Homotopy invariants of manifolds: Poincare duality and the signature formulae

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1. Classical Hirzebruch formula

2. Finite dimensional representation

3. Underlying algebraic Poincare complex. Abstract generalization

4. Novikov’s conjecture

5. Topological invariance of rational Pontryagin classes

6. Manifolds with proper actions
Outline

1. Classical Hirzebruch formula
2. Finite dimensional representation
3. Underlying algebraic Poincare complex. Abstract generalization
4. Novikov’s conjecture
5. Topological invariance of rational Pontryagin classes
6. Manifolds with proper actions
H. Poincare in his prominent paper ”Analysis situs” (1895)


represented the first systematic study of topology.
Especially he discovered so called Poincare duality that says ”the Betti numbers equidistant from the ends are equal”.
This fundamental property of closed manifolds has very large generalizations up to homotopy invariants of non simply connected manifolds.
Let $\beta_k$ be $k^{th}$ Betti number of a closed oriented compact manifold $M$, $\dim M = n$.

**Theorem [H. Poincare]**

$$\beta_k = \beta_{n-k}.$$
H. Poincare has not presented strict concept of the Betti numbers and, especially, strict proof of the Poincare duality. New things were required to create: the homology groups (E. Noether, 1926, L. Vietoris, 1927),

Leopold Vietoris., *Ueber den höheren Zusammenhang von kompakten Räumen. Proceedings of the Section on Sciences,, KNAW, 29:1008–13, 1926.,*

the cohomology groups (J. Alexander, A.N. Kolomogorov, 1934), duality between them (L.S. Pontryagin).
After that the Poincare duality has been gotten more exact and clear sense:

\[ H_k(M) \approx H^{n-k}(M). \]

or to be more precise

\[ H_k(M; \mathbb{Z}) \approx H^{n-k}(M; \mathbb{Z}). \] (1)

The crucial understanding here is that this isomorphism is not abstract but is generated by a natural geometric operation.
Signature and the Hirzebruch formula

Namely, this isomorphism is generated by the intersection operation with the fundamental homology class \([M]\):

\[
\cap [M] : H^{n-k}(M; \mathbb{Z}) \longrightarrow H_k(M; \mathbb{Z}).
\] (2)

Passing to rational coefficients \(\mathbb{Q}\) the intersection operation generates a bilinear form

\[
\cap [M] : \text{Hom}(H_{n-k}(M), \mathbb{Q}) \xrightarrow{\sim} H_k(M; \mathbb{Q})
\]
which has an additional invariant — the signature of the quadratic form in the case $\dim M = 4k$ for orientable manifold $M$: the isomorphism

$$\cap[M] : \text{Hom}(H_{2k}(M), Q) \xrightarrow{\cong} H_{2k}(M; Q)$$

defines a non degenerated symmetric form

$$< \bullet, \bullet >[M] : H_{2k}(M) \times H_{2k}(M) \rightarrow \mathbb{Q}.$$
Signature and the Hirzebruch formula
Definition of signature of manifold

By definition the signature of $M$ is the number

$$\text{sign}(M) \overset{\text{def}}{=} \text{sign} \left( H_{2k}(M); \langle \bullet, \bullet \rangle_M \right).$$

If $\dim M \neq 4k$, then one supposes that

$$\text{sign}(M) \overset{\text{def}}{=} 0.$$
Signature and the Hirzebruch formula

Remark

The homology and cohomology groups are homotopy invariants including all homological operations, in particular the intersection operation of homologies with cohomologies. Hence the signature $\text{sign} M$ is a homotopy invariant, that is does not depend of the choice of simplicial structure on manifold.
More important is invariance of the signature with respect to the bordism relation:

