Magnetic maps

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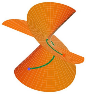

History

2012 (Zlatibor):
Killing `magnetic trajectories` in 3-dimensional Riemannian manifolds

2014 (Vrnjačka Banja):
On some periodic `magnetic curves`

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2016 (Zlatibor): `magnetic trajectories` $\longrightarrow$ **magnetic maps**

This talk is based on two papers written in collaboration with

Jun-ichi Inoguchi (Japan).

Background

(M, g) ($\dim M = n \geq 2$)

magnetic field: F - closed 2-form on M

Lorentz force Φ: $g(\Phi(X), Y) = F(X, Y)$, X, Y tangent to M

Background

A smooth curve γ in (M, g, F) is called

a flowline of the dynamical system associated with F

or simply:

magnetic curve of (M, g, F)

if its velocity vector field γ' satisfies the Lorentz equation:

$$\nabla_{\gamma'}\gamma' = \Phi(\gamma')$$

Magnetic curves vs. geodesics

• For trivial magnetic field $F = 0$ (the magnetic field is absent) magnetic curves correspond to geodesics of (M, g).

• Geodesics are characterized as critical points of the energy action.

Magnetic curves of (M, g, F) can be also viewed (at least locally) as the solutions of a variational principle.

• The existence and uniqueness of geodesics remain true for magnetic curves.

• Property for magnetic curves: $\frac{\mathrm{d}}{\mathrm{d}t} g(\gamma', \gamma') = 0$. In particular, a magnetic curve is called normal if it has unit energy, i.e. $||\gamma'|| = 1$.

Geodesics

... are given by a second order nonlinear differential equation: **Euler-Lagrange equation of motions**.

More precisely, a *geodesic* γ in a Riemannian manifold (M, g) is characterized as critical point of the **kinetic energy** (also called the **action integral**)

$$E(\gamma) = \int \frac{1}{2}|\gamma'(s)|^2\, ds$$

Harmonic maps

The notion of geodesic is generalized to maps between Riemannian manifolds.

A map $f : (N, h) \to (M, g)$ between Riemannian manifolds is said to be **harmonic** if it is a critical point of the energy functional:

$$E(f) = \int_N \frac{1}{2}|df|^2 dv_h$$

under compactly supported variations. The Euler-Lagrange equation of this variational problem is given by

$$\tau(f) = \mathrm{div}\, df.$$

Here $\tau(f)$ is called the tension field of f.

Magnetic curves: variational approach

A special case: F is an exact 2-form, namely, there exists a 1-form ω, usually called the **potential 1-form**, such that $F = d\omega$.

For a curve $\gamma : [a, b] \longrightarrow M$ consider the functional

$$LH(\gamma) = \int_a^b \left(\frac{1}{2} \langle \gamma'(t), \gamma'(t) \rangle + \omega(\gamma'(t)) \right) dt.$$

It is often called the **Landau Hall functional** for the curve γ with the potential 1-form ω.

Magnetic curves: variational approach

Consider a variation of γ:

$\Gamma : [a,b] \times (-v, v) \longrightarrow M, \quad \Gamma(t,0) = \gamma(t)$

Magnetic curves: variational approach

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$\Gamma : [a, b] \times (-\upsilon, \upsilon) \longrightarrow M, \quad \Gamma(t, 0) = \gamma(t)$

Simplify the notations: $\gamma_\epsilon : [a, b] \longrightarrow M$, $\gamma_\epsilon(t) = \Gamma(t, \epsilon)$

The variation vector on γ: $V = \frac{\partial \gamma_\epsilon}{\partial \epsilon} : [a, b] \longrightarrow M$, that is $V(a) = V(b) = 0$.

Magnetic curves: variational approach

In order to find the critical points of the functional LH we compute:

$$\frac{d}{d\epsilon} LH(\gamma_\epsilon)\Big|_{\epsilon=0} = -\int_a^b g\big(\nabla_{\gamma'}\gamma' - \phi(\gamma'), V\big)\,dt.$$

Magnetic curves: variational approach

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The critical points of the LH functional are solutions of the equation $\frac{d}{d\epsilon} LH(\gamma_\epsilon)\big|_{\epsilon=0} = 0$ which is equivalent to the **Lorentz equation**.

The Landau Hall functional for maps

Let $f : N \longrightarrow M$ be a smooth maps between two Riemannian manifolds (N, h) of dimension n and (M, g) of dimension m.

