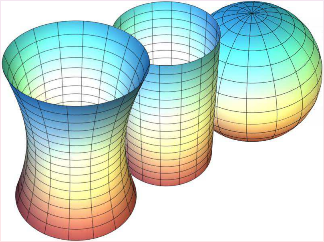
The Einstein-Hilbert type action on foliated metric-affine manifolds

Vladimir Rovenski
Department of Mathematics, University of Haifa

XIX Geometrical seminar, Zlatribor

September 2, 2016

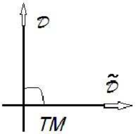


Contents

- (M^n, g) – a **pseudo-Riemannian manifold**
- ∇ – the **Levi-Civita connection**
- **Linear connection** $\bar\nabla$ is ∇ shifted by **contortion tensor**: $\mathfrak{T} := \bar\nabla - \nabla$.
  Particular cases of $\bar\nabla$:
  (a) Riemann-Cartan geometry: the metric property $\bar\nabla g = 0$.
  (b) Torsionless $\bar\nabla$ with symmetric tensor $\bar\nabla g$ arise in statistical structures of differential geometry.
- The **torsion tensor** $\mathcal{S}$ of $\bar\nabla$ is represented as $2\,\mathcal{S}(X,Y) = \mathfrak{T}(X,Y) - \mathfrak{T}(Y,X)$.
- $\bar R_{X,Y} = \bar\nabla_Y \bar\nabla_X - \bar\nabla_X \bar\nabla_Y + \bar\nabla_{[X,Y]}$ – the **curvature tensor**

Define several tensors for one of distributions; similar tensors for the 2nd distribution are denoted using $\widetilde{\ }$ or $^{\perp}$ notation.

- $\widetilde{\mathcal{D}}$ – n-dimensional **distribution**, **subbundle** of TM
- $\mathcal{D}$ – p-dimensional distribution, g-**orthogonal** to $\widetilde{\mathcal{D}}$
- $(\,\cdot\,)^{\top}$, $(\,\cdot\,)^{\perp}$ – **projections** of TM onto $\widetilde{\mathcal{D}}$ and $\mathcal{D}$.



$\widetilde{\mathcal{D}}$-<u>related tensors</u>, $X, Y \in \widetilde{\mathcal{D}}$:

- $T(X, Y) = \frac{1}{2}\,[X, Y]^{\perp}$ – the **integrability tensor**
- $h(X, Y) = \frac{1}{2}(\nabla_X Y + \nabla_Y X)^{\perp}$ – **2- nd fundamental form**
- $H = \operatorname{Tr}_g h \in \mathcal{D}$ – the **mean curvature vector**
- $\mathrm{S}_{\rm ex} = g(H, H) - \langle h, h\rangle$ – the **extrinsic scalar curvature**

The $(1,1)$-tensors A and $T^{\sharp}$ correspond to h and T:

$$g(A_Z(X), Y) = g(h(X, Y), Z), \quad g(T^{\sharp}_Z(X), Y) = g(T(X, Y), Z).$$

The mixed scalar curvature

From now, a distribution $\widetilde{\mathcal{D}} \subset TM$ is fixed.

- $\{E_a \in \Gamma(\widetilde{\mathcal{D}}),\ \mathcal{E}_i \in \Gamma(\mathcal{D})\}$ a local orthonormal frame, and $\varepsilon_i = g(\mathcal{E}_i, \mathcal{E}_i) = \pm 1$, $\varepsilon_a = g(E_a, E_a) = \pm 1$.
- $\bar{\mathrm{S}}_{\rm mix} = \frac{1}{2}\sum_{a,i}\varepsilon_a\varepsilon_i\,[g(\bar{R}(E_a,\mathcal{E}_i)E_a,\ \mathcal{E}_i) + g(\bar{R}(\mathcal{E}_i, E_a)\mathcal{E}_i,\ E_a)]$ – the **mixed scalar curvature**,