**Theorem**

Let $M$ be an orientable manifold, $\dim M = 4k$. Assume that there is an orientable manifold with boundary $W$, $\dim W = 4k + 1$, and $M \approx \partial W$. Then

$$\text{sign} M = 0.$$
The theorem means that the signature of the manifold can be expressed in terms of the characteristic numbers of the manifold that is known as the Hirzebruch formula. The Hirzebruch formula is an expression of the relation between signature of the manifold $M$ and some characteristic number of the same manifold $M$. Namely the Hirzebruch formula says that for $4k$-dimensional orientable compact closed manifold $M$ the following equality holds

$$\text{sign} M = 2^{2k} \langle L(M), [M] \rangle.$$  \hspace{1cm} (3)
Here

\[ L(M) = \prod_j \frac{t_j/2}{\text{th}(t_j/2)} \]

is the Hirzebruch characteristic class defined by formal generators \( t_j \) such that

\[ \sigma_k(t_1, \ldots, t_n) = c_k(cTM) \in H^{2k}(M; \mathbb{Z}), \]

where \( \sigma_k \) is elementary symmetric polynomial.
One of the way to prove the Hirzebruch formula is to consider so called signature elliptic operator on the de Rham complex of differential forms on the manifold $M$:

$$0 \to \Omega_0(M) \xrightarrow{d} \Omega_1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega_{4k}(M) \to 0.$$
The $\cup$-product is induced by the external multiplication of differential forms, so the Poincare duality is defined by the formula

$$\langle \omega_1, \omega_2 \rangle = \int_X \omega_1 \wedge \omega_2.$$
Using a Riemannian metric on the manifold $M$, $(ω₁, ω₂)$, one can define the bounded operator $\ast :$

$$(ω₁, ω₂) = \int_X ω₁ \wedge *ω₂,$$

$* : \Omega_k(X) \longrightarrow \Omega_{n-k}(X).$
Signature operator

Put

\[ \alpha = i^k(k+1) \ast. \]

\[ \alpha d\alpha = d^*; \quad \alpha^2 = 1. \]

Let

\[ \Omega^+(M) = \text{Ker} (\alpha - 1); \quad \Omega^-(M) = \text{Ker} (\alpha + 1). \]

Then

\[ (d + d^*) (\Omega^+(M)) \subset \Omega^-(M). \]
Put $D = d + d^*$ . One can consider elliptic operator

$$D : \Omega^+(M) \longrightarrow \Omega^-(M).$$

**Theorem**

$\text{index } D = \text{sign} M.$
The Atiyah–Singer index formula for elliptic operators gives us the Hirzebruch formula:

$$\text{sign} M = \text{index} (D) = \langle T(M) \text{ch} \sigma(D), [M] \rangle = 2^{2k} \langle L(M), [M] \rangle.$$
Consider de Rham complex on $M$ with values in a flat vector bundle $\xi^\rho$:

$$0 \longrightarrow \Omega_0(M, \xi^\rho) \xrightarrow{d} \Omega_1(M, \xi^\rho) \xrightarrow{d} \cdots \xrightarrow{d} \Omega_{4k}(M, \xi^\rho) \longrightarrow 0.$$ 

Homologies of the de Rham complex coincides with cohomologies $H^*(M, \xi^\rho)$:

$$H^*(M, \xi^\rho) = (\text{Ker } d) / (\text{Im } d).$$
Local system of coefficients

Consider the elliptic operator

\[ D \otimes \xi^\rho = (d + d^*) \otimes \xi^\rho : \Omega^+(M, \xi^\rho) \rightarrow \Omega^-(M, \xi^\rho). \]

Then for some representations \( \rho \) one can establish

\[ \text{sign}_\rho(M) = \text{index} (D \otimes \xi^\rho). \]

Using Atiyah-Singer formula

\[ \text{index} (D \otimes \xi) = 2^{2k} \langle L(M) \text{ch} \xi, [M] \rangle, \]

for any vector bundle \( \xi \) over the manifold \( M \) we obtain

**Theorem**

\[ \text{sign}_\rho(M) = 2^{2k} \langle L(M) \text{ch} \xi^\rho, [M] \rangle. \]
Consider a finite dimensional representation

\[ \rho : \pi \rightarrow U(N). \]

Using the representation \( \rho \) one can construct several things:
Using the representation $\rho$ one can construct several things:

