Liouville-type theorems for twisted and warped products and their applications

Stepanov Sergey
Finance University
under the Government of Russian Federation
e-mail: ***s.e.stepanov@mail.ru***

XIX Geometrical Seminar
August 28-September 4, 2016
Zlatibor, Serbia

Introduction

In the present report we will prove Liouville-type non-existence theorems for complete twisted and warped products of Riemannian manifolds which generalize similar results for compact manifolds (see [PR] and [ON, p. 205-211]). To do this, we will use a generalization of the Bochner technique (see [P]).

[PR] Ponge R., Reckziegel H., Twisted products in pseudo-Riemannian geometry, Geom. Dedic., 48:1 (1993), 15-25.

[ON] O'Neill B., Semi-Riemannian geometry with applications to relativity, Academic Press, San Diego, 1983.

[P] Pigola S., Rigoli M., Setti A.G., Vanishing and Finiteness Results in Geometric Analysis. A Generalization of the Bochner Technique, Birkhäuser Verlag AG, Berlin, 2008.

1. Double-twisted products and twisted products

The double-twisted product ${}_{\lambda_1}M_1 \times_{\lambda_2} M_2$ of the Riemannian manifolds (M_1, g_1) and (M_2, g_2) is the manifold $M = M_1 \times M_2$ equipped with the Riemannian metric $g = \lambda_1^2 g_1 \oplus \lambda_2^2 g_2$ where the strictly positive functions $\lambda_1 : M_1 \times M_2 \to \mathbb{R}$ and $\lambda_2 : M_1 \times M_2 \to \mathbb{R}$ are called twisted functions.

For $\lambda_1 = 1$ and $\lambda_1 = \lambda_2 = 1$ we have the twisted product $M_1 \times_{\lambda_2} M_2$ and the direct product $(M_1 \times M_2,\ g_1 \oplus g_2)$, respectively.

The manifold ${}_{\lambda_1}M_1 \times_{\lambda_2} M_2$ carries two orthogonal complementary totally umbilical foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ with the mean curvature vectors $\xi_1 = -\pi_{2*}(grad \log \lambda_1)$ and $\xi_2 = -\pi_{1*}(grad \log \lambda_2)$ for the natural projection $\pi_{i*} : T(M_1 \times M_2) \to TM_i$ (see [S1] and [S2]).

We have proved in [S1] and [S2] the following relation

$$div\ (\xi_1 + \xi_2) = -s_{\text{mix}} + \frac{m-1}{m}\|\xi_1\|^2 + \frac{n-m-1}{n-m}\|\xi_2\|^2 \qquad (*)$$

where $m = \dim \mathcal{F}_1$ and $n - m = \dim \mathcal{F}_2$.

[S1] Stepanov S.E., A class of Riemannian almost-product structures, Soviet Mathematics (Izv. VUZ), 33:7 (1989), 51-59.

[S2] Stepanov S.E., Riemannian almost product manifolds and submersions, Journal of Mathematical Sciences (NY), 99:6 (2000), 1788-1831.

In the above formula s_{mix} denotes the mixed scalar curvature of ${}_{\lambda_1}M_1 \times_{\lambda_2} M_2$ which defined as the scalar function

$$s_{\text{mix}} = \sum_{a=1}^{m} \sum_{\alpha=m+1}^{n} \sec(E_a, E_\alpha)$$

where $\sec(E_a, E_\alpha)$ is the mixed sectional curvature in direction of the two-plane $\pi = \operatorname{span}\{E_a, E_\alpha\}$ for the local orthonormal frames $\{E_1, \ldots, E_m\}$ and $\{E_{m+1}, \ldots, E_n\}$ tangent to $\mathcal{F}_1$ and $\mathcal{F}_2$ (see [R]), respectively.

[R] Rocamora A.H., Some geometric consequences of the Weitzenböck formula on Riemannian almost-product manifolds; weak-harmonic distributions, Illinois Journal of Mathematics, 32:4 (1988), 654-671.

We recall here a generalized Green's divergence theorem.

Proposition (see [CSC]; [C]). *Let X be a smooth vector field on a connected complete and oriented Riemannian manifold* (M, g), *such that the norm* $\| X \| \in L^1(M, g)$. *If* $div\, X \geq 0$ (*or* $div\, X \leq 0$) *everywhere on* (M, g)*, then* $div\, X = 0$.

[CSC] Caminha A., Souza P., Camargo F., Complete foliations of space forms by hypersufaces, Bull. Braz. Math. Soc., New Series, 41:3 (2010), 339-353.

