Polyhedral products and commutator subgroups of right-angled Artin and Coxeter groups

joint with Yakov Veryovkin

Taras Panov

Lomonosov Moscow State University

XIX Geometrical Seminar
Zlatibor, Serbia, 29 August–3 September 2016

1. Preliminaries

Polyhedral product

$(\boldsymbol{X}, \boldsymbol{A}) = \{(X_1, A_1), \ldots, (X_m, A_m)\}$ a sequence of pairs of spaces, $A_i \subset X_i$.

1. Preliminaries

Polyhedral product

$(\boldsymbol{X}, \boldsymbol{A}) = \{(X_1, A_1), \ldots, (X_m, A_m)\}$ a sequence of pairs of spaces, $A_i \subset X_i$.

$\mathcal{K}$ a simplicial complex on $[m] = \{1, 2, \ldots, m\}$, $\quad \varnothing \in \mathcal{K}$.

Polyhedral product

$(\boldsymbol{X}, \boldsymbol{A}) = \{(X_1, A_1), \ldots, (X_m, A_m)\}$ a sequence of pairs of spaces, $A_i \subset X_i$.

$\mathcal{K}$ a simplicial complex on $[m] = \{1, 2, \ldots, m\}$, $\quad \varnothing \in \mathcal{K}$.

Given $I = \{i_1, \ldots, i_k\} \subset [m]$, set

$$(\boldsymbol{X}, \boldsymbol{A})^I = Y_1 \times \cdots \times Y_m \qquad \text{where } Y_i = \begin{cases} X_i & \text{if } i \in I, \\ A_i & \text{if } i \notin I. \end{cases}$$

1. Preliminaries

Polyhedral product

$(\boldsymbol{X}, \boldsymbol{A}) = \{(X_1, A_1), \ldots, (X_m, A_m)\}$ a sequence of pairs of spaces, $A_i \subset X_i$.

$\mathcal{K}$ a simplicial complex on $[m] = \{1, 2, \ldots, m\}$, $\quad \varnothing \in \mathcal{K}$.

Given $I = \{i_1, \ldots, i_k\} \subset [m]$, set

$$(\boldsymbol{X}, \boldsymbol{A})^I = Y_1 \times \cdots \times Y_m \qquad \text{where } Y_i = \begin{cases} X_i & \text{if } i \in I, \\ A_i & \text{if } i \notin I. \end{cases}$$

The $\mathcal{K}$-polyhedral product of $(\boldsymbol{X}, \boldsymbol{A})$ is

$$(\boldsymbol{X}, \boldsymbol{A})^{\mathcal{K}} = \bigcup_{I \in \mathcal{K}} (\boldsymbol{X}, \boldsymbol{A})^I = \bigcup_{I \in \mathcal{K}} \Big(\prod_{i \in I} X_i \times \prod_{j \notin I} A_j \Big).$$

1. Preliminaries

Polyhedral product

$(\boldsymbol{X}, \boldsymbol{A}) = \{(X_1, A_1), \dots, (X_m, A_m)\}$ a sequence of pairs of spaces, $A_i \subset X_i$.

$\mathcal{K}$ a simplicial complex on $[m] = \{1, 2, \dots, m\}$, $\varnothing \in \mathcal{K}$.

Given $I = \{i_1, \dots, i_k\} \subset [m]$, set

$$(\boldsymbol{X}, \boldsymbol{A})^I = Y_1 \times \dots \times Y_m \qquad \text{where } Y_i = \begin{cases} X_i & \text{if } i \in I, \\ A_i & \text{if } i \notin I. \end{cases}$$

The $\mathcal{K}$-polyhedral product of $(\boldsymbol{X}, \boldsymbol{A})$ is

$$(\boldsymbol{X}, \boldsymbol{A})^{\mathcal{K}} = \bigcup_{I \in \mathcal{K}} (\boldsymbol{X}, \boldsymbol{A})^I = \bigcup_{I \in \mathcal{K}} \Big(\prod_{i \in I} X_i \times \prod_{j \notin I} A_j \Big).$$

Notation: $(X, A)^{\mathcal{K}} = (\boldsymbol{X}, \boldsymbol{A})^{\mathcal{K}}$ when all $(X_i, A_i) = (X, A)$;
$\boldsymbol{X}^{\mathcal{K}} = (\boldsymbol{X}, pt)^{\mathcal{K}}$, $X^{\mathcal{K}} = (X, pt)^{\mathcal{K}}$.

Categorical approach

Category of faces $\text{CAT}(\mathcal{K})$.
Objects: simplices $I \in \mathcal{K}$. Morphisms: inclusions $I \subset J$.

Categorical approach

Category of faces $\mathrm{CAT}(\mathcal{K})$.
Objects: simplices $I \in \mathcal{K}$. Morphisms: inclusions $I \subset J$.

TOP the category of topological spaces.
Define the $\mathrm{CAT}(\mathcal{K})$-diagram

$$\begin{aligned} \mathcal{D}_{\mathcal{K}}(\boldsymbol{X}, \boldsymbol{A})\colon \mathrm{CAT}(\mathcal{K}) &\longrightarrow \mathrm{TOP}, \\ I &\longmapsto (\boldsymbol{X}, \boldsymbol{A})^I, \end{aligned}$$

Categorical approach

Category of faces $\text{CAT}(\mathcal{K})$.
Objects: simplices $I \in \mathcal{K}$. Morphisms: inclusions $I \subset J$.

TOP the category of topological spaces.
Define the $\text{CAT}(\mathcal{K})$-diagram

$$\begin{aligned}\mathcal{D}_{\mathcal{K}}(\boldsymbol{X}, \boldsymbol{A})\colon \text{CAT}(\mathcal{K}) &\longrightarrow \text{TOP},\\ I &\longmapsto (\boldsymbol{X}, \boldsymbol{A})^I,\end{aligned}$$

which maps the morphism $I \subset J$ of $\text{CAT}(\mathcal{K})$ to the inclusion of spaces $(\boldsymbol{X}, \boldsymbol{A})^I \subset (\boldsymbol{X}, \boldsymbol{A})^J$.

Categorical approach

Category of faces $\mathrm{CAT}(\mathcal{K})$.
Objects: simplices $I \in \mathcal{K}$. Morphisms: inclusions $I \subset J$.

TOP the category of topological spaces.
Define the $\mathrm{CAT}(\mathcal{K})$-diagram

$$\begin{aligned} \mathcal{D}_{\mathcal{K}}(\boldsymbol{X}, \boldsymbol{A}) \colon \mathrm{CAT}(\mathcal{K}) &\longrightarrow \mathrm{TOP}, \\ I &\longmapsto (\boldsymbol{X}, \boldsymbol{A})^I, \end{aligned}$$

which maps the morphism $I \subset J$ of $\mathrm{CAT}(\mathcal{K})$ to the inclusion of spaces $(\boldsymbol{X}, \boldsymbol{A})^I \subset (\boldsymbol{X}, \boldsymbol{A})^J$.

Then we have

$$(\boldsymbol{X}, \boldsymbol{A})^{\mathcal{K}} = \operatorname{colim} \mathcal{D}_{\mathcal{K}}(\boldsymbol{X}, \boldsymbol{A}) = \operatorname*{colim}_{I \in \mathcal{K}} (\boldsymbol{X}, \boldsymbol{A})^I.$$

Example

Let $(X, A) = (S^1, pt)$, where S^1 is a circle. Then

$$(S^1)^{\mathcal{K}} = \bigcup_{I \in \mathcal{K}} (S^1)^I \subset (S^1)^m.$$

Example

Let $(X, A) = (S^1, pt)$, where S^1 is a circle. Then

$$(S^1)^{\mathcal{K}} = \bigcup_{I\in\mathcal{K}} (S^1)^I \subset (S^1)^m.$$

When $\mathcal{K} = \{\varnothing, \{1\}, \ldots, \{m\}\}$ (m disjoint points), the polyhedral product $(S^1)^{\mathcal{K}}$ is the wedge $(S^1)^{\vee m}$ of m circles.

