Topology of non-compact integrable Hamiltonian systems: first steps

Stanislav Nikolaienko
Lomonosov Moscow State University

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Integrable Hamiltonian system

Definition

A *Hamiltonian system with n degrees of freedom* is a triple $(M^{2n}, \omega, v = \operatorname{sgrad} H)$ where

- M^{2n} is a smooth manifold;
- ω is a symplectic structure on M^{2n} (closed non-degenerate 2-form);
- H is smooth function on M^{2n};
- $\operatorname{sgrad} H = \omega^{-1} dH$ is a Hamiltonian vector field.

Definition

A Hamiltonian system $(M^{2n}, \omega, v = \operatorname{sgrad} H)$ is said to be *Liouville integrable* if there exist n first integrals $f_1, \ldots, f_n$ such that

- $\{f_i, f_j\} = 0$ for any i, j;
- the differentials $df_1, \ldots, df_n$ are linearly independent almost everywhere on M^{2n};
- the vector fields $\operatorname{sgrad} f_1, \ldots, \operatorname{sgrad} f_n$ are complete (i.e. the natural parameter on their integral trajectories is defined on the whole real axis).

Liouville foliation

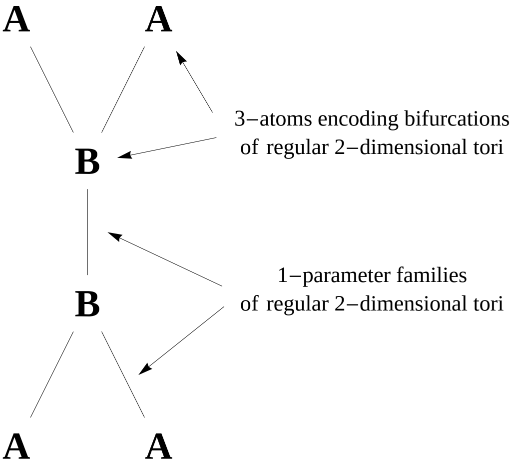
Definition

The decomposition of the manifold M^{2n} into connected components of the level sets $\{f_1 = c_1, \dots, f_n = c_n\}$ is called the *Liouville foliation* corresponding to a given Liouville integrable Hamiltonian system.

In the typical case almost all the leaves of the Liouville foliation are closures of the integral trajectories.
Problem: to describe the topology of the Liouville foliation of a given integrable Hamiltonian system.
Classical Liouville theorem $\rightarrow$ almost all the leaves (regular leaves) are diffeomorfic to $\mathbb{R}^n/\mathbb{Z}^k$, and in their neighbourhood the foliation is trivial. There exist also critical leaves corresponding to bifurcations of regular leaves.

2 degrees of freedom, compact case

$Q^3 = \{H = h\}$ – 3-dimensional surface of constant energy with the structure of the Liouville foliation. Fomenko invariant (molecule) is obtained by considering the space of leaves and indicating the topological types of bifurcations of regular leaves.



1 degree of freedom, compact case

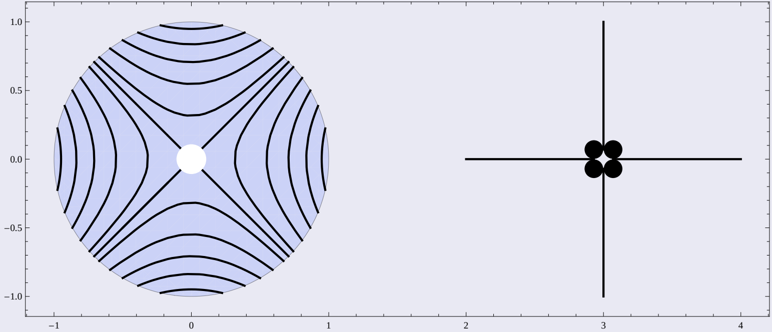
(M^2, H) – 2-dimensional manifold with a foliation into connected components of the level sets of the function H.
Regular leaves = circles.
A molecule is constructed in the same way, and it is a complete invariant of the Liouville foliation.

1 degree of freedom, non-compact case

Regular leaves = circles and lines.

Example

$M^2 = \{(x, y) \mid 0 < x^2 + y^2 < 1\}$, $f(x, y) = x^2 - y^2$.