[C] Caminha A., The geometry of closed conformal vector fields on Riemannian spaces, Bull. Braz. Math. Soc., New Series, 42:2 (2011), 277-300.

At the same time, for $s_{\mathrm{mix}} \leq 0$ from the above formula we obtain

$$div\ \left(\xi_1 + \xi_2\right) \geq 0.$$

If we assume that ${}_{\lambda_1}M_1 \times_{\lambda_2} M_2$ is a connected complete and oriented manifold and $\|\xi_1 + \xi_2\| \in L^1(M, g)$ then

$$div\ \left(\xi_1 + \xi_2\right) = 0$$

by the above proposition. In this case, from the above formula we obtain the equalities $\xi_1 = \xi_2 = 0$.

Then the following Liouville-type non-existence theorem holds. It generalizes a similar theorem for compact double-twisted products Riemannian manifolds (see [NR]).

Theorem 1. *Let* ${}_{\lambda_1}M_1 \times_{\lambda_2} M_2$ *be a connected complete and oriented* double-twisted product *of some Riemannian manifolds* (M_1, g_1) *and* (M_2, g_2). *If its mixed scalar curvature* s_{mix} *is non-positive and*

$$\left\| \pi_{2*}(grad \log \lambda_1) + \pi_{1*}(grad \log \lambda_2) \right\| \in L^1(M, g),$$

then the twisted functions λ_1 *and* λ_2 *are positive constants* C_1 *and* C_2, *respectively, and therefore,* (M, g) *is the direct product* $(M_1 \times M_2, \bar{g}_1 \oplus \bar{g}_2)$ *for* $\bar{g}_1 = C_1^2 g_1$ *and* $\bar{g}_1 = C_2^2 g_2$.

[NR] Naveira A.M., Rocamora A.H., A geometrical obstruction to the existence of two totally umbilical complementary foliations in compact manifolds, Differential Geometrical Methods in Mathematical Physics, Lecture Notes in Mathematics 1139 (1985), 263-279.

If ${}_{\lambda_1}M_1 \times_{\lambda_2} M_2$ is a Cartan-Hadamard manifold (see [P, p. 90]), i.e. a complete, noncompact simply connected Riemannian manifold of nonpositive sectional curvature, then we have

Corollary 1. *If a Cartan-Hadamard manifold* (M, g) *is a doubly twisted product* ${}_{\lambda_1}M_1 \times_{\lambda_2} M_2$ *such that*

$$\left\| \pi_{2*}(grad \log \lambda_1) + \pi_{1*}(grad \log \lambda_2) \right\| \in L^1(M, g),$$

then the twisted functions λ_1 *and* λ_2 *are positive constants* C_1 *and* C_2*, respectively, and therefore,* (M, g) *is the direct product* $(M_1 \times M_2, \bar{g}_1 \oplus \bar{g}_2)$ *for* $\bar{g}_1 = C_1^2 g_1$ *and* $\bar{g}_1 = C_2^2 g_2$*.*

For a twisted product $M_1 \times_{\lambda_2} M_2$ the foliation $\mathcal{F}_1$ is totally geodesic and therefore, the following theorem is a corollary of the theorem in [BW] where consider complete Riemannian manifold with two orthogonal complementary foliations one of which has a totally geodesic and geodesically complete leaves.

Theorem 2. *If a twisted product* $M_1 \times_{\lambda_2} M_2$ *is a complete and simply connected Riemannian manifold and its mixed sectional curvature is nonnegative then it is isometric to a direct product* $M_1 \times M_2$.

[BW] Brito F., Walczak P.G., Totally geodesic foliations with integrable normal bundles, Bol. Soc. Bras. Mat., 17:1 (1986), 41-46.

2. Twisted products and projective submersions

Let (M, g) and $(\overline{M}, \overline{g})$ be Riemannian manifolds of dimension n and m such that $n > m$. A surjective map $f:(M,g)\to(\overline{M},\overline{g})$ is a projective submersion if it has maximal rank m at any point x of M and if for an arbitrary geodesic γ in (M, g) its image $f(\gamma)$ is a geodesic in $(\overline{M}, \overline{g})$ too (see [S2]).

If $f:(M,g)\to(\overline{M},\overline{g})$ is a projective submersion then (M, g) carries two orthogonal complementary totally geodesic and totally umbilical foliations $Ker\, f_*$ and $(Ker\, f_*)^{\perp}$, respectively (see [S2]).

[S2] Stepanov S.E., Riemannian almost product manifolds and submersions, Journal of Mathematical Sciences (NY), 99:6 (2000), 1788-1831.