Example

Let $(X, A) = (S^1, pt)$, where S^1 is a circle. Then

$$(S^1)^{\mathcal{K}} = \bigcup_{I \in \mathcal{K}} (S^1)^I \subset (S^1)^m.$$

When $\mathcal{K} = \{\varnothing, \{1\}, \ldots, \{m\}\}$ (m disjoint points), the polyhedral product $(S^1)^{\mathcal{K}}$ is the wedge $(S^1)^{\vee m}$ of m circles.

When $\mathcal{K}$ consists of all proper subsets of $[m]$ (the boundary $\partial\Delta^{m-1}$ of an $(m-1)$-dimensional simplex), $(S^1)^{\mathcal{K}}$ is the fat wedge of m circles; it is obtained by removing the top-dimensional cell from the m-torus $(S^1)^m$.

Example

Let $(X, A) = (S^1, pt)$, where S^1 is a circle. Then

$$(S^1)^{\mathcal{K}} = \bigcup_{I \in \mathcal{K}} (S^1)^I \subset (S^1)^m.$$

When $\mathcal{K} = \{\varnothing, \{1\}, \ldots, \{m\}\}$ (m disjoint points), the polyhedral product $(S^1)^{\mathcal{K}}$ is the wedge $(S^1)^{\vee m}$ of m circles.

When $\mathcal{K}$ consists of all proper subsets of $[m]$ (the boundary $\partial\Delta^{m-1}$ of an $(m-1)$-dimensional simplex), $(S^1)^{\mathcal{K}}$ is the fat wedge of m circles; it is obtained by removing the top-dimensional cell from the m-torus $(S^1)^m$.

For a general $\mathcal{K}$ on m vertices, $(S^1)^{\vee m} \subset (S^1)^{\mathcal{K}} \subset (S^1)^m$.

Example

Let $(X, A) = (\mathbb{R}, \mathbb{Z})$. Then

$$\mathcal{L}_{\mathcal{K}} := (\mathbb{R}, \mathbb{Z})^{\mathcal{K}} = \bigcup_{I \in \mathcal{K}} (\mathbb{R}, \mathbb{Z})^I \subset \mathbb{R}^m.$$

Example

Let $(X, A) = (\mathbb{R}, \mathbb{Z})$. Then

$$\mathcal{L}_{\mathcal{K}} := (\mathbb{R}, \mathbb{Z})^{\mathcal{K}} = \bigcup_{I \in \mathcal{K}} (\mathbb{R}, \mathbb{Z})^{I} \subset \mathbb{R}^m.$$

When $\mathcal{K}$ consists of m disjoint points, $\mathcal{L}_{\mathcal{K}}$ is a grid in $\mathbb{R}^m$ consisting of all lines parallel to one of the coordinate axis and passing though integer points.

Example

Let $(X, A) = (\mathbb{R}, \mathbb{Z})$. Then

$$\mathcal{L}_{\mathcal{K}} := (\mathbb{R}, \mathbb{Z})^{\mathcal{K}} = \bigcup_{I \in \mathcal{K}} (\mathbb{R}, \mathbb{Z})^I \subset \mathbb{R}^m.$$

When $\mathcal{K}$ consists of m disjoint points, $\mathcal{L}_{\mathcal{K}}$ is a grid in $\mathbb{R}^m$ consisting of all lines parallel to one of the coordinate axis and passing though integer points.

When $\mathcal{K} = \partial\Delta^{m-1}$, the complex $\mathcal{L}_{\mathcal{K}}$ is the union of all integer hyperplanes parallel to coordinate hyperplanes.

Example

Let $(X, A) = (\mathbb{R}P^\infty, pt)$, where $\mathbb{R}P^\infty = B\mathbb{Z}_2$. Then

$$(\mathbb{R}P^\infty)^{\mathcal{K}} = \bigcup_{I \in \mathcal{K}} (\mathbb{R}P^\infty)^I \subset (\mathbb{R}P^\infty)^m.$$

Example

Let $(X, A) = (\mathbb{R}P^\infty, pt)$, where $\mathbb{R}P^\infty = B\mathbb{Z}_2$. Then

$$(\mathbb{R}P^\infty)^{\mathcal{K}} = \bigcup_{I \in \mathcal{K}} (\mathbb{R}P^\infty)^I \subset (\mathbb{R}P^\infty)^m.$$

Example

Let $(X, A) = (D^1, S^0)$, where $D^1 = [-1, 1]$ and $S^0 = \{1, -1\}$. The real moment-angle complex is

$$\mathcal{R}_{\mathcal{K}} := (D^1, S^0)^{\mathcal{K}} = \bigcup_{I \in \mathcal{K}} (D^1, S^0)^I.$$

Example

Let $(X, A) = (\mathbb{R}P^\infty, pt)$, where $\mathbb{R}P^\infty = B\mathbb{Z}_2$. Then

$$(\mathbb{R}P^\infty)^\mathcal{K} = \bigcup_{I \in \mathcal{K}} (\mathbb{R}P^\infty)^I \subset (\mathbb{R}P^\infty)^m.$$

Example

Let $(X, A) = (D^1, S^0)$, where $D^1 = [-1, 1]$ and $S^0 = \{1, -1\}$. The real moment-angle complex is

$$\mathcal{R}_\mathcal{K} := (D^1, S^0)^\mathcal{K} = \bigcup_{I \in \mathcal{K}} (D^1, S^0)^I.$$

It is a cubic subcomplex in the m-cube $(D^1)^m = [-1, 1]^m$.

Example

Let $(X, A) = (\mathbb{R}P^\infty, pt)$, where $\mathbb{R}P^\infty = B\mathbb{Z}_2$. Then

$$(\mathbb{R}P^\infty)^{\mathcal{K}} = \bigcup_{I \in \mathcal{K}} (\mathbb{R}P^\infty)^I \subset (\mathbb{R}P^\infty)^m.$$

Example

Let $(X, A) = (D^1, S^0)$, where $D^1 = [-1, 1]$ and $S^0 = \{1, -1\}$. The real moment-angle complex is

$$\mathcal{R}_{\mathcal{K}} := (D^1, S^0)^{\mathcal{K}} = \bigcup_{I \in \mathcal{K}} (D^1, S^0)^I.$$

It is a cubic subcomplex in the m-cube $(D^1)^m = [-1, 1]^m$.

When $\mathcal{K}$ consists of m disjoint points, $\mathcal{R}_{\mathcal{K}}$ is the 1-dimensional skeleton of the cube $[-1, 1]^m$. When $\mathcal{K} = \partial\Delta^{m-1}$, $\mathcal{R}_{\mathcal{K}}$ is the boundary of the cube $[-1, 1]^m$. Also, $\mathcal{R}_{\mathcal{K}}$ is a topological manifold when $|\mathcal{K}|$ is a sphere.

The four polyhedral products above are related by the two homotopy fibrations

$$(\mathbb{R},\mathbb{Z})^{\mathcal{K}} = \mathcal{L}_{\mathcal{K}} \longrightarrow (S^1)^{\mathcal{K}} \longrightarrow (S^1)^m,$$

$$(D^1,S^0)^{\mathcal{K}} = \mathcal{R}_{\mathcal{K}} \longrightarrow (\mathbb{R}P^\infty)^{\mathcal{K}} \longrightarrow (\mathbb{R}P^\infty)^m.$$

By analogy with the polyhedral product of spaces $\boldsymbol{X}^{\mathcal{K}} = \operatorname{colim}_{I \in \mathcal{K}} \boldsymbol{X}^I$, we may consider the following construction of a discrete group.

By analogy with the polyhedral product of spaces $\boldsymbol{X}^{\mathcal{K}} = \mathrm{colim}_{I \in \mathcal{K}} \boldsymbol{X}^I$, we may consider the following construction of a discrete group.

Graph product

$\boldsymbol{G} = (G_1, \ldots, G_m)$ a sequence of m discrete groups, $G_i \neq \{1\}$.

By analogy with the polyhedral product of spaces $\boldsymbol{X}^{\mathcal{K}} = \operatorname{colim}_{I \in \mathcal{K}} \boldsymbol{X}^I$, we may consider the following construction of a discrete group.