$$\bar{S}_{\rm mix} = S_{\rm mix} + Q_1 + Q_2, \qquad (1)$$

$$\mathrm{S}_{\rm mix} = \mathrm{S}_{\rm ex} + \widetilde{\mathrm{S}}_{\rm ex} + \langle T, T\rangle + \langle \tilde{T}, \tilde{T}\rangle + {\rm div}(H + \tilde{H}), \qquad (2)$$

where (2) is due P. Walczak (1990), and

$$2Q_1 = \sum\nolimits_{i,a}\varepsilon_i\varepsilon_a\big[g(((\nabla_i\mathfrak{T})_a - (\nabla_a\mathfrak{T})_i + [\mathfrak{T}_i, \mathfrak{T}_a])E_a, \mathcal{E}_i)\big],$$

$$2Q_2 = \sum\nolimits_{i,a}\varepsilon_i\varepsilon_a\big[g(((\nabla_a\mathfrak{T})_i - (\nabla_i\mathfrak{T})_a + [\mathfrak{T}_a, \mathfrak{T}_i])\mathcal{E}_i, E_a)\big],$$

If $\widetilde{\mathcal{D}}$ is spanned by a unit vector field N then $\mathrm{S}_{\rm mix} = \varepsilon_N\,{\rm Ric}_N$.

Let $\widetilde{\mathcal{D}}$ be spanned by a vector field N with $\varepsilon_N = g(N,N) = \pm 1$, $\tilde{T}^\sharp_N$ corresponds to the integrability tensor $\tilde{T}$ of $\mathcal{D}$; $\tilde{A}_N$ the shape operator of $\mathcal{D}$. Set $\tilde{\tau}_i = \operatorname{Tr}\tilde{A}^i_N$ $(i=1,2)$.
The **mixed E.-H. action** on (M,g,N) is defined in [1] by

$$J_{\widetilde{\mathcal{D}},\Omega}:\ g \mapsto \int_\Omega \Big\{\frac{1}{2\mathfrak{a}}\,(\varepsilon_N\,\mathrm{Ric}_N - 2\,\Lambda) + \mathcal{L}(g)\Big\}\,\mathrm{d}\,\mathrm{vol}_g\ . \qquad (3)$$

Λ – a cosmological constant, $\mathfrak{a}$ – a coupling constant, $\mathcal{L} = \mathcal{L}(g)$ – Lagrangian describing the matter contents,
$\Omega \subset M$ – a relatively compact domain in (M,g).

Example (A Lorentzian manifold (M,g), $\dim M = 4$:)

(i) (M^4,g) with a timelike vector field N is called a **space-time**.
(ii) **Globally hyperbolic space-times** (M^4,g) come equipped with a codimension-one foliation, since $\exists$ time function $\mathbb{T}: M \to \mathbb{R}$, $M = \mathbb{R}\times\hat{S}$, where for each t, $\{t\}\times\hat{S}$ is a Cauchy's surface.

Let $\tilde T = 0$. The **mixed Ricci tensor** is defined in [1] by

$$\mathrm{Ric}_{\mathcal D} := \mathrm{Ric}^{\perp}_{\mathcal D} + \mathrm{Ric}^{/}_{\mathcal D} + \mathrm{Ric}^{\top}_{\mathcal D},$$

where

$$(\mathrm{Ric}_{\mathcal D})^{\perp}(X,Y) = \big(\nabla_N\,\tilde h_{sc} - \tilde\tau_1 \tilde h_{sc}\big)(X,Y), \quad X, Y \perp N,$$
$$(\mathrm{Ric}_{\mathcal D})^{/}(X,N) = \mathrm{Ric}^{1}_{\mathcal D}(g)(N,X) = \mathrm{div}(\tilde A_N(X)),$$
$$(\mathrm{Ric}_{\mathcal D})^{\top}(N,N) = -\,\mathrm{div}\,H,$$

and $\tilde h_{sc} = \varepsilon_N\langle \tilde h, N\rangle$. Put $\mathrm{Scal}_{\mathcal D} := \mathrm{Tr}_g(\mathrm{Ric}_{\mathcal D})$.