1. The space of leaves is not Hausdorff.
2. There are no critical leaves, but the foliation is not trivial.

1 degree of freedom, non-compact case

Definition

We call two leaves Γ and Γ' equivalent if there exists a sequence of leaves $\Gamma_1 = \Gamma, \Gamma_2, \dots, \Gamma_n = \Gamma'$ such that for any i, $1 \le i \le n-1$, the leaves Γ_i and Γ_{i+1} are not separable in the space of leaves. It is convenient to introduce a new definition of a leave as a union of leaves from a certain equivalence class.

Definition

We say that a leaf is *regular* if the foliation in its small enough neighbourhood is trivial. Otherwise we call a leaf *bifurcational*.

1 degree of freedom, non-compact case

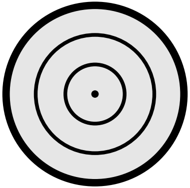
Under the assumption that the number of bifurcational leaves is finite, the topology of the foliation can again be described in terms of a molecule, i.e. a graph with vertices-atoms encoding bifurcations.

Theorem

Two Hamiltonian systems of non-compact type with one degree of freedom are Liouville equivalent (i.e. there exists a diffeomorphism between their phase spaces preserving the structure of the Liouville foliation) if and only if their molecules coincide.

Minimax 2-atoms

Compact case:



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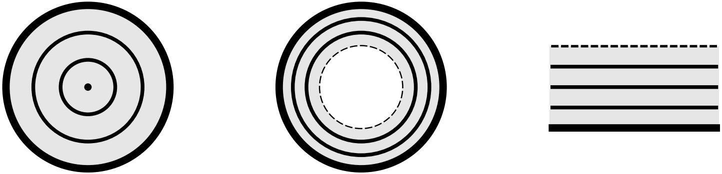
Non-compact case:



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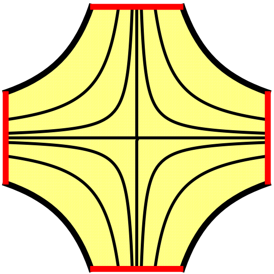
Non-minimax 2-atoms

Every compact non-minimax 2-atom can be glued from elementary pieces of the following type:

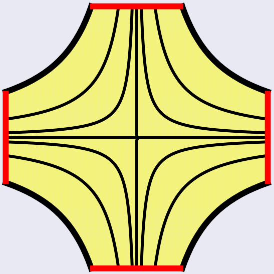


saddle

Non-minimax 2-atoms

Theorem

Every non-compact non-minimax 2-atom can be glued from elementary pieces of the following 3 types:



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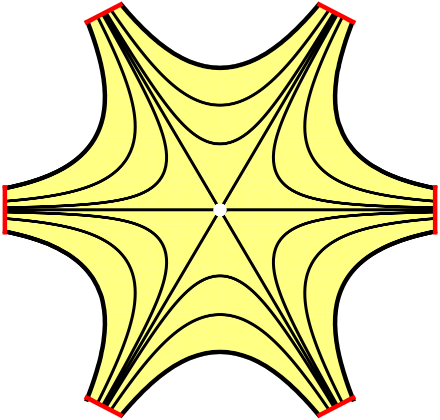
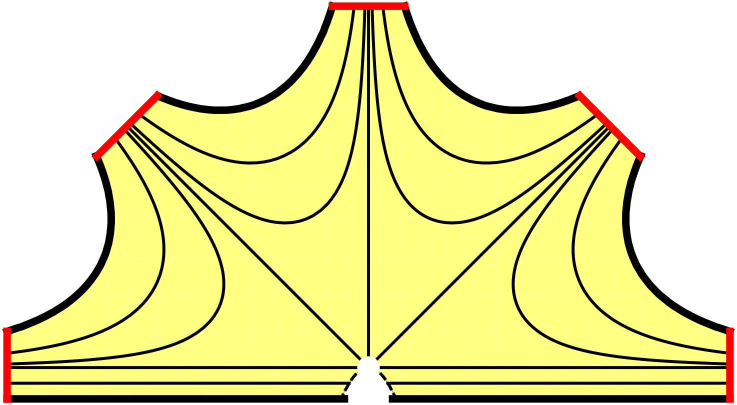
Non-minimax 2-atoms



sun with 2n rays

Non-minimax 2-atoms



semi–sun with n rays

Thank you!