Graph product

$\boldsymbol{G} = (G_1, \ldots, G_m)$ a sequence of m discrete groups, $G_i \neq \{1\}$.

Given $I = \{i_1, \ldots, i_k\} \subset [m]$, set

$$\boldsymbol{G}^I = \Big\{(g_1, \ldots, g_m) \in \prod_{k=1}^{m} G_k \colon\ g_k = 1 \quad \text{for } k \notin I\Big\}.$$

By analogy with the polyhedral product of spaces $\boldsymbol{X}^{\mathcal{K}} = \operatorname{colim}_{I \in \mathcal{K}} \boldsymbol{X}^I$, we may consider the following construction of a discrete group.

Graph product

$\boldsymbol{G} = (G_1, \ldots, G_m)$ a sequence of m discrete groups, $G_i \neq \{1\}$.

Given $I = \{i_1, \ldots, i_k\} \subset [m]$, set

$$\boldsymbol{G}^I = \Big\{(g_1, \ldots, g_m) \in \prod_{k=1}^{m} G_k \colon \; g_k = 1 \quad \text{for } k \notin I\Big\}.$$

Then consider the following $\mathrm{CAT}(\mathcal{K})$-diagram of groups:

$$\mathcal{D}_{\mathcal{K}}(\boldsymbol{G}) \colon \mathrm{CAT}(\mathcal{K}) \longrightarrow \mathrm{GRP}, \qquad I \longmapsto \boldsymbol{G}^I,$$

which maps a morphism $I \subset J$ to the canonical monomorphism $\boldsymbol{G}^I \to \boldsymbol{G}^J$.

By analogy with the polyhedral product of spaces $\boldsymbol{X}^{\mathcal{K}} = \operatorname{colim}_{I\in\mathcal{K}} \boldsymbol{X}^I$, we may consider the following construction of a discrete group.

Graph product

$\boldsymbol{G} = (G_1, \ldots, G_m)$ a sequence of m discrete groups, $G_i \neq \{1\}$.

Given $I = \{i_1, \ldots, i_k\} \subset [m]$, set

$$\boldsymbol{G}^I = \Bigl\{(g_1, \ldots, g_m) \in \prod_{k=1}^{m} G_k \colon \; g_k = 1 \quad \text{for } k \notin I\Bigr\}.$$

Then consider the following $\mathrm{CAT}(\mathcal{K})$-diagram of groups:

$$\mathcal{D}_{\mathcal{K}}(\boldsymbol{G}) \colon \mathrm{CAT}(\mathcal{K}) \longrightarrow \mathrm{GRP}, \qquad I \longmapsto \boldsymbol{G}^I,$$

which maps a morphism $I \subset J$ to the canonical monomorphism $\boldsymbol{G}^I \to \boldsymbol{G}^J$.

The graph product of the groups $G_1, \ldots, G_m$ is

$$\boldsymbol{G}^{\mathcal{K}} = \operatorname{colim}^{\mathrm{GRP}} \mathcal{D}_{\mathcal{K}}(\boldsymbol{G}) = \operatorname{colim}^{\mathrm{GRP}}_{I\in\mathcal{K}} \boldsymbol{G}^I.$$

The graph product $\mathbf{G}^{\mathcal{K}}$ depends only on the 1-skeleton (graph) of $\mathcal{K}$. Namely, The graph product $\boldsymbol{G}^{\mathcal{K}}$ depends only on the 1-skeleton (graph) of $\mathcal{K}$. Namely,

Proposition

The is an isomorphism of groups

$$\boldsymbol{G}^{\mathcal{K}} \cong \mathop{\bigstar}_{k=1}^{m} G_k \big/ (g_i g_j = g_j g_i \ \text{ for } g_i \in G_i,\ g_j \in G_j,\ \{i,j\} \in \mathcal{K}),$$

where $\bigstar_{k=1}^{m} G_k$ *denotes the free product of the groups* G_k*.*

Example

Let $G_i = \mathbb{Z}$. Then $\boldsymbol{G}^{\mathcal{K}}$ is the right-angled Artin group

$$RA_{\mathcal{K}} = F(g_1, \ldots, g_m) \big/ (g_i g_j = g_j g_i \text{ for } \{i, j\} \in \mathcal{K}),$$

where $F(g_1, \ldots, g_m)$ is a free group with m generators.

Example

Let $G_i = \mathbb{Z}$. Then $\boldsymbol{G}^{\mathcal{K}}$ is the right-angled Artin group

$$RA_{\mathcal{K}} = F(g_1, \dots, g_m) / (g_i g_j = g_j g_i \text{ for } \{i, j\} \in \mathcal{K}),$$

where $F(g_1, \dots, g_m)$ is a free group with m generators.

When $\mathcal{K}$ is a full simplex, we have $RA_{\mathcal{K}} = \mathbb{Z}^m$. When $\mathcal{K}$ is m points, we obtain a free group of rank m.

Example

Let $G_i = \mathbb{Z}$. Then $\boldsymbol{G}^{\mathcal{K}}$ is the right-angled Artin group

$$RA_{\mathcal{K}} = F(g_1, \ldots, g_m) \big/ (g_i g_j = g_j g_i \text{ for } \{i, j\} \in \mathcal{K}),$$

where $F(g_1, \ldots, g_m)$ is a free group with m generators.

When $\mathcal{K}$ is a full simplex, we have $RA_{\mathcal{K}} = \mathbb{Z}^m$. When $\mathcal{K}$ is m points, we obtain a free group of rank m.

Example

Let $G_i = \mathbb{Z}_2$. Then $\boldsymbol{G}^{\mathcal{K}}$ is the right-angled Coxeter group

$$RC_{\mathcal{K}} = F(g_1, \ldots, g_m) \big/ (g_i^2 = 1,\ g_i g_j = g_j g_i \text{ for } \{i, j\} \in \mathcal{K}).$$

2. Classifying spaces

The homotopy fibrations $\mathcal{L}_\mathcal{K} \to (S^1)^\mathcal{K} \to (S^1)^m$ and $\mathcal{R}_\mathcal{K} \to (\mathbb{R}P^\infty)^\mathcal{K} \to (\mathbb{R}P^\infty)^m$ are generalised as follows.

2. Classifying spaces

The homotopy fibrations $\mathcal{L}_{\mathcal{K}} \to (S^1)^{\mathcal{K}} \to (S^1)^m$ and $\mathcal{R}_{\mathcal{K}} \to (\mathbb{R}P^\infty)^{\mathcal{K}} \to (\mathbb{R}P^\infty)^m$ are generalised as follows.

Proposition

There is a homotopy fibration

$$(E\boldsymbol{G}, \boldsymbol{G})^{\mathcal{K}} \longrightarrow (B\boldsymbol{G})^{\mathcal{K}} \longrightarrow \prod_{k=1}^{m} BG_k.$$

2. Classifying spaces

The homotopy fibrations $\mathcal{L}_\mathcal{K} \to (S^1)^\mathcal{K} \to (S^1)^m$ and $\mathcal{R}_\mathcal{K} \to (\mathbb{R}P^\infty)^\mathcal{K} \to (\mathbb{R}P^\infty)^m$ are generalised as follows.

Proposition

There is a homotopy fibration

$$(E\boldsymbol{G}, \boldsymbol{G})^\mathcal{K} \longrightarrow (B\boldsymbol{G})^\mathcal{K} \longrightarrow \prod_{k=1}^{m} BG_k.$$

A missing face (a minimal non-face) of $\mathcal{K}$ is a subset $I \subset [m]$ such that $I \notin \mathcal{K}$, but $J \in \mathcal{K}$ for each $J \subsetneq I$.

2. Classifying spaces

The homotopy fibrations $\mathcal{L}_\mathcal{K} \to (S^1)^\mathcal{K} \to (S^1)^m$ and $\mathcal{R}_\mathcal{K} \to (\mathbb{R}P^\infty)^\mathcal{K} \to (\mathbb{R}P^\infty)^m$ are generalised as follows.