So, if $f:(M,g)\to(\overline{M},\overline{g})$ is a projective submersion then (M, g) is a locally isometric to a twisted product $M_1 \times_{\lambda_2} M_2$. The converse is also true (see [S3]).

Theorem 3. *Let* $M_1 \times_{\lambda_2} M_2$ *be a twisted product of some Riemannian manifolds* (M_1, g_1) *and* (M_2, g_2). *Then the natural projection* $\pi_2 : M_1 \times M_2 \to M_2$ *is a local projective submersion from* $M_1 \times_{\lambda_2} M_2$ *to* $(M_2, \overline{g}_2)$ *for* $\overline{g}_2 = \lambda_2^2 g_2$.

[S3] Stepanov S.E., On the global theory of projective mappings, Mathematical Notes, 58:1 (1995), 752-756.

Then the following theorem is a corollary of our Theorem 2.

Theorem 3. *Let* (M, g) *is a simply connected complete Riemannian manifold and* $f:(M,g)\to(\bar{M},\bar{g})$ *be a projective submersion onto another* m*-dimensional* $(m < n)$ *Riemannian manifold* $(\bar{M},\bar{g})$. *If the foliations* $Ker\, f_*$ *has geodesically complete leaves, then* (M, g) *is isometric to a twisted product* $M_1 \times_{\lambda_2} M_2$ *such that the leaves of* $Ker\, f_*$ *and* $(Ker\, f_*)^{\perp}$ *correspond to the canonical foliations of* $M_1 \times M_2$.

Another statement follows directly from our Theorem 1.

Corollary 2. *Let (M, g) be an n-dimensional complete and simply connected Riemannian manifold with non-negative sectional curvature. If (M, g) admits a projective submersion $f:(M,g)\to(\overline{M},\overline{g})$ onto another m-dimensional ($m < n$) Riemannian manifold $(\overline{M},\overline{g})$. then it is isometric to a direct product $(M_1 \times M_2,\ g_1 \oplus g_2)$ of some Riemannian manifolds (M_1, g_2) and (M_1, g_2).*

3. Double warped products and warped products

A double-warped product manifold (M, g) is a double-twisted product manifold ${}_{\lambda_1}M_1 \times_{\lambda_2} M_2$ where $\lambda_1 : M_2 \to \mathbb{R}$ and $\lambda_2 : M_1 \to \mathbb{R}$ are positive smooth functions (see [U]).

In this case, the mean curvature vectors of the orthogonal complementary foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ have the forms

$$\xi_1 = -grad \log \lambda_1 \qquad \text{and} \qquad \xi_2 = -grad \log \lambda_2 .$$

[U] Ünal B., Doubly warped products, Differential Geometry and its Applications, 15 (2001), 253-263.

Then the formula $(*)$ can be rewriten in the form

$$\Delta \log(\lambda_1 \lambda_2) = -s_{\mathrm{mix}} + \frac{m-1}{m} \| grad \log \lambda_1 \|^2 + \frac{n-m-1}{n-m} \| grad \log \lambda_2 \|^2 .$$

Therefore, if $s_{\mathrm{mix}} \leq 0$ then from the above formula we obtain $\Delta \log(\lambda_1 \lambda_2) \geq 0$, i.e. $\log(\lambda_1 \lambda_2)$ is a subharmonic function.

We recall that on a complete Riemannian manifold (M, g) each subharmonic function $f : M \to \mathbb{R}$ whose gradient has integrable norm on (M, g) must actually be harmonic, i.e. $\Delta f = 0$ (see [Y]).

[Y] Yau S.T., Some function-theoretic properties of complete Riemannian manifolds and their applications to geometry, Indiana Univ. Math. J., 25 (1976), 659-670.

Then we conclude that the following theorem holds..

Theorem 3. *Let* (M, g) *be a complete double-warped product* ${}_{\lambda_1}M_1 \times_{\lambda_2} M_2$ *of* (M_1, g_1) *and* (M_2, g_2) *such that* $s_{\mathrm{mix}} \le 0$. *If the gradient of* $\log(\lambda_1 \lambda_2)$ *has integrable norm*, *then* $\lambda_1 = C_1$ *and* $\lambda_2 = C_2$ *for some positive constants* C_1 *and* C_2 *and therefore,* (M, g) *is a direct product of* $(M_1, \bar{g}_1)$ *and* $(M_2, \bar{g}_2)$ *for* $\bar{g}_i = C_i g_i$.