Theorem ($\varepsilon_N = -1$, see [1])

Let $\widetilde{\mathcal D} = \mathrm{Span}(N)$ with integrable $\mathcal D$. If g is critical for (3) then

$$\mathrm{Ric}_{\mathcal D} - \frac{1}{2}\,\mathrm{Scal}_{\mathcal D}\cdot g - \mathrm{Ric}_N\left(N^\flat\otimes N^\flat + \frac{1}{2}\,g\right) + \Lambda\, g = \mathfrak{a}\,\Theta\,. \qquad (4)$$

Here Θ – **stress-energy tensor**, $\Theta_{ij} = -2\,\frac{\partial\mathcal L}{\partial g_{ij}} + \mathcal L\, g_{ij}$.

(A **Riemannian flow** *of N obeys (4) with $\Theta = \Lambda = 0$).*

Goals of:
[1] E. Barletta, S. Dragomir, V. Rovenski and M. Soret:
Mixed gravitational field equations on globally hyperbolic spacetimes, Class. Quantum Grav., 30 (2013), 26 pp.

– sufficient conditions, e.g. isoparametric time-function and totally umbilical $\mathcal{D}$, for the existence of solutions to (4) in empty space.
– the conservation law, analog of $\operatorname{div}\Theta = 0$ in relativity.
– linearization and solution of (4) for empty space.
– equations of motion (geodesics) of a particle in the 'gravitational field' governed by (4).
– after the linearization of motion equations about the Minkowski metric, the coupling constant $\mathfrak{a} = 2\pi k/c^2$ has been determined, where k is the gravitational constant.

The mixed Einstein-Hilbert action

M with a smooth distribution $\widetilde{\mathcal{D}}$ of arbitrary dimension.
The **mixed Einstein-Hilbert action** on $(M, g, \widetilde{\mathcal{D}}, \bar\nabla = \nabla + \mathfrak{T})$

$$J_{\widetilde{\mathcal{D}},\Omega} : (g, \mathfrak{T}) \mapsto \int_\Omega \Big\{ \frac{1}{2\mathfrak{a}} (\mathrm{S}_{\rm mix} - 2\Lambda) + \mathcal{L}(g, \mathfrak{T}) \Big\} \mathrm{d}\,\mathrm{vol}_g \,. \qquad (5)$$

$J_{\rm mix}(g) = \int_\Omega \mathrm{S}_{\rm mix}\, \mathrm{d}\,\mathrm{vol}_g$ – the **gravitational part** of the action.

Question:

– *Euler-Lagrange equations of* $J_{\widetilde{\mathcal{D}},\Omega}$*?*
– *their solutions: critical metrics and connections ?*

A **variation of connection** has the form $\bar\nabla^t = \nabla + \mathfrak{T}^t, \ |t| < \varepsilon$.

Proposition (E.-L. equations)

Let $(M, \widetilde{\mathcal{D}}, \mathcal{D}, g)$ *be a pseudo-Riemannian almost product manifold. Then the E.-L. equations for* (5) *with* $\Lambda = \Theta = 0$, *as a functional on the space of all connections on* M, *are the following:*

$$-2\tilde{T}(\mathcal{E}_i, \mathcal{E}_j) - (\mathfrak{T}_j^*\mathcal{E}_i + \mathfrak{T}_i\mathcal{E}_j)^\top = 0,$$

$$\sum\nolimits_i \varepsilon_i(\mathfrak{T}_i^*\mathcal{E}_i)^\top = \tilde{H}, \quad \text{for } n > 1,$$

$$\sum\nolimits_i \varepsilon_i(\mathfrak{T}_i\mathcal{E}_i)^\top = -\tilde{H}, \quad \text{for } n > 1,$$

$$\mathfrak{T}_i^\top - (\mathfrak{T}_i^\top)^* = 2T_i^\sharp,$$

$$\sum\nolimits_j \varepsilon_j g((\mathfrak{T}_j + \mathfrak{T}_j^*)\mathcal{E}_j,\, \mathcal{E}_i)\mathrm{Id}_{\widetilde{\mathcal{D}}} - \mathfrak{T}_i^\top - (\mathfrak{T}_i^\top)^* = 0,$$

$$h = (1/n)Hg^\top,$$

$$\sum\nolimits_j \varepsilon_j(\mathfrak{T}_j\mathcal{E}_j - \mathfrak{T}_j^*\mathcal{E}_j)^\perp = \frac{2}{n}(n-1)H,$$

and 7 dual equations.