Let ξ be a global vector field on N and ω be a 1-form on M.

The energy of f is $E(f) = \dfrac{1}{2}\displaystyle\int_N |df|^2 dv_h.$

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Let us define the following functional for f associated to ξ and ω

$$LH(f) \;=\; E(f) + \int_N \omega(df(\xi))dv_h.$$

First variation for the Landau Hall functional

A smooth variation $\{\mathcal{F}_\epsilon\}$ of f means a smooth map $\mathcal{F} : N \times I \longrightarrow M$, such that $\mathcal{F}(p, 0) = f(p)$. For the sake of simplicity we use to write $f_\epsilon(p) = \mathcal{F}(p, \epsilon)$.

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Definition. The map f is called **magnetic** with respect to ξ and ω if it is a critical point of the Landau Hall integral defined above, i.e. the first variation

$$\frac{d}{d\epsilon} LH(f_\epsilon)\Big|_{\epsilon=0}$$

is zero.

Magnetic maps

Theorem (Inoguchi, M. - 2014)

Let $f : (N, h) \longrightarrow (M, g)$ be a magnetic map with respect to ξ and ω. Then f satisfies the Lorentz equation

$$\tau(f) = \phi(f_* \xi).$$

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Remark (remove assumptions)

A magnetic map is defined without assumptions N compact and F exact.

Examples of magnetic maps

1. A *constant map* $f : N \longrightarrow M$ is magnetic with respect to any $\xi \in \chi(N)$ and any closed 2-form F on M.

Examples of magnetic maps

2. Let $N = [a, b]$, and t be the parameter on N. Take $h = dt^2$ and $\xi = \frac{d}{dt}$. If F is a magnetic field on M and γ a magnetic curve on M corresponding to F, then γ is a magnetic map associated to ξ and F. This allows us to say that **magnetic maps extend magnetic curves.**

Examples of magnetic maps

3. In the absence of a magnetic field the magnetic equation becomes $\tau(f) = 0$; hence f is a harmonic map. Therefore one may say that **magnetic maps extend harmonic maps.**

Isometric immersions

Let $f : (N, h) \longrightarrow (M, g)$ be an isometric immersion between two Riemannian manifolds N and M.

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Isometric immersions

Let $f:(N,h)\longrightarrow(M,g)$ be an isometric immersion between two Riemannian manifolds N and M. Then, the tension field $\boxed{\tau(f)=n\mathbf{H}}$, where $\mathbf{H}$ is the mean curvature vector field of N in M. We have the following

Proposition (new form of the magnetic equation)

If ξ is a global vector field on N and ϕ is a Lorentz force on M, then f is magnetic if and only if

$$\mathbf{H}=\frac{1}{n}\,\phi(f_*\xi).$$

Composition

It is known that the composition of two harmonic maps is **not**, in general a harmonic map. Yet, if

$\psi : (N, h) \longrightarrow (M, g)$ is a harmonic map and

$f : (M, g) \longrightarrow (\widetilde{M}, \widetilde{g})$ is totally geodesic,

then $f \circ \psi$ is harmonic.

Composition

1. Let $f : (M, g) \longrightarrow (\widetilde{M}, \widetilde{g})$ be a **totally geodesic** submanifold. Let ξ be a vector field on M and let F be a magnetic field on $\widetilde{M}$. Suppose that f is a **magnetic map** with respect to ξ and F.

If γ is an **integral curve** of ξ such that

$\tilde{\gamma} = f \circ \gamma$ is a **normal magnetic curve** for F,

then γ is a **geodesic.**

2. Let $f : (M, g) \longrightarrow (\widetilde{M}, \widetilde{g})$ be a **totally umbilical** submanifold, namely $\sigma(X, Y) = g(X, Y)H$. Let ξ be a vector field on M such that its **integral curves are geodesics** and let F be a magnetic field on $\widetilde{M}$.

Suppose that f is a **magnetic map** with respect to ξ and F.

If γ is an integral curve of ξ,

then $f \circ \gamma$ is a **magnetic curve** for $\frac{1}{n}F$, where $n = \dim M$.