Proposition

There is a homotopy fibration

$$(E\boldsymbol{G}, \boldsymbol{G})^\mathcal{K} \longrightarrow (B\boldsymbol{G})^\mathcal{K} \longrightarrow \prod_{k=1}^{m} BG_k.$$

A missing face (a minimal non-face) of $\mathcal{K}$ is a subset $I \subset [m]$ such that $I \notin \mathcal{K}$, but $J \in \mathcal{K}$ for each $J \subsetneq I$.

$\mathcal{K}$ a flag complex if each of its missing faces consists of two vertices. Equivalently, $\mathcal{K}$ is flag if any set of vertices of $\mathcal{K}$ which are pairwise connected by edges spans a simplex.

2. Classifying spaces

The homotopy fibrations $\mathcal{L}_\mathcal{K} \to (S^1)^\mathcal{K} \to (S^1)^m$ and $\mathcal{R}_\mathcal{K} \to (\mathbb{R}P^\infty)^\mathcal{K} \to (\mathbb{R}P^\infty)^m$ are generalised as follows.

Proposition

There is a homotopy fibration

$$(E\mathbf{G}, \mathbf{G})^\mathcal{K} \longrightarrow (B\mathbf{G})^\mathcal{K} \longrightarrow \prod_{k=1}^{m} BG_k.$$

A missing face (a minimal non-face) of $\mathcal{K}$ is a subset $I \subset [m]$ such that $I \notin \mathcal{K}$, but $J \in \mathcal{K}$ for each $J \subsetneq I$.

$\mathcal{K}$ a flag complex if each of its missing faces consists of two vertices. Equivalently, $\mathcal{K}$ is flag if any set of vertices of $\mathcal{K}$ which are pairwise connected by edges spans a simplex.

Every flag complex $\mathcal{K}$ is determined by its 1-skeleton $\mathcal{K}^1$.

Theorem

Let $\mathbf{G}^{\mathcal{K}}$ *be a graph product group.*

Theorem

Let $\mathbf{G}^{\mathcal{K}}$ *be a graph product group.*

1. $\pi_1((B\mathbf{G})^{\mathcal{K}}) \cong \mathbf{G}^{\mathcal{K}}$.

Theorem

Let $\mathbf{G}^{\mathcal{K}}$ be a graph product group.

1. $\pi_1((B\mathbf{G})^{\mathcal{K}}) \cong \mathbf{G}^{\mathcal{K}}$.
2. *Both spaces $(B\mathbf{G})^{\mathcal{K}}$ and $(E\mathbf{G}, \mathbf{G})^{\mathcal{K}}$ are aspherical if and only if $\mathcal{K}$ is flag. Hence, $B(\mathbf{G}^{\mathcal{K}}) = (B\mathbf{G})^{\mathcal{K}}$ whenever $\mathcal{K}$ is flag.*

Theorem

Let $\mathbf{G}^{\mathcal{K}}$ *be a graph product group.*

1. $\pi_1((B\mathbf{G})^{\mathcal{K}}) \cong \mathbf{G}^{\mathcal{K}}$.
2. *Both spaces* $(B\mathbf{G})^{\mathcal{K}}$ *and* $(E\mathbf{G}, \mathbf{G})^{\mathcal{K}}$ *are aspherical if and only if* $\mathcal{K}$ *is flag. Hence,* $B(\mathbf{G}^{\mathcal{K}}) = (B\mathbf{G})^{\mathcal{K}}$ *whenever* $\mathcal{K}$ *is flag.*
3. $\pi_i((B\mathbf{G})^{\mathcal{K}}) \cong \pi_i((E\mathbf{G}, \mathbf{G})^{\mathcal{K}})$ *for* $i \geqslant 2$.

Theorem

Let $\mathbf{G}^{\mathcal{K}}$ be a graph product group.

1. $\pi_1((B\mathbf{G})^{\mathcal{K}}) \cong \mathbf{G}^{\mathcal{K}}$.
2. *Both spaces* $(B\mathbf{G})^{\mathcal{K}}$ *and* $(E\mathbf{G}, \mathbf{G})^{\mathcal{K}}$ *are aspherical if and only if* $\mathcal{K}$ *is flag. Hence,* $B(\mathbf{G}^{\mathcal{K}}) = (B\mathbf{G})^{\mathcal{K}}$ *whenever* $\mathcal{K}$ *is flag.*
3. $\pi_i((B\mathbf{G})^{\mathcal{K}}) \cong \pi_i((E\mathbf{G}, \mathbf{G})^{\mathcal{K}})$ *for* $i \geqslant 2$.
4. $\pi_1((E\mathbf{G}, \mathbf{G})^{\mathcal{K}})$ *is isomorphic to the kernel of the canonical projection* $\mathbf{G}^{\mathcal{K}} \to \prod_{k=1}^{m} G_k$.

Theorem

Let $\mathbf{G}^{\mathcal{K}}$ *be a graph product group.*

1. $\pi_1((B\mathbf{G})^{\mathcal{K}}) \cong \mathbf{G}^{\mathcal{K}}$.
2. *Both spaces* $(B\mathbf{G})^{\mathcal{K}}$ *and* $(E\mathbf{G},\mathbf{G})^{\mathcal{K}}$ *are aspherical if and only if* $\mathcal{K}$ *is flag. Hence,* $B(\mathbf{G}^{\mathcal{K}}) = (B\mathbf{G})^{\mathcal{K}}$ *whenever* $\mathcal{K}$ *is flag.*
3. $\pi_i((B\mathbf{G})^{\mathcal{K}}) \cong \pi_i((E\mathbf{G},\mathbf{G})^{\mathcal{K}})$ *for* $i \geqslant 2$.
4. $\pi_1((E\mathbf{G},\mathbf{G})^{\mathcal{K}})$ *is isomorphic to the kernel of the canonical projection* $\mathbf{G}^{\mathcal{K}} \to \prod_{k=1}^{m} G_k$.

Proof

(1) Proceed inductively by adding simplices to $\mathcal{K}$ one by one and use van Kampen's Theorem. The base of the induction is $\mathcal{K}$ consisting of m disjoint points. Then $(B\mathbf{G})^{\mathcal{K}}$ is the wedge $BG_1 \vee \cdots \vee BG_m$, and $\pi_1((B\mathbf{G})^{\mathcal{K}})$ is the free product $G_1 \star \cdots \star G_m$.

Proof

(2) To see that $B(\mathbf{G}^{\mathcal{K}}) = (B\mathbf{G})^{\mathcal{K}}$ when $\mathcal{K}$ is flag, consider the map

$$\operatorname{colim}_{I\in\mathcal{K}} B\mathbf{G}^I = (B\mathbf{G})^{\mathcal{K}} \to B(\mathbf{G}^{\mathcal{K}}). \tag{1}$$

Proof

(2) To see that $B(\boldsymbol{G}^{\mathcal{K}}) = (B\boldsymbol{G})^{\mathcal{K}}$ when $\mathcal{K}$ is flag, consider the map

$$\operatorname{colim}_{I\in\mathcal{K}} B\boldsymbol{G}^{I} = (B\boldsymbol{G})^{\mathcal{K}} \to B(\boldsymbol{G}^{\mathcal{K}}). \qquad (1)$$

According to [PRV], the homotopy fibre of (1) is $\operatorname{hocolim}_{I\in\mathcal{K}} \boldsymbol{G}^{\mathcal{K}}/\boldsymbol{G}^{I}$, which is homeomorphic to the identification space

$$\bigl(B\mathrm{CAT}(\mathcal{K}) \times \boldsymbol{G}^{\mathcal{K}}\bigr)/\sim . \qquad (2)$$

Proof

(2) To see that $B(\boldsymbol{G}^{\mathcal{K}}) = (B\boldsymbol{G})^{\mathcal{K}}$ when $\mathcal{K}$ is flag, consider the map

$$\operatorname{colim}_{I\in\mathcal{K}} B\boldsymbol{G}^{I} = (B\boldsymbol{G})^{\mathcal{K}} \to B(\boldsymbol{G}^{\mathcal{K}}). \qquad (1)$$

According to [PRV], the homotopy fibre of (1) is $\operatorname{hocolim}_{I\in\mathcal{K}} \boldsymbol{G}^{\mathcal{K}}/\boldsymbol{G}^{I}$, which is homeomorphic to the identification space

$$\bigl(B\mathrm{CAT}(\mathcal{K}) \times \boldsymbol{G}^{\mathcal{K}}\bigr)/\sim\,. \qquad (2)$$

Here $B\mathrm{CAT}(\mathcal{K})$ is homeomorphic to the cone on $|\mathcal{K}|$. The equivalence relation $\sim$ is defined as follows: $(x, gh)\sim(x, g)$ whenever $h \in \boldsymbol{G}^{I}$ and $x \in B(I\downarrow\mathrm{CAT}(\mathcal{K}))$, where $I\downarrow\mathrm{CAT}(\mathcal{K})$ is the *undercategory*, and $B(I\downarrow\mathrm{CAT}(\mathcal{K}))$ is homeomorphic to the star of I in $\mathcal{K}$.