Palatini type variational principle

Proposition

The Levi-Civita connection of a metric $g \in \mathrm{Riem}(M, \widetilde{\mathcal{D}}, \mathcal{D})$ *is critical for for* (5) *as a functional on the space of all linear connections on* M *if and only if*

$$g(\tilde{H},\, E_b\delta_{ac} - E_c\delta_{ab}) = x\mathfrak{a}\, S_{bc;a}, \qquad (6a)$$

$$g(H,\, \mathcal{E}_j\delta_{ik} - \mathcal{E}_k\delta_{ij}) = \mathfrak{a}\, S_{jk;i}, \qquad (6b)$$

$$-g(h(E_a, E_b) - H\delta_{ab},\, \mathcal{E}_i) = \mathfrak{a}\,(S_{bi;a} + S_{ab;i}/2), \qquad (6c)$$

$$-g(\tilde{h}(\mathcal{E}_i, \mathcal{E}_j) - \tilde{H}\delta_{ij},\, E_a) = \mathfrak{a}\,(S_{ja;i} - S_{ij;a}/2), \qquad (6d)$$

$$2\, g(\tilde{T}^\sharp_a \mathcal{E}_j, \mathcal{E}_i) = \mathfrak{a}\, S_{ij;a}, \qquad (6e)$$

$$2\, g(T^\sharp_i E_b, E_a) = \mathfrak{a}\, S_{ab;i}, \qquad (6f)$$

and the spin tensor S *should satisfy the following compatibility conditions when* $\mathfrak{a} \neq 0$*:*

$$S_{bc;a} = -S_{cb;a},\ S_{jk;i} = -S_{kj;i},\ S_{ja;i} = -S_{aj;i},\ S_{bi;a} = -S_{ib;a}, \qquad (7a)$$

$$\sum\nolimits_a (S_{ai;a} + S_{aa;i}/2) = 0, \quad \sum\nolimits_i (S_{ia;i} + S_{ii;a}/2) = 0. \qquad (7b)$$

Palatini type variational principle (cont.)

Example

Consider a spin fluid $S^{\eta}_{\mu\nu}(x) = \lambda^{\eta}(x)\, s_{\mu\nu}$, with skew-symmetric spin: $s_{\mu\nu} = -s_{\nu\mu}$, Obviously, (7a) are satisfied, and (7b) reads

$$\sum\nolimits_a \lambda_a\, s_{ai} = 0, \quad \sum\nolimits_i \lambda_i\, s_{ai} = 0,$$

from which nonzero $\lambda = (\lambda_\mu)$ can be found when $\dim M > 2$.

One may examine smaller spaces of varied connections, e.g., metric connections, corresponding to Riemann-Cartan manifolds.

Corollary

Let $n + p > 2$. The Levi-Civita connection of g is critical for the gravitational part of (5) on the space of all metric connections if and only if both distributions are totally geodesic and integrable, i.e., M splits.

From now assume $\mathfrak{T}=0$.

We consider variations $\{g_t\in \mathrm{Riem}(M):\ |t|<\varepsilon\}$ of metric $g_0=g$ on M such that the induced infinitesimal variations, presented by a symmetric $(0,2)$-tensor $B_t\equiv\partial g_t/\partial t$, are supported in a relatively compact domain Ω in M.
If $\widetilde{\mathcal D}$ and $\mathcal D$ are fixed and their orthogonality is preserved by (g_t) then we have **adapted variation** of metric, see [2, 3]. Adapted variations preserving metric on $\mathcal D$ are called $g^\top$**-variations**.