Theorem 3 complements the result of [GO] where was proved that if $s_{\mathrm{mix}} \ge 0$ of a complete double-warped product ${}_{\lambda_1}M_1 \times_{\lambda_2} M_2$ then λ_1 and λ_2 are constants.

[GO] Gutierrez M., Olea B., Semi-Riemannian manifolds with a doubly warped structure, Revista Matematica Iberoamericana, 28:1 (2012), 1-24.

The manifold $M_1 \times_{\lambda_2} M_2$ with a smooth positive function $\lambda_2 : M_1 \to \mathbb{R}$ is called a warped product (see [ON, p. 206]). In this case, the well known curvature identity holds (see [ON, p. 211])

$$\pi_1^* Ric = Ric_1 - \frac{n-m}{\lambda_2} Hess(\lambda_2).$$

From this identity we obtain

$$\Delta_1 \lambda_2 = \frac{1}{n-m} \lambda_2 \left(s_1 - trace_{g_1} \left(\pi_1^* Ric \right) \right)$$

where $trace_{g_1} \pi_1^* Ric = \sum_{a=1}^{m} Ric(E_a, E_a)$.

[ON] O'Neill B., Semi-Riemannian geometry with applications to relativity, Academic Press, San Diego, 1983.

If we assume that $s_1 \geq trace_{g_1} \pi_1^* Ric$ then from the above formula we obtain $\Delta_1 \lambda_2 \geq 0$ and therefore, $\lambda_2 : M_1 \to \mathbb{R}$ is a subharmonic non-negative function.

It is well known that Yau showed in [Y] that every non-negative L^p -subharmonic function on a complete Riemannian manifold must be constant for any $p > 1$.

Summarizing the above arguments we can formulate the theorem.

[Y] Yau S.T., Some function-theoretic properties of complete Riemannian manifolds and their applications to geometry, Indiana Univ. Math. J., 25 (1976), 659-670.

Theorem 4. *Let* (M, g) *be a warped product* $M_1 \times_{\lambda_2} M_2$ *of two Riemannian manifolds* (M_1, g_1) *and* (M_2, g_2) *such that* (M_1, g_1) *is a complete manifold and* $s_1 \geq trace_{g_1} \pi_1^* Ric$ *for the scalar curvature* s_1 *of* (M_1, g_1) *and for the Ricci tensor Ric of* $M_1 \times_{\lambda_2} M_2$*. If* $\int_{M_1} \lambda_2{}^p dV_{g_1} < \infty$ *for some* $p > 1$ *then* $\lambda_2 = C_2$ *for some positive constant* C_2 *and therefore,* (M, g) *is the direct product* $M_1 \times M_2$ *of* (M_1, g_1) *and* $(M_2, \bar{g}_2)$ *for* $\bar{g}_2 = C_2 g_2$*.*

If the warped product $M_1 \times_{\lambda_2} M_2$ is an n-dimensional ($n \geq 3$) Einstein manifold, i.e. $Ric = \frac{s}{n} g$ for the constant scalar curvature s of $M_1 \times_{\lambda_2} M_2$, then

$$\Delta_1 \lambda_2 = \frac{1}{n-m} \lambda_2 \left(s_1 - \frac{m}{n} s \right).$$

In this case, we can formulate a generalization of the main theorem on an Einstein warped product with compact M_1 from the paper [KK].

[KK] Kim D.-S., Kim Y.H., Compact Einstein warped product spaces with non-positive scalar curvature, Proceedings of the American Mathematical Society, 131:8 (2003), 2573-2576.

Corollary 4. *Let* $M_1 \times_{\lambda_2} M_2$ *be an n-dimensional* ($n \geq 3$) *Einstein warped product of two Riemannian manifolds* (M_1, g_1) *and* (M_2, g_2) *such that* (M_1, g_1) *is an m-dimensional complete manifold and* $s_1 \geq \frac{m}{n} s$ *for the scalar curvature* s_1 *of* (M_1, g_1) *and for the constant scalar curvature s of* $M_1 \times_{\lambda_2} M_2$. *If* $\int_{M_1} \lambda_2{}^p dV_{g_1} < \infty$ *for some* $p \neq 1$ *then* $s_1 = \frac{m}{n} s = constant$ *and* $\lambda_2 = C_2$ *for some positive constant* C_2 *and therefore,* (M, g) *is the direct product* $M_1 \times M_2$ *of* (M_1, g_1) *and* $(M_2, \bar{g}_2)$ *for* $\bar{g}_2 = C_2 g_2$.

Thanks a lot for your attention!