Proof

(2) To see that $B(\boldsymbol{G}^{\mathcal{K}}) = (B\boldsymbol{G})^{\mathcal{K}}$ when $\mathcal{K}$ is flag, consider the map

$$\operatorname{colim}_{I\in\mathcal{K}} B\boldsymbol{G}^I = (B\boldsymbol{G})^{\mathcal{K}} \to B(\boldsymbol{G}^{\mathcal{K}}). \tag{1}$$

According to [PRV], the homotopy fibre of (1) is $\operatorname{hocolim}_{I\in\mathcal{K}} \boldsymbol{G}^{\mathcal{K}}/\boldsymbol{G}^I$, which is homeomorphic to the identification space

$$\bigl(B\mathrm{CAT}(\mathcal{K}) \times \boldsymbol{G}^{\mathcal{K}}\bigr)/\sim . \tag{2}$$

Here $B\mathrm{CAT}(\mathcal{K})$ is homeomorphic to the cone on $|\mathcal{K}|$. The equivalence relation $\sim$ is defined as follows: $(x,gh)\sim(x,g)$ whenever $h \in \boldsymbol{G}^I$ and $x \in B(I\downarrow\mathrm{CAT}(\mathcal{K}))$, where $I\downarrow\mathrm{CAT}(\mathcal{K})$ is the *undercategory*, and $B(I\downarrow\mathrm{CAT}(\mathcal{K}))$ is homeomorphic to the star of I in $\mathcal{K}$.
When $\mathcal{K}$ is a flag complex, the identification space (2) is contractible by [PRV]. Therefore, the map (1) is a homotopy equivalence, which implies that $(B\boldsymbol{G})^{\mathcal{K}}$ is aspherical when $\mathcal{K}$ is flag.

Proof

Assume now that $\mathcal{K}$ is not flag. Choose a missing face $J = \{j_1, \ldots, j_k\} \subset [m]$ with $k \geqslant 3$ vertices. Let $\mathcal{K}_J = \{I \in \mathcal{K} \colon I \subset J\}$.

Proof

Assume now that $\mathcal{K}$ is not flag. Choose a missing face $J = \{j_1, \ldots, j_k\} \subset [m]$ with $k \geqslant 3$ vertices. Let $\mathcal{K}_J = \{I \in \mathcal{K} \colon I \subset J\}$.

Then $(B\boldsymbol{G})^{\mathcal{K}_J}$ is the fat wedge of the spaces $\{BG_j, j \in J\}$, and it is a retract of $(B\boldsymbol{G})^{\mathcal{K}}$.

Proof

Assume now that $\mathcal{K}$ is not flag. Choose a missing face $J = \{j_1, \ldots, j_k\} \subset [m]$ with $k \geqslant 3$ vertices. Let $\mathcal{K}_J = \{I \in \mathcal{K} \colon I \subset J\}$.

Then $(B\boldsymbol{G})^{\mathcal{K}_J}$ is the fat wedge of the spaces $\{BG_j, j \in J\}$, and it is a retract of $(B\boldsymbol{G})^{\mathcal{K}}$.

The homotopy fibre of the inclusion $(B\boldsymbol{G})^{\mathcal{K}_J} \to \prod_{j \in J} BG_j$ is $\Sigma^{k-1} G_{j_1} \wedge \cdots \wedge G_{j_k}$, a wedge of $(k-1)$-dimensional spheres.

Proof

Assume now that $\mathcal{K}$ is not flag. Choose a missing face $J = \{j_1, \ldots, j_k\} \subset [m]$ with $k \geqslant 3$ vertices. Let $\mathcal{K}_J = \{I \in \mathcal{K} \colon I \subset J\}$.

Then $(B\boldsymbol{G})^{\mathcal{K}_J}$ is the fat wedge of the spaces $\{BG_j, j \in J\}$, and it is a retract of $(B\boldsymbol{G})^{\mathcal{K}}$.

The homotopy fibre of the inclusion $(B\boldsymbol{G})^{\mathcal{K}_J} \to \prod_{j \in J} BG_j$ is $\Sigma^{k-1} G_{j_1} \wedge \cdots \wedge G_{j_k}$, a wedge of $(k-1)$-dimensional spheres.
Hence, $\pi_{k-1}((B\boldsymbol{G})^{\mathcal{K}_J}) \neq 0$ where $k \geqslant 3$.
Thus, $(B\boldsymbol{G})^{\mathcal{K}_J}$ and $(B\boldsymbol{G})^{\mathcal{K}}$ are non-aspherical.

Proof

Assume now that $\mathcal{K}$ is not flag. Choose a missing face $J = \{j_1, \ldots, j_k\} \subset [m]$ with $k \geqslant 3$ vertices. Let $\mathcal{K}_J = \{I \in \mathcal{K}\colon I \subset J\}$.

Then $(B\boldsymbol{G})^{\mathcal{K}_J}$ is the fat wedge of the spaces $\{BG_j, j \in J\}$, and it is a retract of $(B\boldsymbol{G})^{\mathcal{K}}$.

The homotopy fibre of the inclusion $(B\boldsymbol{G})^{\mathcal{K}_J} \to \prod_{j\in J} BG_j$ is $\Sigma^{k-1} G_{j_1} \wedge \cdots \wedge G_{j_k}$, a wedge of $(k-1)$-dimensional spheres.
Hence, $\pi_{k-1}((B\boldsymbol{G})^{\mathcal{K}_J}) \neq 0$ where $k \geqslant 3$.
Thus, $(B\boldsymbol{G})^{\mathcal{K}_J}$ and $(B\boldsymbol{G})^{\mathcal{K}}$ are non-aspherical.

The rest of the proof (the asphericity of $(E\boldsymbol{G}, \boldsymbol{G})^{\mathcal{K}}$ and statements (3) and (4)) follow from the homotopy exact sequence of the fibration $(E\boldsymbol{G}, \boldsymbol{G})^{\mathcal{K}} \to (B\boldsymbol{G})^{\mathcal{K}} \to \prod_{k=1}^{m} BG_k$.

Specialising to the cases $G_k = \mathbb{Z}$ and $G_k = \mathbb{Z}_2$ respectively we obtain:

Specialising to the cases $G_k = \mathbb{Z}$ and $G_k = \mathbb{Z}_2$ respectively we obtain:

Corollary

Let $RA_{\mathcal{K}}$ be a right-angled Artin group.

1. $\pi_1((S^1)^{\mathcal{K}}) \cong RA_{\mathcal{K}}$.
2. *Both* $(S^1)^{\mathcal{K}}$ *and* $\mathcal{L}_{\mathcal{K}} = (\mathbb{R}, \mathbb{Z})^{\mathcal{K}}$ *are aspherical iff* $\mathcal{K}$ *is flag.*
3. $\pi_i((S^1)^{\mathcal{K}}) \cong \pi_i(\mathcal{L}_{\mathcal{K}})$ *for* $i \geqslant 2$.
4. $\pi_1(\mathcal{L}_{\mathcal{K}})$ *is isomorphic to the commutator subgroup* $RA'_{\mathcal{K}}$.