A tensor $B=\partial_t\,g_t\in\mathrm{Sym}^2(M)$ is orthogonally decomposed $B=B^\perp+B^/+B^\top$, with symmetric tensors

$$B^\top(X,Y):=B(X^\top,Y^\top),$$
$$B^\perp(X,Y):=B(X^\perp,Y^\perp),$$
$$B^/(X,Y):=\tfrac12(B(X^\top,Y^\perp)+B(X^\perp,Y^\top)).$$

Lemma

Let a local $(\widetilde{\mathcal{D}}, \mathcal{D})$*-adapted and orthonormal for* $t = 0$ *frame* $\{E_a(t), \mathcal{E}_i(t)\}$ *be evolved by* $g_t \in \mathrm{Riem}(M)$, $|t| < \varepsilon$, *according to*

$$\partial_t E_a = -(1/2)\, B_t^\sharp(E_a)^\top, \quad \partial_t \mathcal{E}_i = -\frac{1}{2}\, B_t^\sharp(\mathcal{E}_i)^\perp - B_t^\sharp(\mathcal{E}_i)^\top. \quad (8)$$

Then, for all t, $\{E_a(t), \mathcal{E}_i(t)\}$ *is a* g_t*-orthonormal frame adapted to* $\widetilde{\mathcal{D}}$*, i.e.,* $\{E_a(t)\}$ *belong to* $\widetilde{\mathcal{D}}$.

Definition

Define symmetric $(0,2)$-tensors $\alpha, \theta, \tilde\delta$ by

$$2\,\alpha(X,Y) = A_{X^\perp}(Y^\top) + A_{Y^\perp}(X^\top),$$
$$2\,\theta(X,Y) = T^\sharp_{X^\perp}(Y^\top) + T^\sharp_{Y^\perp}(X^\top),$$
$$2\,\tilde\delta_Z(X,Y) = g(\nabla_{X^\top}Z,\, Y^\perp) + g(\nabla_{Y^\top}Z,\, X^\perp).$$

Define (1,1)-tensors $\mathcal{T} := \sum_i \varepsilon_i\,(T^\sharp_i)^2, \quad \mathcal{K} = \sum_i \varepsilon_i\,[T^\sharp_i, A_i]$,
and a (0,2)-tensor $\operatorname{div} h = \sum_\nu \varepsilon_\nu\, g(\nabla_\nu\, h,\, e_\nu)$.
For any $(0,2)$-tensors P, Q and S, define a tensor $\Upsilon_{P,Q}$ by

$$\langle\Upsilon_{P,Q}, S\rangle = \sum_{\lambda,\nu}\varepsilon_\lambda\varepsilon_\nu[S(P(e_\lambda,e_\nu),Q(e_\lambda,e_\nu)) + S(Q(e_\lambda,e_\nu),P(e_\lambda,e_\nu))]$$

where $\{e_\lambda\}$ is an orthonormal basis and $\varepsilon_\lambda = g(e_\lambda, e_\lambda) \in \{-1, 1\}$.

Proposition (For any variation g_t we have)

$$\partial_t\,\langle\tilde h,\tilde h\rangle = \langle \operatorname{div}\tilde h - 4\Upsilon_{\tilde\alpha,\theta} + \widetilde{\mathcal{K}}^\flat - (1/2)\,\Upsilon_{\tilde h,\tilde h},\ B\rangle - \operatorname{div}\langle\tilde h, B\rangle,$$

$$\partial_t\, g(\tilde H,\tilde H) = \langle (\operatorname{div}\tilde H)\,g^\perp + 4\langle\theta,\tilde H\rangle - \tilde H^\flat\otimes\tilde H^\flat, B\rangle - \operatorname{div}((\operatorname{Tr}_{\mathcal D} B^\sharp)\tilde H),$$

