrija, Grafiti i Matematički fakultet, 1994.

[3] Somerville: An introduction ... 1929.

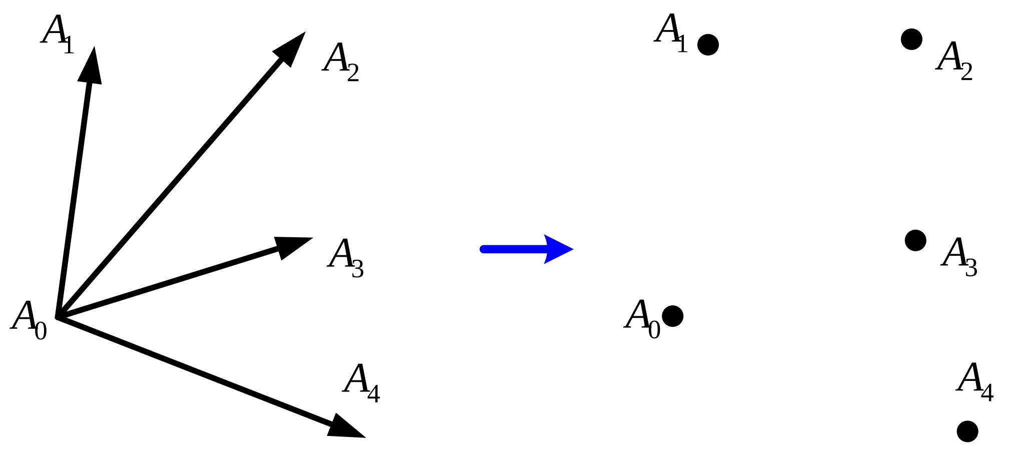
- In the beginning geometry of *n*-dimensional spaces was considered dominantly by analytical (vector) method. *Somerville* gave geometrical point of view.

10. Extension to n Dimensions. We may now extend these ideas straightaway to n dimensions, and at the same time acquire both greater generality and greater succinctness in expression. The series point, line, plane, hyperplane (or as it is more explicitly termed *three-flat*), . . ., n-flat are regions determined by one, two, three, four, . . ., $n + 1$ points, and having zero, one, two, three, . . ., n dimensions, i.e. an r-flat is determined by $r + 1$ points, and every p-flat ($p < r$) which is determined by $p + 1$ of these points lies entirely in the r-flat. We shall suppose that the n-flat, or space of n dimensions, contains all the points. A p-flat, or hyperplane of p dimensions will be denoted by S_p. A flat space is also called a *linear space*.

[3] Somerville: An introduction ... 1929.

... A flat space is also called a *linear space.*

11. Independent Points. If $p + 1$ points uniquely determine a p-flat they must not be contained in the same $(p - 1)$-flat. Also no r of them $(r \gtrless p)$ must be contained in the same $(r - 2)$-flat, for this $(r - 2)$-flat, which is determined by $r - 1$ points, together with the remaining $p + 1 - r$ points, would determine a $(p - 1)$-flat. We shall call a system of $p + 1$ points, no r of which lie in the same $(r - 2)$-flat, a system of *linearly independent* points. Any $p + 1$ points of a p-flat, if they are linearly independent, can be chosen to determine the p-flat.



[6] Lopandić: Geometrija ... 1979.

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Motivation for investigation

- *Theorem*. Every four vectors in space are linearly dependent.
- *Question*: Which postulate from axiomatic system not allow to space to have dimension more the 3?
- Answer have to be somewhere inside of *axioms of incidence*, since in other groups of axioms dimension is either at least 2 or it is not mentioned.

Axioms of incidence:

https://www.imsc.res.in/~kapil/geometry/euclid/node2.html

- Incidence between points and lines:
  - There are at least two distinct points.
  - There is one and only one line that contains two distinct points.
  - Every line contains at least two distinct points.
- Incidence between points and planes:
  - There are three points that do not all lie on the same line.
  - For any three points that do not lie on the same line there is a one and only one plane that contains them.
  - Any plane contains at least three points.
- Incidence between lines and planes:
  - If a line lies on a plane then every point contained in the line lies on that plane.
  - If a line contains two points which lie on a plane then the line lies on the plane.
- Dimensionality of space:
  - If two planes both contain a point then they also contain a line.
  - There are at least four points that do not all lie on the same plane.

[7] Lučić: Euklidska i ... 1994. - *axioms of incidence*

- Incidence between points and lines:

  I.1 Every line contains at least two distinct points.