Specialising to the cases $G_k = \mathbb{Z}$ and $G_k = \mathbb{Z}_2$ respectively we obtain:

Corollary

Let $RA_\mathcal{K}$ be a right-angled Artin group.

1. $\pi_1((S^1)^\mathcal{K}) \cong RA_\mathcal{K}$.
2. *Both* $(S^1)^\mathcal{K}$ *and* $\mathcal{L}_\mathcal{K} = (\mathbb{R}, \mathbb{Z})^\mathcal{K}$ *are aspherical iff* $\mathcal{K}$ *is flag.*
3. $\pi_i((S^1)^\mathcal{K}) \cong \pi_i(\mathcal{L}_\mathcal{K})$ *for* $i \geqslant 2$.
4. $\pi_1(\mathcal{L}_\mathcal{K})$ *is isomorphic to the commutator subgroup* $RA'_\mathcal{K}$.

Corollary

Let $RC_\mathcal{K}$ be a right-angled Coxeter group.

1. $\pi_1((\mathbb{R}P^\infty)^\mathcal{K}) \cong RC_\mathcal{K}$.
2. *Both* $(\mathbb{R}P^\infty)^\mathcal{K}$ *and* $\mathcal{R}_\mathcal{K} = (D^1, S^0)^\mathcal{K}$ *are aspherical iff* $\mathcal{K}$ *is flag.*
3. $\pi_i((\mathbb{R}P^\infty)^\mathcal{K}) \cong \pi_i(\mathcal{R}_\mathcal{K})$ *for* $i \geqslant 2$.
4. $\pi_1(\mathcal{R}_\mathcal{K})$ *is isomorphic to the commutator subgroup* $RC'_\mathcal{K}$.

Example

Let $\mathcal{K}$ be an m-cycle (the boundary of an m-gon).
A simple argument with Euler characteristic shows that $\mathcal{R}_{\mathcal{K}}$ is homeomorphic to a closed orientable surface of genus $(m-4)2^{m-3}+1$.

Example

Let $\mathcal{K}$ be an m-cycle (the boundary of an m-gon).
A simple argument with Euler characteristic shows that $\mathcal{R}_\mathcal{K}$ is homeomorphic to a closed orientable surface of genus $(m-4)2^{m-3}+1$.
(This observation goes back to a 1938 work of Coxeter.)
Therefore, the commutator subgroup of the corresponding right-angled Coxeter group $RC_\mathcal{K}$ is a surface group.

Example

Let $\mathcal{K}$ be an m-cycle (the boundary of an m-gon).
A simple argument with Euler characteristic shows that $\mathcal{R}_\mathcal{K}$ is homeomorphic to a closed orientable surface of genus $(m-4)2^{m-3}+1$.
(This observation goes back to a 1938 work of Coxeter.)
Therefore, the commutator subgroup of the corresponding right-angled Coxeter group $RC_\mathcal{K}$ is a surface group.

Similarly, when $|\mathcal{K}| \cong S^2$ (which is equivalent to $\mathcal{K}$ being the boundary of a 3-dimensional simplicial polytope), $\mathcal{R}_\mathcal{K}$ is a 3-dimensional manifold.
Therefore, the commutator subgroup of the corresponding $RC_\mathcal{K}$ is a 3-manifold group.

3. The structure of the commutator subgroups

We have

$$\operatorname{Ker}\Big(\boldsymbol{G}^{\mathcal{K}} \to \prod_{k=1}^{m} G_k\Big) = \pi_1((E\boldsymbol{G}, \boldsymbol{G})^{\mathcal{K}}).$$

3. The structure of the commutator subgroups

We have

$$\mathrm{Ker}\Big(\boldsymbol{G}^{\mathcal{K}} \to \prod_{k=1}^{m} G_k\Big) = \pi_1((E\boldsymbol{G}, \boldsymbol{G})^{\mathcal{K}}).$$

In the case of right-angled Artin or Coxeter groups (or when each G_k is abelian), the group above is the commutator subgroup $(\boldsymbol{G}^{\mathcal{K}})'$.

3. The structure of the commutator subgroups

We have

$$\operatorname{Ker}\Big(\mathbf{G}^{\mathcal{K}} \to \prod_{k=1}^{m} G_k\Big) = \pi_1((E\mathbf{G}, \mathbf{G})^{\mathcal{K}}).$$

In the case of right-angled Artin or Coxeter groups (or when each G_k is abelian), the group above is the commutator subgroup $(\mathbf{G}^{\mathcal{K}})'$.

We want to study the group $\pi_1((E\mathbf{G}, \mathbf{G})^{\mathcal{K}})$, identify the class of simplicial complexes $\mathcal{K}$ for which this group is free, and describe a generator set.

3. The structure of the commutator subgroups

We have

$$\operatorname{Ker}\Big(\boldsymbol{G}^{\mathcal{K}} \to \prod_{k=1}^{m} G_k\Big) = \pi_1((E\boldsymbol{G}, \boldsymbol{G})^{\mathcal{K}}).$$

In the case of right-angled Artin or Coxeter groups (or when each G_k is abelian), the group above is the commutator subgroup $(\boldsymbol{G}^{\mathcal{K}})'$.

We want to study the group $\pi_1((E\boldsymbol{G}, \boldsymbol{G})^{\mathcal{K}})$, identify the class of simplicial complexes $\mathcal{K}$ for which this group is free, and describe a generator set.

A graph Γ is called chordal (in other terminology, triangulated) if each of its cycles with $\geqslant 4$ vertices has a chord.

3. The structure of the commutator subgroups

We have

$$\mathrm{Ker}\Big(\boldsymbol{G}^{\mathcal{K}} \to \prod_{k=1}^{m} G_k\Big) = \pi_1((E\boldsymbol{G},\boldsymbol{G})^{\mathcal{K}}).$$

In the case of right-angled Artin or Coxeter groups (or when each G_k is abelian), the group above is the commutator subgroup $(\boldsymbol{G}^{\mathcal{K}})'$.

We want to study the group $\pi_1((E\boldsymbol{G},\boldsymbol{G})^{\mathcal{K}})$, identify the class of simplicial complexes $\mathcal{K}$ for which this group is free, and describe a generator set.

A graph Γ is called chordal (in other terminology, triangulated) if each of its cycles with $\geqslant 4$ vertices has a chord.

By a result of Fulkerson–Gross, a graph is chordal if and only if its vertices can be ordered in such a way that, for each vertex i, the lesser neighbours of i form a complete subgraph. (A perfect elimination order.)

Theorem

The following conditions are equivalent:

Theorem

The following conditions are equivalent:

1. $\mathrm{Ker}(\boldsymbol{G}^{\mathcal{K}} \to \prod_{k=1}^{m} G_k)$ *is a free group;*

Theorem

The following conditions are equivalent:

1. $\mathrm{Ker}(\boldsymbol{G}^{\mathcal{K}} \to \prod_{k=1}^{m} G_k)$ *is a free group;*
2. $(E\boldsymbol{G}, \boldsymbol{G})^{\mathcal{K}}$ *is homotopy equivalent to a wedge of circles;*

Theorem

The following conditions are equivalent:

1. $\mathrm{Ker}(\boldsymbol{G}^{\mathcal{K}} \to \prod_{k=1}^{m} G_k)$ *is a free group;*
2. $(E\boldsymbol{G}, \boldsymbol{G})^{\mathcal{K}}$ *is homotopy equivalent to a wedge of circles;*
3. $\mathcal{K}^1$ *is a chordal graph.*

Theorem

The following conditions are equivalent:

1. $\mathrm{Ker}(\boldsymbol{G}^{\mathcal{K}} \to \prod_{k=1}^{m} G_k)$ *is a free group;*
2. $(E\boldsymbol{G}, \boldsymbol{G})^{\mathcal{K}}$ *is homotopy equivalent to a wedge of circles;*
3. $\mathcal{K}^1$ *is a chordal graph.*

Proof

(2)⇒(1) Because $\mathrm{Ker}\Big(\boldsymbol{G}^{\mathcal{K}} \to \prod_{k=1}^{m} G_k\Big) = \pi_1((E\boldsymbol{G}, \boldsymbol{G})^{\mathcal{K}})$.