Magnetic maps to Euclidean spaces

Let $f : N \longrightarrow \mathbb{E}^m$ be a smooth map from the Riemannian manifold (N, h) to the Euclidean space $\mathbb{E}^m$. If $(f^1, \ldots, f^m)$ are the components of the map f and if ϕ is a Lorentz force on $\mathbb{E}^m$, then the Lorentz equation can be written as

$$(\Delta f^1, \ldots, \Delta f^m) = \phi(f_* \xi)$$

where Δ denotes the Beltrami-Laplace operator on functions on N, namely

$$\Delta f^\alpha = \sum_{i=1}^{n} \Big(e_i e_i(f^\alpha) - ({}^h\nabla_{e_i} e_i)(f^\alpha) \Big)$$

for any $\alpha = 1, \ldots, m$, where $\{e_i\}_{i=1,\ldots,n}$ is an orthonormal basis on N.

Magnetic maps in almost contact geometry

Example 1.

Let $(M, \varphi, \xi, \eta, g)$ be an almost contact metric manifold.

The identity map $\mathbf{1}_M : M \longrightarrow M$ is a magnetic map with respect to ξ and $F = d\eta$ if and only if

$$\iota_\xi d\eta = 0.$$

Magnetic maps in almost contact geometry

Example 2.

Let $f : M_1 \longrightarrow M_2$ be a φ-holomorphic map between almost contact metric manifolds.

Theorem. [S. Ianuş, A.M. Pastore - Ann. Math. Blaise Pascal, 1995]
Any φ-holomorphic map between contact metric manifolds is a harmonic map.

Magnetic maps in almost contact geometry

Example 2.

Let $f : M_1 \longrightarrow M_2$ be a φ-holomorphic map between almost contact metric manifolds.

Suppose that the fundamental 2-form Ω_2 is closed.

Then f is a magnetic map with respect to ξ_1 and a magnetic field $F = q\Omega_2$ if and only if f is a harmonic map.

Magnetic maps in almost contact geometry

Example 3.

Let (N, h) be a Riemannian manifold and ξ be a *regular* vector field on N, i.e. the action of its 1-parameter group (of isometries) on N is simply transitive. Denote by $M = N/\xi$ the orbit space. Then, the projection $\pi : N \longrightarrow M$ is a magnetic map (w.r.t. ξ and arbitrary magnetic field F on M) if and only if π is a harmonic map.

Tangent bundle of a Riemannian manifold

(M, g) a Riemannian manifold of dimension n

$\pi : T(M) \longrightarrow M$ its tangent bundle

Decomposition of $T_{\mathfrak{u}} T(M)$:

$$T_{\mathfrak{u}} T(M) = V_{\mathfrak{u}} T(M) \oplus H_{\mathfrak{u}} T(M)$$

where $V_{\mathfrak{u}} T(M) = \ker \pi_{*,\mathfrak{u}}$ is the vertical space
and $H_{\mathfrak{u}} T(M)$ is the horizontal space in $\mathfrak{u}$ obtained by using ∇

Tangent bundle of a Riemannian manifold

We have:

the horizontal distribution HTM
the vertical distribution VTM

the direct sum decomposition

$$TTM = HTM \oplus VTM$$

If $X \in \mathfrak{X}(M)$, denote by

X^H the horizontal lift
X^V the vertical lift

of X to $T(M)$.

Tangent bundle of a Riemannian manifold

Sasaki metric:

$$g_S(X^H, Y^H) = g_S(X^V, Y^V) = g(X, Y) \circ \pi, \; g_S(X^H, Y^V) = 0$$

an almost complex structure:

$$J_S X^H = X^V, \; J_S X^V = -X^H, \text{ for all } X \in \mathfrak{X}(M)$$

Classical result: $(T(M), g_S, J_S)$ is an almost Käherian manifold.

Tangent bundle of a Riemannian manifold

Classical result: $(T(M), g_S, J_S)$ is an almost Käherian manifold.

Hence, the Kähler 2-form $\Omega_S = g_S(J_S\,\cdot,\ \cdot)$ may be considered as a magnetic field on $T(M)$.

When a vector field is a magnetic map?

Let $\xi \in \mathfrak{X}(M)$ be thought as a map from (M, g) to $(T(M), g_S, J_S)$.

Compute the differential of this map: $\xi_{*,p} : T_pM \longrightarrow T_{(p,\xi(p))}T(M)$.

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Well known result:

The map $\xi : (M, g) \longrightarrow (T(M), g_S)$ is an isometric immersion if and only if $\nabla\xi = 0$.

When a vector field is a magnetic map?