  I.2 There is at least one line that contains two points.

  I.3 There is at most one line that contains two distinct points.

- Incidence between points and planes:

  I.4 Every plane contains at least three non-collinear points. (**Def:** *collinear points*: points that lie on the same line; collinear ≡ 1-linearly dependent ; non-collinear ≡ 1-linearly independent)

  I.5 There is at least one plane that contains three points.

  I.6 There is at most one plane that contains three non-collinear points.

- Incidence between lines and planes:

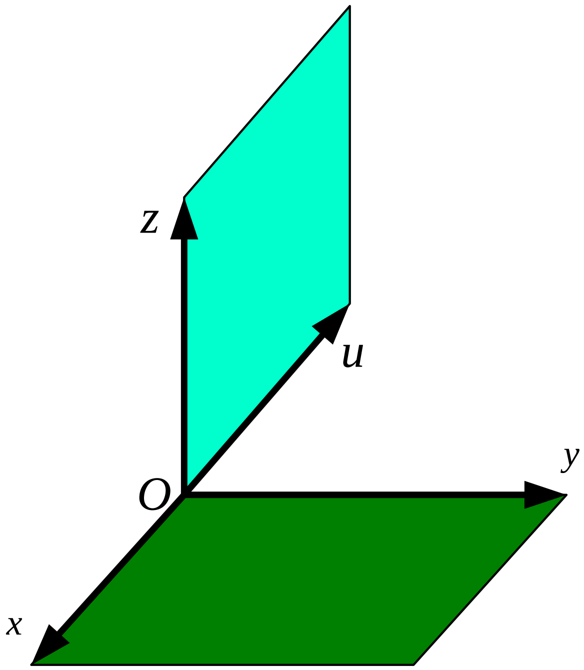
  I.7 If a line contains two distinct points which lie on a plane, then each point from that line lies in the plane. (**Def:** A line lie in a plane if all points lying in that line lies in the plane.)

- Dimensionality of space:

  I.8 If two distinct planes contain a common point, then they contain at least one other common point.

  I.9 There are at least four non-coplanar points. (**Def:** *coplanar points*: points that lie on the same plane; coplanar ≡ 2-linearly dependent ; non- coplanar ≡ 2-linearly independent)

Postulate **I.8** in 4-dimensional space



$$xOy \cap zOu = \{O\}$$

I.8’: If two distinct planes *lying in the same hyperplane*, contain a common point, then they contain at least one other common point.

Def: A plane (a line) lie in a hyperplane if all points lying in that plane (that line) lies in the plane.

I.8*: If some hyperplane and some plane contain a common point, then they contain at least one other common point.

Axiomatic system in dimension 4

I.1 Every line contains at least two distinct points.

I.2 There is at least one line that contains two points.

I.3 There is at most one line that contains two distinct points.

I.4 Every plane contains at least three non-collinear points.

I.5 There is at least one plane that contains three points.

I.6 There is at most one plane that contains three non-collinear points.

I.4* Every hyperplane contains at least four non-coplanar points.

I.5* There is at least one hyperplane that contains four points.

I.6* There is at most one hyperplane that contains four non-coplanar points.

I.7 If a line contains two distinct points which lie on a plane, then each point from that line lies in the plane.

I.7* If a line contains two distinct points which lie on a hyperplane, then each point from that line lies in the hyperplane.

I.8' If two distinct planes lying in the same hyperplane, contain a common point, then they contain at least one other common point.

I.8* If some hyperplane and some plane contain a common point, then they contain at least one other common point.

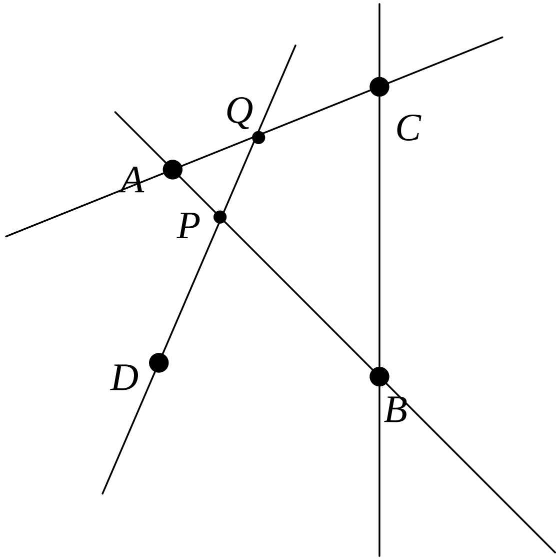
I.9' There are at least five 3-linearly independent points. (**Def:** *3-linearly dependent points*: points that lie on the same hyperplane)

Theorem 1. If a plane contains three non-collinear points which lie on a hyperplaneplane, then each point from that plane lies in the hyperplaneplane.

