H-contact unit tangent sphere bundles

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- $(\tilde{M}, \tilde{g})$ a compact, orientable Riemannian manifold.
- *Energy* of a map $f : (\tilde{M}, \tilde{g}) \to (N, h)$ is $E(f) = \frac{1}{2} \int_M \|df\|_h^2 dv_{\tilde{g}}$.
- *Energy of a unit vector field V* on $(\tilde{M}, \tilde{g})$ is the energy of the corresponding map between $(\tilde{M}, \tilde{g})$ and its tangent sphere bundle equipped with the Sasaki metric:

$$E(V) = \frac{m}{2} \mathrm{Vol}(\tilde{M}, \tilde{g}) + \frac{1}{2} \int_M \|\nabla V\|^2 dv_{\tilde{g}},$$

  where $m = \dim M$ [Wood, 1997].
- A unit vector field is *harmonic* if it is a critical point for the energy functional E on the set of all unit vector fields of $\tilde{M}$.
- First variation: local condition (compactness and orientability not required).

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Definition

A $(2n-1)$-dimensional manifold $\bar{M}$ equipped with structure tensors $(\bar{g}, \phi, \xi, \eta)$ satisfying

- η is a 1-form such that $\eta \wedge (d\eta)^{n-1} \neq 0$ (such a form is called a *contact form*).
- ξ is the (unique) vector field such that $\eta(\xi) = 1$ and $d\eta(\xi, X) = 0$ (called the *characteristic vector field*).
- ϕ is a $(1,1)$-tensor field such that $\phi^2 X = -X + \eta(X)\xi$.
- $\bar{g}$ is a Riemannian metric such that

$$\eta(X) = \bar{g}(X, \xi), \quad d\eta(X, Y) = \bar{g}(X, \phi Y).$$

is called a *contact metric manifold*.

As the consequence ξ is unit and geodesic.
A contact metric manifold whose characteristic vector field ξ is harmonic is called an *H-contact manifold*.
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An n-dimensional Riemannian manifold (M, g) is *2-stein* if there exist two functions $f_1, f_2 : M \to \mathbb{R}$ such that for every $p \in M$ and every vector X tangent to M at p we have

$$\operatorname{Tr} R_X = f_1(p)\|X\|^2, \qquad \operatorname{Tr}(R_X^2) = f_2(p)\|X\|^4,$$

where R_X is the Jacobi operator at p.

- Any 2-stein manifold is Einstein.
- f_1 is constant when $n \geq 3$ (Schur's Theorem), and f_2 is constant when $n \geq 5$ (e.g., [Besse, 1978]).
- In dimension $n = 4$, the structure of the curvature tensor of a 2-stein manifold is well known [Sekigawa-Vanhecke, 1986]; any four-dimensional 2-stein manifold is pointwise Osserman [Besse, 1978], hence self-dual (up to the choice of orientation) [Blažić-Gilkey, 2005], and as such, is either hyperkähler or quaternionic Kähler depending on the scalar curvature (zero or non-zero respectively).
- In dimension $n = 5$, one can be much more specific: any 2-stein manifold either has constant curvature, or up to scaling, is locally isometric either to the symmetric space $SU(3)/SO(3)$ or to its non-compact dual $SL(3)/SO(3)$ [N, 2005].

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Let (M, g) be an n-dimensional Riemannian manifold, with ∇, R and ρ the Levi-Civita connection, the Riemann curvature tensor and the Ricci tensor respectively.
Suppose that the unit tangent sphere bundle T_1M of a equipped with the standard contact metric structure is H-contact.
By the earlier results we can assume that $n \geq 5$.

Proposition (Calvaruso-Perrone, 2007)

The unit tangent sphere bundle T_1M of a Riemannian manifold (M, g) is H-contact if and only if the following two conditions are satisfied.

- *The Ricci tensor is* Codazzi, *that is, for arbitrary vector fields X, Y and Z on M we have*

$$\nabla_X \rho(Y, Z) = \nabla_Y \rho(X, Z).$$

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Theorem (Derdzinski-Shen, 1983)

Let $E_1, \ldots, E_N$ be the eigenspaces of a symmetric Codazzi tensor at $p \in M$. Then for $1 \leq \lambda, \mu, \nu \leq N$,

$$R(E_\lambda, E_\mu)E_\nu = 0, \quad \text{when } \nu \notin \{\lambda, \mu\}.$$

Condition 2:

Define an algebraic curvature tensor $\mathcal{R}$ by

$$\mathcal{R}(X,Y,Z,W) = R(X,Y,Z,W) - 2(\langle X,Z\rangle\langle Y,W\rangle - \langle X,W\rangle\langle Y,Z\rangle).$$

Lemma

Condition 2 is equivalent to the fact that the algebraic curvature tensor $\mathcal{R}$ satisfies the equation

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Proposition

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