$$\partial_t\,\langle h,h\rangle = \langle \operatorname{div} h + \mathcal{K}^\flat - 2\,(\operatorname{div}\alpha)_{\,|\widetilde{\mathcal D}\times\mathcal D} - 2\,(\operatorname{div}\alpha)_{\,|\mathcal D\times\widetilde{\mathcal D}} - 2\Upsilon_{\alpha,\tilde\alpha+\tilde\theta}$$
$$- (1/2)\Upsilon_{h,h},\ B\rangle + 2\operatorname{div}\langle\alpha, B\rangle - \operatorname{div}\langle h, B\rangle,$$

$$\partial_t\, g(H,H) = \langle 2\langle\tilde\theta - \tilde\alpha,\ H\rangle + 2\,\mathrm{Sym}(H^\flat\otimes\tilde H^\flat) - 2\,\tilde\delta_H + (\operatorname{div} H)\,g^\top$$
$$- H^\flat\otimes H^\flat,\ B\rangle + 2\operatorname{div}((B^\sharp H)^\top) - \operatorname{div}((\operatorname{Tr}_{\widetilde{\mathcal D}} B^\sharp)H),$$

$$\partial_t\,\langle\tilde T,\tilde T\rangle = \langle 2\,\tilde{\mathcal T}^\flat + 2\Upsilon_{\tilde\theta,\theta-\alpha} - 2\,(\operatorname{div}\tilde\theta)_{|\widetilde{\mathcal D}\times\mathcal D} - 2\,(\operatorname{div}\tilde\theta)_{|\mathcal D\times\widetilde{\mathcal D}}$$
$$+ (1/2)\Upsilon_{\tilde T,\tilde T},\ B\rangle + 2\operatorname{div}\langle\tilde\theta,\ B\rangle,$$

$$\partial_t\,\langle T,T\rangle = \langle (1/2)\Upsilon_{T,T} + 2\,\mathcal{T}^\flat,\ B\rangle.$$

The formulas are applicable for various curvature functionals on a pseudo-Riemannian manifold with a distribution, e.g. for the **total extrinsic scalar curvature** of $\widetilde{\mathcal D}$.

Definition

The symmetric $(0,2)$-tensor $\mathrm{Ric}_{\mathcal{D}} = (\mathrm{Ric}_{\mathcal{D}})^{\perp} + (\mathrm{Ric}_{\mathcal{D}})^{/} + (\mathrm{Ric}_{\mathcal{D}})^{\top}$ with the trace $\mathrm{Scal}_{\mathcal{D}} = \mathrm{Tr}_g(\mathrm{Ric}_{\mathcal{D}})$ and the components

$$(\mathrm{Ric}_{\mathcal{D}})^{\perp} = \operatorname{div}\tilde h + \widetilde{\mathcal{K}}^{\flat} - 2\,\tilde{\mathcal{T}}^{\flat} + H^{\flat}\otimes H^{\flat} - \frac{1}{2}\Upsilon_{h,h} - \frac{1}{2}\Upsilon_{T,T} + \eta\, g^{\perp},$$

$$\frac{1}{2}(\mathrm{Ric}_{\mathcal{D}})^{/} = \tilde\delta_H + (\operatorname{div}(\tilde\theta-\alpha))_{\,|\widetilde{\mathcal{D}}\times\mathcal{D}} + (\operatorname{div}(\tilde\theta-\alpha))_{\,|\mathcal{D}\times\widetilde{\mathcal{D}}} - 2\langle\theta,\tilde H\rangle$$
$$- \langle\tilde\theta-\tilde\alpha, H\rangle - 2\Upsilon_{\tilde\alpha,\theta} - \Upsilon_{\alpha,\tilde\alpha} - \Upsilon_{\tilde\theta,\theta} - \mathrm{Sym}(H^{\flat}\otimes\tilde H^{\flat}),$$

$$(\mathrm{Ric}_{\mathcal{D}})^{\top} = \operatorname{div} h + \mathcal{K}^{\flat} - 2\,\mathcal{T}^{\flat} + \tilde H^{\flat}\otimes\tilde H^{\flat} - \frac{1}{2}\Upsilon_{\tilde h,\tilde h} - \frac{1}{2}\Upsilon_{\tilde T,\tilde T} + \mu\, g^{\top}$$

is called the **mixed Ricci curvature** of $g\in\mathrm{Riem}(M,\widetilde{\mathcal{D}},\mathcal{D})$. Here

$$\eta = -\frac{n-1}{p+n-2}\operatorname{div}(\tilde H - H),\quad \mu = \frac{p-1}{p+n-2}\operatorname{div}(\tilde H - H). \quad (9)$$