Theorem

The following conditions are equivalent:

1. $\mathrm{Ker}(\boldsymbol{G}^{\mathcal{K}} \to \prod_{k=1}^{m} G_k)$ *is a free group;*
2. $(E\boldsymbol{G}, \boldsymbol{G})^{\mathcal{K}}$ *is homotopy equivalent to a wedge of circles;*
3. $\mathcal{K}^1$ *is a chordal graph.*

Proof

(2)⇒(1) Because $\mathrm{Ker}\Big(\boldsymbol{G}^{\mathcal{K}} \to \prod_{k=1}^{m} G_k\Big) = \pi_1((E\boldsymbol{G}, \boldsymbol{G})^{\mathcal{K}})$.

(3)⇒(2) Use induction and perfect elimination order.

Theorem

The following conditions are equivalent:

1. $\mathrm{Ker}(\boldsymbol{G}^{\mathcal{K}} \to \prod_{k=1}^{m} G_k)$ *is a free group;*
2. $(E\boldsymbol{G}, \boldsymbol{G})^{\mathcal{K}}$ *is homotopy equivalent to a wedge of circles;*
3. $\mathcal{K}^1$ *is a chordal graph.*

Proof

(2)⇒(1) Because $\mathrm{Ker}\Big(\boldsymbol{G}^{\mathcal{K}} \to \prod_{k=1}^{m} G_k\Big) = \pi_1((E\boldsymbol{G}, \boldsymbol{G})^{\mathcal{K}})$.

(3)⇒(2) Use induction and perfect elimination order.

(1)⇒(3) Assume that $\mathcal{K}^1$ is not chordal. Then, for each chordless cycle of length $\geqslant 4$, one can find a subgroup in $\mathrm{Ker}(\boldsymbol{G}^{\mathcal{K}} \to \prod_{k=1}^{m} G_k)$ which is a surface group. Hence, $\mathrm{Ker}(\boldsymbol{G}^{\mathcal{K}} \to \prod_{k=1}^{m} G_k)$ is not a free group.

Corollary

Let $RA_{\mathcal{K}}$ and $RC_{\mathcal{K}}$ be the right-angled Artin and Coxeter groups corresponding to a simplicial complex $\mathcal{K}$.

(a) *The commutator subgroup $RA'_{\mathcal{K}}$ is free if and only if $\mathcal{K}^1$ is a chordal graph.*

(b) *The commutator subgroup $RC'_{\mathcal{K}}$ is free if and only if $\mathcal{K}^1$ is a chordal graph.*

Corollary

Let $RA_{\mathcal{K}}$ and $RC_{\mathcal{K}}$ be the right-angled Artin and Coxeter groups corresponding to a simplicial complex $\mathcal{K}$.

(a) *The commutator subgroup $RA'_{\mathcal{K}}$ is free if and only if $\mathcal{K}^1$ is a chordal graph.*

(b) *The commutator subgroup $RC'_{\mathcal{K}}$ is free if and only if $\mathcal{K}^1$ is a chordal graph.*

Part (a) is the result of Servatius, Droms and Servatius.

Corollary

Let $RA_{\mathcal{K}}$ and $RC_{\mathcal{K}}$ be the right-angled Artin and Coxeter groups corresponding to a simplicial complex $\mathcal{K}$.

(a) *The commutator subgroup $RA'_{\mathcal{K}}$ is free if and only if $\mathcal{K}^1$ is a chordal graph.*

(b) *The commutator subgroup $RC'_{\mathcal{K}}$ is free if and only if $\mathcal{K}^1$ is a chordal graph.*

Part (a) is the result of Servatius, Droms and Servatius.

The difference between (a) and (b) is that the commutator subgroup $RA'_{\mathcal{K}}$ is infinitely generated, unless $RA_{\mathcal{K}} = \mathbb{Z}^m$, while the commutator subgroup $RC'_{\mathcal{K}}$ is finitely generated. We elaborate on this in the next theorem.

Let $(g, h) = g^{-1}h^{-1}gh$ denote the group commutator of g, h.

Let $(g, h) = g^{-1}h^{-1}gh$ denote the group commutator of g, h.

Theorem

The commutator subgroup $RC'_{\mathcal{K}}$ has a finite minimal generator set consisting of $\sum_{J\subset[m]} \operatorname{rank} \widetilde{H}_0(\mathcal{K}_J)$ iterated commutators

$$(g_j, g_i), \quad (g_{k_1}, (g_j, g_i)), \quad \ldots, \quad (g_{k_1}, (g_{k_2}, \cdots (g_{k_{m-2}}, (g_j, g_i)) \cdots)),$$

where $k_1 < k_2 < \cdots < k_{\ell-2} < j > i$, $k_s \neq i$ for any s, and i is the smallest vertex in a connected component not containing j of the subcomplex $\mathcal{K}_{\{k_1,\ldots,k_{\ell-2},j,i\}}$.

Let $(g,h) = g^{-1}h^{-1}gh$ denote the group commutator of g,h.

Theorem

The commutator subgroup $RC'_{\mathcal{K}}$ *has a finite minimal generator set consisting of* $\sum_{J\subset[m]} \operatorname{rank} \widetilde{H}_0(\mathcal{K}_J)$ *iterated commutators*

$$(g_j,g_i),\quad (g_{k_1},(g_j,g_i)),\quad \ldots,\quad (g_{k_1},(g_{k_2},\cdots(g_{k_{m-2}},(g_j,g_i))\cdots)),$$

where $k_1 < k_2 < \cdots < k_{\ell-2} < j > i$, $k_s \neq i$ *for any* s, *and* i *is the smallest vertex in a connected component not containing* j *of the subcomplex* $\mathcal{K}_{\{k_1,\ldots,k_{\ell-2},j,i\}}$.

Idea of proof

First consider the case $\mathcal{K} = m$ points. Then $\mathcal{R}_{\mathcal{K}}$ is the 1-skeleton of an m-cube and $RC'_{\mathcal{K}} = \pi_1(\mathcal{R}_{\mathcal{K}})$ is a free group of rank $\sum_{\ell=2}^{m}(\ell-1)\binom{m}{\ell}$. It agrees with the total number of nested commutators in the list.

Let $(g,h) = g^{-1}h^{-1}gh$ denote the group commutator of g,h.

Theorem

The commutator subgroup $RC'_{\mathcal{K}}$ has a finite minimal generator set consisting of $\sum_{J\subset[m]} \operatorname{rank} \widetilde{H}_0(\mathcal{K}_J)$ iterated commutators

$$(g_j, g_i), \quad (g_{k_1}, (g_j, g_i)), \quad \ldots, \quad (g_{k_1}, (g_{k_2}, \cdots (g_{k_{m-2}}, (g_j, g_i)) \cdots)),$$

where $k_1 < k_2 < \cdots < k_{\ell-2} < j > i$, $k_s \neq i$ for any s, and i is the smallest vertex in a connected component not containing j of the subcomplex $\mathcal{K}_{\{k_1,\ldots,k_{\ell-2},j,i\}}$.

Idea of proof

First consider the case $\mathcal{K} = m$ points. Then $\mathcal{R}_{\mathcal{K}}$ is the 1-skeleton of an m-cube and $RC'_{\mathcal{K}} = \pi_1(\mathcal{R}_{\mathcal{K}})$ is a free group of rank $\sum_{\ell=2}^{m}(\ell-1)\binom{m}{\ell}$. It agrees with the total number of nested commutators in the list.

Then eliminate the extra nested commutators using the commutation relations $(g_i, g_j) = 1$ for $\{i,j\} \in \mathcal{K}$.