Compact case:
the energy of ξ on M is

$$E(\xi) = \frac{n}{2}\,\mathrm{vol}(M) + \frac{1}{2}\int_M ||\nabla\xi||_g^2 dv_g,$$

The number

$$\mathcal{B}(\xi) = \int_M ||\nabla\xi||_g^2 dv_g$$

is called the *total bending* of the vector field ξ.

Result. $\xi : (M, g) \longrightarrow (T(M), g_S)$ is harmonic if and only if ξ is parallel. In such a case it is an absolute minimum of the energy functional $E(\xi)$.

When a vector field is a magnetic map?

Arbitrary case:

$$\tau(\xi) = -\left\{\left(\mathrm{trace}_g R(\nabla_\bullet \xi, \xi)\bullet\right)^H + \left(\Delta_g \xi\right)^V\right\} \circ \xi$$

Δ_g denotes the rough Laplacian on vector fields:

$$\Delta_g X = -\sum_{k=1}^{n} \left[\nabla_{e_k}\nabla_{e_k}X - \nabla_{\nabla_{e_k}e_k}X\right],$$

When a vector field is a magnetic map?

Theorem (Inoguchi, M.)

Let (M, g) be a Riemannian manifold and $(T(M), g_S, J_S)$ its tangent bundle endowed with the usual almost Kählerian structure.

Let ξ be a vector field on M.

Then ξ is a magnetic map with strength q associated to ξ itself and the Kähler magnetic field Ω_S if and only if the following conditions hold:

$$(*) \qquad \mathrm{trace}_g R(\nabla_\bullet \xi, \xi)\bullet) = q\, \nabla_\xi \xi$$

$$(**) \qquad \Delta_g \xi = -q\, \xi.$$

When a vector field is a magnetic map?

Proof.

The magnetic equation with strength q:

$$\tau(\xi) = q\, J_S(\xi_*\xi), \ q \in \mathbb{R}.$$

We compute

$$J_S(\xi_*\xi) = \xi^V - (\nabla_\xi \xi)^H.$$

Identify the vertical and the horizontal parts, respectively. □

When a vector field is a magnetic map?

Interesting results may be obtained in cases when the curvature tensor has a certain expression.

When a vector field is a magnetic map?

1. M is of constant sectional curvature c:

$$R(X,Y)Z = c\big(g(Y,Z)X - g(X,Z)Y\big), \text{ for all } X,Y,Z \in \mathfrak{X}(M)$$

We obtain:

$$\mathrm{trace}_g R(\nabla_\bullet \xi, \xi)\bullet = c\left[\nabla_\xi \xi - (\mathrm{div}\ \xi)\xi\right].$$

When a vector field is a magnetic map?

$(*)$ becomes

$$(c - q)\nabla_\xi \xi - c(\mathrm{div}\ \xi)\xi = 0.$$

(i) If $c = 0$, that is M is flat, it follows that ξ is self-parallel.

(ii) If $c \neq 0$, then we have

$$\left(1 - \frac{q}{c}\right) \nabla_\xi \xi = (\mathrm{div}\ \xi)\xi$$

Hence, for $q = c$, the vector field ξ is divergence free.

When a vector field is a magnetic map?

2. $M = M(c)$ is a Sasakian space form

$$
\begin{aligned}
R(X,Y)Z = \tfrac{c+3}{4}\big(g(Y,Z)X - g(X,Z)Y\big) \\
+\tfrac{c-1}{4}\Big(\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi \\
+g(Z,\varphi Y)\varphi X - g(Z,\varphi X)\varphi Y + 2g(X,\varphi Y)\varphi Z\Big)
\end{aligned}
$$

When a vector field is a magnetic map?

$(*)$ is automatically satisfied.

For $(**)$ compute:

$$\Delta_g \xi = 2n\xi$$

Proposition (Inoguchi, M.)

The vector field ξ is magnetic with the strength $q = -2n$.

[1] INOGUCHI J. AND MUNTEANU M.I., *Magnetic maps*, Int. J. Geom. Methods Mod. Phys. **11** 6 (2014), art. 1450058, (22 pages).

[2] INOGUCHI J. AND MUNTEANU M.I., *New examples of magnetic maps involving tangent bundles,* Rendiconti Sem Matematico (Universita e Politecnico di Torino), (accepted).

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Thank you for attention

T h a n k y o u f o r a t t e n t i o n !