Theorem (Mixed gravitational field equations)

A metric $g \in \mathrm{Riem}(M, \widetilde{\mathcal{D}}, \mathcal{D})$ *is critical for the action* (5) *if and only if g is a solution to the Euler-Lagrange equations*

$$\mathrm{Ric}_{\mathcal{D}} - (\mathbf{1}/\mathbf{2})\,\mathrm{Scal}_{\mathcal{D}} \cdot \mathbf{g} + \mathbf{\Lambda}\,\mathbf{g} = \mathfrak{a}\,\mathbf{\Theta}. \tag{10}$$

The following symmetric $(0, 2)$-tensor:

$$\mathrm{G}_{\mathcal{D}} := \mathrm{Ric}_{\mathcal{D}} - (1/2)\,\mathrm{Scal}_{\mathcal{D}} \cdot g$$

is referred to as the **mixed Einstein tensor**. Note that

$$\mathrm{Tr}_g\,\mathrm{G}_{\mathcal{D}} = \frac{2 - n - p}{2}\,\mathrm{Scal}_{\mathcal{D}},$$

and for $n = p = 1$ we have $\mathrm{G}_{\mathcal{D}} = 0$.

Corollary (The mixed field equations for a flow)

Let $\widetilde{\mathcal{D}}$ *be spanned by a unit vector field* N. *Then* g *is critical for the action* (3) *if and only if* (10) *holds, or, in components,*

$$\nabla_N \tilde h_{sc} - \tilde\tau_1 \tilde h_{sc} - \varepsilon_N \big(2(\tilde T^\sharp_N)^2 + [\tilde A_N,\ \tilde T^\sharp_N]\big)^\flat + \langle \tilde T, \tilde T\rangle - \frac{1}{2}\big(\varepsilon_N(2\,N(\tilde\tau_1) - \tilde\tau_1^2 - \tilde\tau_2) - 2\Lambda\big)\, g^\perp = \mathfrak{a}\,\Theta_{\,|\mathcal{D}\times\mathcal{D}},$$

$$(\operatorname{div}^\perp \tilde T^\sharp_N)^\sharp + 2\,\tilde T^\sharp_N(H) = \mathfrak{a}\,\Theta_{\,|\mathcal{D}\times\widetilde{\mathcal{D}}},$$

$$(1/2)\,\varepsilon_N(\tilde\tau_1^2 - \tilde\tau_2) - (3/2)\,\langle \tilde T, \tilde T\rangle + \Lambda = \mathfrak{a}\,\Theta_{N,N}\,,$$

where $\Theta_{N,N} : M \to \mathbb{R}$ *is defined by* $\Theta_{\,|\widetilde{\mathcal{D}}\times\widetilde{\mathcal{D}}} = \Theta_{N,N}\, g^\top$.

These equations for a **space-time** generalize result in [1] for a **globally hyperbolic space-time** (i.e., $\tilde T = 0$).