Proof. Let A, B, C be non-collinear points which lie on a plane α and hyperplane Π;

- By I.7 each point E lying on lines AB, AC, BC also lies on Π.
- Let D be any point from α which is not lying on AB, AC, BC and P be any point lying on AB such that $B(A, P, B)$. Such P exists by Th. 2.4 from [7] (consequence of axioms of order).

1. $C \in DP$: There are two points P and C of DP lying on $\alpha \Rightarrow$ all points from DP are lying on $\Pi \Rightarrow D$ is lying on Π.
2. $C \notin DP$: By the Pasch's postulate ((plane) axiom of order - II.7 in [7]) DP either intersect line AC in Q such that $B(A, Q, C)$ or BC in R such that $B(B, R, C)$. Then by I.7 points P, Q or P, R are two points from DP are lying on α and consequently on $\Pi \Rightarrow D$ is lying on Π.



Axiomatic system in dimension *n*

I(*n*).1 For each $m \leq n$-1, every *m*-flat contains at least *m*+1 *m*-linearly independent points.

I(*n*).2 For each $m \leq n$-1, there is at least one *m*-flat that contains *m*+1 points.

I(*n*).3 For each $m \leq n$-1, there is at most one *m*-flat that contains *m*+1 *m*-linearly independent points.

I(*n*).4 For each $m \leq n$-1, if a line contains two distinct points which lie on a *m*-flat, then each point from that line lies in the *m*-flat.

I(*n*).5 For each $m \leq n$-1, if some *m*-flat and some 2-flat lying in the same (*m*+1)-flat, contain a common point, then they contain at least one other common point.

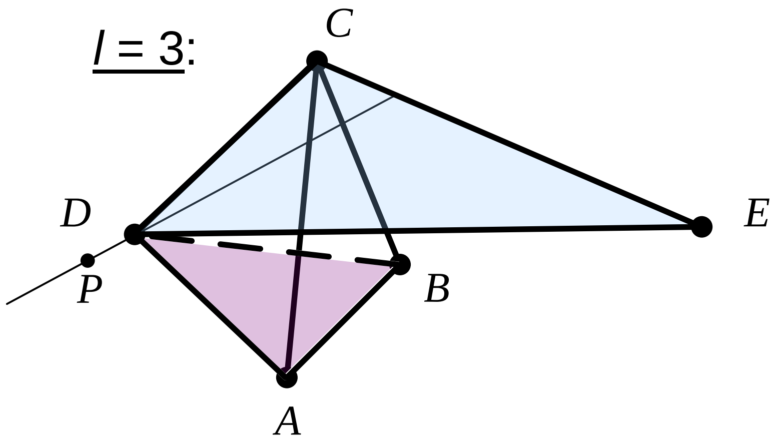
I(*n*).6 There are at least *n*+1 (*n*-1)-linearly independent points.

Def: *m-linearly dependent points*: points that lie on the same *m*-flat.

Theorem 2. For each $l < m \leq n$-1, if a l-flat π contains l+1 l -linearly independent points which lie on a m-flat Π, then each point from that l-flat lies in the m-flat.

Proof. By the mathematical induction on l.

<u>l = 3</u>:



α: A, B, D; β: D, C, E

$C \notin \alpha$

$D \in \alpha \cap \beta \Rightarrow \exists\ P \neq D,\ P \in \alpha \cap \beta$

$D, C, P \in \pi$, $\Pi \Rightarrow \underline{E \in \Pi}$

Theorem 3. For each l, $m \leq n$-1, if some m-flat π and some l-flat α lying in the same (m+l-1)-flat Π, contain a common point A, then they contain at least one other common point B.

Proof. Let P, $Q \in \alpha$ and A, P, Q are non-coplanar. If no one of P, Q are in intersection $\alpha \cap \pi$, then there exists 2-flat β which contain A, P, Q. By I(n).5 and Theorem 2, $\beta \subseteq \alpha$, $\exists B \neq A$, $B \in \beta \cap \pi \Rightarrow B \in \alpha \cap \pi$.

Way we need such axiomatic system?

- Historical resons
- Investigation of Euclidean and hyperbolic geometry
- By vectors we can define only direct isometry transformations.

For the further work

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Thank you for your attention!