Idea of proof

To see that the given generating set is minimal, argue as follows. The first homology group $H_1(\mathcal{R}_\mathcal{K})$ is $RC'_\mathcal{K}/RC''_\mathcal{K}$. On the other hand,

$$H_1(\mathcal{R}_\mathcal{K}) \cong \sum_{J\subset[m]} \widetilde{H}_0(\mathcal{K}_J).$$

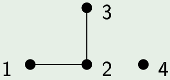
Idea of proof

To see that the given generating set is minimal, argue as follows. The first homology group $H_1(\mathcal{R}_\mathcal{K})$ is $RC'_\mathcal{K}/RC''_\mathcal{K}$. On the other hand,

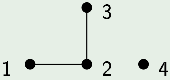
$$H_1(\mathcal{R}_\mathcal{K}) \cong \sum_{J\subset[m]} \widetilde{H}_0(\mathcal{K}_J).$$

Hence, the number of generators in the abelian group $H_1(\mathcal{R}_\mathcal{K}) \cong RC'_\mathcal{K}/RC''_\mathcal{K}$ is $\sum_{J\subset[m]} \mathrm{rank}\, \widetilde{H}_0(\mathcal{K}_J)$, and the latter number agrees with the number of iterated commutators in the in generator set for $RC'_\mathcal{K}$ constructed above.

Example

Let $\mathcal{K} =$ 

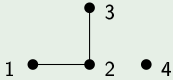
Example

Let $\mathcal{K} =$ 

Then the commutator subgroup $RC'_{\mathcal{K}}$ is free with the following basis:

$$(g_3, g_1),\ (g_4, g_1),\ (g_4, g_2),\ (g_4, g_3),$$
$$(g_2, (g_4, g_1)),\ (g_3, (g_4, g_1)),\ (g_1, (g_4, g_3)),\ (g_3, (g_4, g_2)),$$
$$(g_2, (g_3, (g_4, g_1))).$$

Example

Let $\mathcal{K} =$ 

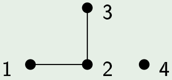
Then the commutator subgroup $RC'_{\mathcal{K}}$ is free with the following basis:

$$(g_3, g_1),\ (g_4, g_1),\ (g_4, g_2),\ (g_4, g_3),$$
$$(g_2, (g_4, g_1)),\ (g_3, (g_4, g_1)),\ (g_1, (g_4, g_3)),\ (g_3, (g_4, g_2)),$$
$$(g_2, (g_3, (g_4, g_1))).$$

Example

Let $\mathcal{K}$ be an m-cycle with $m \geqslant 4$ vertices.
Then $\mathcal{K}^1$ is not a chordal graph, so the group $RC'_{\mathcal{K}}$ is not free.

Example

Let $\mathcal{K} =$ 

Then the commutator subgroup $RC'_{\mathcal{K}}$ is free with the following basis:

$$(g_3, g_1),\ (g_4, g_1),\ (g_4, g_2),\ (g_4, g_3),$$
$$(g_2, (g_4, g_1)),\ (g_3, (g_4, g_1)),\ (g_1, (g_4, g_3)),\ (g_3, (g_4, g_2)),$$
$$(g_2, (g_3, (g_4, g_1))).$$

Example

Let $\mathcal{K}$ be an m-cycle with $m \geqslant 4$ vertices.
Then $\mathcal{K}^1$ is not a chordal graph, so the group $RC'_{\mathcal{K}}$ is not free.

In fact, $\mathcal{R}_{\mathcal{K}}$ is an orientable surface of genus $(m-4)2^{m-3}+1$, so $RC'_{\mathcal{K}} \cong \pi_1(\mathcal{R}_{\mathcal{K}})$ is a one-relator group.

The are similar results of Grbic, P., Theriault and Wu describing the commutator subalgebra of the graded Lie algebra given by

$$L_{\mathcal{K}} = FL\langle u_1, \ldots, u_m\rangle \big/ \big([u_i, u_i] = 0,\ [u_i, u_j] = 0 \text{ for } \{i, j\} \in \mathcal{K}\big),$$

The are similar results of Grbic, P., Theriault and Wu describing the commutator subalgebra of the graded Lie algebra given by

$$L_{\mathcal{K}} = FL\langle u_1, \ldots, u_m\rangle \big/ \big([u_i, u_i] = 0,\ [u_i, u_j] = 0 \text{ for } \{i, j\} \in \mathcal{K}\big),$$

where $FL\langle u_1, \ldots, u_m\rangle$ is the free graded Lie algebra on generators u_i of degree one, and $[a, b] = -(-1)^{|a||b|}[b, a]$ denotes the graded Lie bracket.

The are similar results of Grbic, P., Theriault and Wu describing the commutator subalgebra of the graded Lie algebra given by

$$L_{\mathcal{K}} = FL\langle u_1, \ldots, u_m\rangle \big/ \big([u_i, u_i] = 0,\ [u_i, u_j] = 0 \text{ for } \{i, j\} \in \mathcal{K}\big),$$

where $FL\langle u_1, \ldots, u_m\rangle$ is the free graded Lie algebra on generators u_i of degree one, and $[a, b] = -(-1)^{|a||b|}[b, a]$ denotes the graded Lie bracket.

The commutator subalgebra is the kernel of the Lie algebra homomorphism $L_{\mathcal{K}} \to CL\langle u_1, \ldots, u_m\rangle$ to the commutative (trivial) Lie algebra.

The are similar results of Grbic, P., Theriault and Wu describing the commutator subalgebra of the graded Lie algebra given by

$$L_{\mathcal{K}} = FL\langle u_1, \ldots, u_m \rangle \big/ \big([u_i, u_i] = 0,\ [u_i, u_j] = 0 \text{ for } \{i, j\} \in \mathcal{K}\big),$$

where $FL\langle u_1, \ldots, u_m \rangle$ is the free graded Lie algebra on generators u_i of degree one, and $[a, b] = -(-1)^{|a||b|}[b, a]$ denotes the graded Lie bracket.

The commutator subalgebra is the kernel of the Lie algebra homomorphism $L_{\mathcal{K}} \to CL\langle u_1, \ldots, u_m \rangle$ to the commutative (trivial) Lie algebra.

The graded Lie algebra $L_{\mathcal{K}}$ is a graph product similar to the right-angled Coxeter group $RC_{\mathcal{K}}$.

The are similar results of Grbic, P., Theriault and Wu describing the commutator subalgebra of the graded Lie algebra given by

$$L_{\mathcal{K}} = FL\langle u_1, \ldots, u_m \rangle \big/ \big([u_i, u_i] = 0,\ [u_i, u_j] = 0 \text{ for } \{i, j\} \in \mathcal{K} \big),$$

where $FL\langle u_1, \ldots, u_m \rangle$ is the free graded Lie algebra on generators u_i of degree one, and $[a, b] = -(-1)^{|a||b|}[b, a]$ denotes the graded Lie bracket.

The commutator subalgebra is the kernel of the Lie algebra homomorphism $L_{\mathcal{K}} \to CL\langle u_1, \ldots, u_m \rangle$ to the commutative (trivial) Lie algebra.

The graded Lie algebra $L_{\mathcal{K}}$ is a graph product similar to the right-angled Coxeter group $RC_{\mathcal{K}}$.

It has a similar colimit decomposition, with each $G_i = \mathbb{Z}_2$ replaced by the trivial Lie algebra $CL\langle u \rangle = FL\langle u \rangle / ([u, u] = 0)$ and the colimit taken in the category of graded Lie algebras.

References

[1] T. Panov, N. Ray and R. Vogt. *Colimits, Stanley-Reiner algebras, and loop spaces*, in: "Categorical Decomposition Techniques in Algebraic Topology" (G. Arone et al eds.), Progress in Mathematics, vol. 215, Birkhauser, Basel, 2004, pp. 261–291; arXiv:math.AT/0202081.

[2] J. Grbic, T. Panov, S. Theriault and J. Wu. *The homotopy types of moment-angle complexes for flag complexes.* Trans. of the Amer. Math. Soc. **368** (2016), no. 9, 6663–6682; arXiv:1211.0873.

[3] T. Panov and Ya. Veryovkin. *Polyhedral products and commutator subgroups of right-angled Artin and Coxeter groups.* Preprint (2016); arXiv:1603.06902.