For the flow of N with integrable normal distribution, the mixed Ricci tensor $\mathrm{Ric}_{\mathcal{D}}(g)$ has components

$$(\mathrm{Ric}_{\mathcal{D}})^{\perp}(g) = \nabla_N \tilde{h}_{sc} - \tilde{\tau}_1 \tilde{h}_{sc},$$
$$(\mathrm{Ric}_{\mathcal{D}})^{/}(g) = 0,$$
$$(\mathrm{Ric}_{\mathcal{D}})^{\top}(g) = \varepsilon_N (N(\tilde{\tau}_1) - \tilde{\tau}_2)\, g^{\top}. \tag{11}$$

Comparing (10):

$$\mathrm{Ric}_{\mathcal{D}} - \frac{1}{2}\,\mathrm{Scal}_{\mathcal{D}} \cdot g + \Lambda\, g = \mathfrak{a}\,\Theta$$

with (4) of [1]:

$$\mathrm{Ric}_{\mathcal{D}} - \frac{1}{2}\,\mathrm{Scal}_{\mathcal{D}} \cdot g - \mathrm{Ric}_N \left(N^\flat \otimes N^\flat + \frac{1}{2}\, g\right) + \Lambda\, g = \mathfrak{a}\,\Theta,$$

we see that (9) is the best choice of (η, μ) in definition of the mixed Ricci tensor.

Proposition (Double-twisted products, $\phi, \psi \in C^\infty(M_1 \times M_2)$)

Let $\bar g = \psi\, g^\top + \phi\, g^\perp$, *where* $g = g^\top + g^\perp$ *is a product metric. Then* $\bar g$ *is critical for the action* (5) *if and only if the following hold:*

$$(\mathbf{F_1} + \Lambda)\, g^\perp + \mathbf{F_2}\, (\nabla^\perp \phi)^\flat \otimes (\nabla^\perp \phi)^\flat = \mathfrak{a}\, \Theta_{\,|\mathcal{D}\times\mathcal{D}}\,,$$

$$\mathbf{F_3}\, (\nabla^\top \phi)^\flat \otimes (\nabla^\perp \psi)^\flat + \mathbf{F_4}\, (\mathrm{Hess}_\psi)_{\,|\widetilde{\mathcal{D}}\times\mathcal{D}} = \mathfrak{a}\, \Theta_{\,|\widetilde{\mathcal{D}}\times\mathcal{D}}\,,$$

$$(\mathbf{F_5} + \Lambda)\, g^\top + \mathbf{F_6}\, (\nabla^\top \phi)^\flat \otimes (\nabla^\top \phi)^\flat = \mathfrak{a}\, \Theta_{\,|\widetilde{\mathcal{D}}\times\widetilde{\mathcal{D}}}\,,$$

where F_5 *and* F_6 *are 'dual' to* F_1 *and* F_2, *and*

$$\mathbf{F_1} = \frac{n(n-1)}{4}\psi^{-2}(\nabla^\perp\psi)^2 + \frac{4-p}{4}(p-1)\psi^{-1}\phi^{-1}(\nabla^\top\phi)^2$$

$$-(p-1)\psi^{-1}\Delta^\top\phi + \frac{2-n}{2}(p-1)\psi^{-2} g(\nabla^\top\psi, \nabla^\top\phi),$$

$$\mathbf{F_3} = -\frac{n-1}{2}\psi^{-1}\Big(\frac{p-2}{2}\phi^{-1} - \psi^{-1}\Big),$$

$$\mathbf{F_2} = -\frac{n(n-1)}{2}\phi^{-2}\psi^{-2}, \qquad \mathbf{F_4} = \frac{1-n}{2}\psi^{-1}.$$

Bibliography

[1] E. Barletta, S. Dragomir, V. Rovenski and M. Soret: **Mixed gravitational field equations on globally hyperbolic spacetimes**, Class. Quantum Grav., 30 (2013), 26 pp.

[2] E. Barletta, S. Dragomir, and V. Rovenski, **The mixed Einstein-Hilbert action and extrinsic geometry of foliated manifolds**, preprint, 2014, arXiv:1405.6011

[3] V. Rovenski, and T. Zawadzki, **The Einstein-Hilbert type action on foliated pseudo-Riemannian manifolds**, preprint, 2016, arXiv:1604.00985

[4] V. Rovenski, and T. Zawadzki, **Variations of the total mixed scalar curvature of a distribution**, preprint, 2016, arXiv:1609.09409

THANK YOU !