- The flat (complex) vector bundle $\xi^\rho$ over $B\pi$, induced by the representation $\rho$.
- The flat (complex) vector bundle $\xi^\rho_M$ over $M$ induced by the same representation $\rho$, $\xi^\rho_M = f_M^* \xi^\rho$.
- The cohomology groups $H^{2k}(M, \rho)$ with the local system of coefficients induced by the representation $\rho$

$$H^{2k}(M, \rho) = H^{2k}(X, \xi^\rho_M).$$
Finite dimensional representation

Unitary case

The $\cup$ -product induces non degenerated Hermitian form in the group $H^{2k}(M, \rho)$:

$$H^{2k}(M, \xi^\rho_M) \times H^{2k}(M, \xi^\rho_M) \xrightarrow{\cup} H^{4k}(M, \xi^\rho_M \otimes \xi^\rho_M) \xrightarrow{\langle \cdot, \cdot \rangle} H^{4k}(X, \mathbb{C}) \approx \mathbb{C}.$$

The signature of this form we shall denote by

$$\text{sign}_{\rho M} = \text{sign} \left( H^{2k}(M, \rho), \cup \right).$$
It is easy to check that

\[
\text{sign}_\rho M = 2^k \langle L(M) \text{ch}\xi^\rho_M, [M] \rangle.
\]

Since \(\xi^\rho\) is flat bundle one has \(\text{ch}\xi^\rho = \dim\xi^\rho = N\). The formula looks useless since both left side and right side of the formula coincide with classical one up to an integer factor \(N\). Nevertheless this case might be useful for further generalizations.
Consider a representation

$$\rho : \pi \rightarrow U(p, q)$$

into the matrix group $U(p, q)$ that preserves an indefinite Hermitian non-degenerated form of the type $(p, q)$. Then again one can construct the operation of the type $\cup$ which generates a non-degenerated Hermitian form into middle cohomologies $H^{2k}(M; \rho)$. 
On the other hand the flat vector bundle $\xi^\rho$ can be split into the direct sum

$$\xi^\rho = \xi^\rho_+ \oplus \xi^\rho_-,$$

such that on each summand the form is definite (positively and negatively). Then the Hirzebruch formula has the following form

$$\text{sign}_\rho M = 2^{2k} \left\langle L(M) \text{ch}_M^*(\xi^\rho_+ - \xi^\rho_-), [M] \right\rangle,$$
Here the Chern character of the bundles $\xi^\rho_{\pm}$ may be non trivial. The first example was considered by Lusztig

Finite dimensional representation
Case of skew Hermitian structure

Taking a skew Hermitian form $\varphi$ on $\mathbb{C}^N$ and the matrix group $\text{Sp}(N)$ which preserves this form one can consider a representation

$$\rho : \pi \longrightarrow \text{Sp}(N)$$

and flat vector (complex) bundle $\xi^\rho_X$. If $\dim X = 4k + 2$ then in the middle dimension one has non degenerated Hermitian form in the group $H^{2k+1}(X, \rho)$ generated by the $\cup$–product:

$$H^{2k+1}(X, \xi^\rho_X) \times H^{2k+1}(X, \xi^\rho_X) \xrightarrow{\cup}$$

$$\xrightarrow{\cup} H^{4k+2}(X, \xi^\rho_X \otimes \xi^\rho_X) \xrightarrow{\langle \cdot, \cdot \rangle} H^{4k+2}(X, \mathbb{C}) \approx \mathbb{C}.$$
Finite dimensional representation
Case of skew Hermitian structure

The flat vector bundle $\xi^\rho_X$ can be split into the direct sum $\xi^\rho_X = \xi^\rho_+ \oplus \xi^\rho_-$, such that on each summand the Hermitian form $i \cdot \varphi$ is definite (positively and negatively). Then again the Hirzebruch formula has the following form

$$\text{sign}_\rho X = 2^k \left\langle L(X) \text{ch}(\xi^\rho_+ - \xi^\rho_-), [X] \right\rangle,$$

