Lagrangian submanifolds in the Nearly Kähler $S^3 \times S^3$ from minimal surfaces in S^3

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The nearly Kähler $S^3 \times S^3$

- The nearly Kähler structure of $S^3 \times S^3$
- Lagrangian submanifolds
  A 3-dimensional manifold M^3 in $S^3 \times S^3$ is called Lagrangian if J maps the tangent space into the normal space.
- The tensor P, defined as $P(Z)_{(p,q)} = (pV, qU)$,for $Z = (pU, qV)$ plays an important role:
  We can define $\{E_1, E_2, E_3\}$ basis in the tangent space of M and obtain the existence of the angle functions θ_i such that

$$PE_i = \cos 2\theta_i E_i + \sin 2\theta_i JE_i.$$

- $P \neq Q$:
  For $Z = (U, V) \in T_{(p,q))} S^3 \times S^3$, the usual product structure is given by $QZ = Q(U, V) = (-U, V)$.

Problem: Study Lagrangian submanifolds M given by

$$M \xrightarrow{f} S^3 \times S^3 \xrightarrow{\pi_1} \mathbb{S}^3$$
$$g \mapsto (p(g), q(g)) \mapsto p(g)$$

and assume that the map p is nowhere an immersion.

Theorem 1

The above condition is equivalent to one of the angle functions being $\frac{\pi}{3}$.

Next, choose $2\theta_1 = \frac{2\pi}{3}, \quad 2\theta_2 = 2\Lambda + \frac{2\pi}{3}, \quad 2\theta_3 = -2\Lambda + \frac{2\pi}{3}$.

Theorem 2

Assume that M is not totally geodesic. Then $p(M)$ is a minimal surface in S^3.

Elementary properties of orientable minimal surfaces in S^3

Let $p : N \to S^3$ be a minimal surface, not totally geodesic.

- $\langle \partial u, \partial u \rangle = \langle \partial v, \partial v \rangle = 2e^{\omega}$ and $\langle \partial u, \partial v \rangle = 0$,
- $\partial u = p\alpha$, $\partial v = p\beta$, $N = p\frac{\alpha \times \beta}{2e^{\omega}}$.
- the matrix $\left(p, \frac{\partial u}{|\partial u|}, \frac{\partial v}{|\partial v|}, N\right)$ belongs to $SO(4)$
- Let σ be the component of the second fundamental form in the direction of N
- $\frac{\partial}{\partial \bar{z}}(\sigma(\partial z, \partial z)) = 0$, hence $\sigma(\partial z, \partial z)$ is holomorphic
- there are two cases: $\sigma = 0$ or $\sigma(\partial z, \partial z) \neq 0$
- the immersion either admits local isothermal coordinates which satisfy the Sine-Gordon equation ($\Delta\omega = -8\sinh\omega$) or is totally geodesic ($\sigma = 0$).

Denote by $\mathcal{P}$ the lift of the minimal immersion to the immersion of the frame bundle in $SO(4)$, i.e.

$$\mathcal{P} : UN \to SO(4) : v \mapsto \left(p \, v \, \tilde{J}v \, N \right),$$

where UN denotes the unit tangent bundle and $\tilde{J}$ denotes the natural complex structure on an orientable surface.
In terms of our chosen isothermal coordinate this map can be parametrised by

$$\mathcal{P}(u,v,t) = \left(p(u,v), \cos t \frac{p_u}{\mid p_u \mid} + \sin t \frac{p_v}{\mid p_v \mid}, -\sin t \frac{p_u}{\mid p_u \mid} + \cos t \frac{p_v}{\mid p_v \mid}, N(u,v) \right.$$

for some real parameter t. Note that we have the frame equations which state that

$$d\mathcal{P} = \mathcal{P}\Omega^t = -\mathcal{P}\Omega,$$

where Ω is given by

$$\begin{pmatrix} 0 & \sqrt{2}e^{\frac{\omega}{2}}(\cos(t)du+\sin(t)dv) & \sqrt{2}e^{\frac{\omega}{2}}(\cos(t)dv-\sin(t)du) & 0 \\ -\sqrt{2}e^{\frac{\omega}{2}}(\cos(t)du+ \\ +\sin(t)dv) & 0 & \frac{1}{2}(\omega_u dv-\omega_v du)+dt & -\sqrt{2}e^{-\frac{\omega}{2}}(\cos(t)du-\sin(t)dv) \\ -\sqrt{2}e^{\frac{\omega}{2}}(\cos(t)dv- \\ -\sin(t)du) & -\frac{1}{2}(\omega_u dv-\omega_v du)-dt & 0 & \sqrt{2}e^{-\frac{\omega}{2}}(\sin(t)du+\cos(t)dv) \\ 0 & \sqrt{2}e^{-\frac{\omega}{2}}(\cos(t)du-\sin(t)dv) & -\sqrt{2}e^{-\frac{\omega}{2}}(\sin(t)du+\cos(t)dv) & 0 \end{pmatrix}.$$

We can identify M with the frame bundle on the minimal surface by looking at the map $\tilde{P}$:

$$g \in M \mapsto \left(p, \frac{dp(E_2)}{\sin\Lambda}, \frac{dp(E_3)}{\sin\Lambda}, \xi\right)$$

We compute the corresponding matrix $\tilde{\Omega}$, using again $d\tilde{\mathcal{P}} = -\tilde{\mathcal{P}}\tilde{\Omega}$

Remark: The condition for the map $\tilde{P}$ to be an immersion is

$$\frac{1}{\sqrt{3}} + h_{12}^3 \csc(2\Lambda) \neq 0 \quad \text{where} \quad h_{ij}^k = \langle h(E_i, E_j), JE_k \rangle.$$

Therefore, there are three cases that arise:

1. Case 1. $p(M)$ is not a totally geodesic surface and the map $\tilde{\mathcal{P}}$ is an immersion,
2. Case 2. The minimal surface $p(M)$ is not totally geodesic, but the map $\tilde{\mathcal{P}}$ is not an immersion,
3. Case 3. The minimal surface $p(M)$ is totally geodesic, i.e. $\sigma = 0$.

Case 1. $p(M)$ is not a totally geodesic surface and the map $\tilde{\mathcal{P}}$ is an immersion

Write $dp(E_2)$ and $dp(E_3)$ as linear combination of the tangent vectors from the frame bundle:

$$dp(E_2) = \cos x(\cos t\frac{p_u}{|\,p_u\,|} + \sin t\frac{p_v}{|\,p_v\,|}) + \sin x(-\sin t\frac{p_u}{|\,p_u\,|} + \cos t\frac{p_v}{|\,p_v\,|})$$

$$dp(E_3) = -\sin x(\cos t\frac{p_u}{|\,p_u\,|} + \sin t\frac{p_v}{|\,p_v\,|}) + \cos x(-\sin t\frac{p_u}{|\,p_u\,|} + \cos t\frac{p_v}{|\,p_v\,|})$$

- With a change of variable on the frame bundle, we can show that $\mathcal{P} = \tilde{\mathcal{P}}$ and $\tilde{\Omega} = \Omega$.
- Comparing the elements in the two matrices, provides useful information about the invariants on the manifold.

- Finally, the Lagrangian immersion can be reconstructed starting from the minimal surface and any solution for differential equation

$$\Delta\mu = -e^{\mu},$$

where $c_1 = e^{\omega+\mu} - 2$ and $\left(\frac{2\sqrt{3}e^{\omega}}{\tan\Lambda} - 2\sin 2t\right)^2 = c_1 - 2\cos 4t$.

- the other two cases are approached by similar methods.

Thank you!