Continuous family of representations

Consider a continuous family of unitary representations

$$\rho_t : \pi \rightarrow U(N), \quad t \in T,$$

This family generates the family of quadratic forms

$$\left( H^{2k}(X, \rho_t), \cup \right)$$

with constant signature

$$\text{sign}_{\rho_t} X = \text{sign} \left( H^{2k}(X, \rho_t), \cup \right) \in \mathbb{Z}.$$ 

The family of quadratic forms is not continuous. More of that the domains of definition are not the same.
To construct the family of quadratic forms as a continuous family we need to include the family \((H^{2k}(X, \rho_t), \cup)\) into larger space with constant dimension unlike homologies \(H^{2k}(X, \rho_t)\).
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Poincare duality has an algebraic construction as an abstract algebraic Poincare complex (APC) as a commutative diagram

\[ C_0 \leftarrow^{d_1} C_1 \leftarrow^{d_2} C_2 \leftarrow^{d_3} \cdots \leftarrow^{d_{n-1}} C_{n-1} \leftarrow^{d_n} C_n \]

\[ C^n \leftarrow^{d_n^*} C^{n-1} \leftarrow^{d_{n-1}^*} C^{n-2} \leftarrow \cdots \leftarrow^{d_2^*} C^1 \leftarrow^{d_1^*} C^0 \]

where \( C^k \) are free modules over an \(*\)-algebra \( \mathcal{A} \) and \( C^k \overset{\text{def}}{=} \text{Hom}(C_k, \mathcal{A}) \) denote dual modules.
Underlying algebraic Poincare complex. Abstract generalization

Abstract Algebraic Poincare Complex

For the diagram

\[ \begin{array}{ccccccccc}
C_0 & \xleftarrow{d_1} & C_1 & \xleftarrow{d_2} & C_2 & \xleftarrow{d_3} & \cdots & \xleftarrow{d_{n-1}} & C_{n-1} & \xleftarrow{d_n} & C_n \\
\uparrow D_0 & & \uparrow D_1 & & \uparrow D_2 & & \cdots & & \uparrow D_{n-1} & & \uparrow D_n \\
C^n & \xleftarrow{d^*_n} & C^{n-1} & \xleftarrow{d^*_{n-1}} & C^{n-2} & \xleftarrow{d^*_{n-2}} & \cdots & \xleftarrow{d^*_2} & C^1 & \xleftarrow{d^*_1} & C^0 
\end{array} \]

the following properties hold

1. \( d_{k-1}d_k = 0 \),
2. \( d_kD_k + (-1)^{k+1}D_{k-1}d^*_{n-k+1} = 0 \),
3. \( D_k = (-1)^{k(n-k)}D^*_{n-k} \).
4. \( H(D_k) : H(C^{n-k}) \rightarrow H(C_k) \) is isomorphism in homology.
Algebraic Poincare Complex of oriented manifold

Let $M$ be a closed orientable manifold, $\dim M = n$. Fix on $M$ a simplicial structure. Then the manifold $M$ induces the algebraic Poincare complex as following:

- $C_k \overset{\text{def}}{=} C_k(M, \mathcal{R})$, is the group of simplicial chains on $M$ with coefficients in $\mathcal{R}$,
- $C^k \overset{\text{def}}{=} C^k(M, \mathcal{R}) = \text{Hom}(C_k(M, \mathcal{R}), \mathcal{R})$,
- $d_k : C_k \longrightarrow C_{k-1}$ is boundary homomorphism,
- $D_k = \cap[M] : C^{n-k} = \text{Hom}(C_{n-k}(M, \mathcal{R}), \mathcal{R}) \longrightarrow C_k = C_k(M)$, is the intersection operation with fundamental cycle of the manifold $M$.

Here $\mathcal{R}$ is a ring of scalars.
For \( u \in C^{n-k} \), the intersection \( D_k(u) \equiv [M] \cap u \) is defined by natural formula

\[
D_k(u) \overset{\text{def}}{=} \sum_{\sigma} (-1)^{\varepsilon(\sigma)} u(a_0 < \cdots < a_{n-k}) \sigma(a_{n-k} < \cdots < a_n),
\]

where \( \sigma(a_0 < a_1 < \cdots < a_k) \) runs all simplices generated by ordered family of vertices \( a_0 < a_1 < \cdots < a_k \), and

\( \varepsilon(\sigma) = \varepsilon(\sigma(a_0 < \cdots < a_n)) \) denotes the orientation of the simplex \( \sigma(a_0 < a_1 < \cdots < a_n) \), which corresponds to the orientation on the manifold \( M \).
Theorem [H. Poincare]

The theorem states that the homomorphism $D = \{D_k\}$ induces the isomorphism of the homology groups:

\[
\begin{align*}
H_0(M) & \cong D_0 H_1(M) \cong D_1 \cdots \cong D_{n-1} H_n(M) \cong D_n \\
H^n(M) & \cong D_0 H^{n-1}(M) \cong D_1 \cdots \cong D_{n-1} H^1(M) \cong D_n H^0(M)
\end{align*}
\]
**Algebraic Poincare complex of oriented manifold**

If $\mathcal{R}$ is a field and $M$ is compact manifold then one has the isomorphism

$$H^i(M, \mathcal{R}) \cong \text{Hom}(H_i(M, \mathcal{R}), \mathcal{R}).$$

Hence,

$$0 \leftarrow k \quad H_k(M, \mathcal{R}) \quad k \rightarrow n$$

$$\approx \quad D_k$$

$$\cdots \quad \text{Hom}(H_{n-k}(M, \mathcal{R}), \mathcal{R}) \quad \cdots$$
If $M$ is even dimensional, $\dim M = n = 2k$, then

$$D_k : \text{Hom}(H_k(M, \mathcal{R}), \mathcal{R}) \xrightarrow{\cong} H_k(M, \mathcal{R})$$

is isomorphism, and up to the chain homotopy one has

$$D_k = (-1)^{k(n-k)} D^*_{n-k} = (-1)^k D_k^*.$$
In the case when $\dim M = n = 4k$ one has more special relation:

$$D_{2k} : \text{Hom}(H_{2k}(M, \mathcal{R}), \mathcal{R}) \cong H_{2k}(M, \mathcal{R})$$

is self adjoint isomorphism:

$$D_{2k} = D_{2k}^*.$$
In this case the self adjoint isomorphism:

\[ D_{2k} = D_{2k}^* \]

defines the invariant

\[ \text{sign}(M) = \text{sign}(D_{2k}). \]
Poincaré duality for non compact manifolds

Non compact manifolds

For non compact (orientable) manifolds one supposes

- \( C_k \overset{\text{def}}{=} C_k(M, \mathcal{R}) \), the space of chains of the manifold \( M \) with coefficients in the scalar field \( \mathcal{R} \).
- \( C^k \overset{\text{def}}{=} C^0_k(M, \mathcal{R}) = \text{Hom}_0(C_k(M, \mathcal{R}), \mathcal{R}) \), the space of linear functionals with finite supports with respect to natural basis in the group \( C_k \).
Poincare duality for non compact manifolds
Non compact manifolds

One has again the commutative diagram similar to diagrams for algebraic Poincare complexes of compact manifolds:

\[
\begin{align*}
  C_0 & \xleftarrow{d_1} C_1 & \xleftarrow{d_2} C_2 & \cdots & \xleftarrow{d_{n-1}} C_{n-1} & \xleftarrow{d_n} C_n \\
  C^n & \xleftarrow{d^*_n} C^{n-1} & \xleftarrow{d^*_{n-1}} C^{n-2} & \cdots & \xleftarrow{d^*_2} C^1 & \xleftarrow{d^*_1} C^0 \\
  \uparrow D_0 & \quad \uparrow D_1 & \quad \uparrow D_2 & \quad \ldots & \quad \uparrow D_{n-1} & \quad \uparrow D_n \\
  C_0^* & \xrightarrow{D_0} C_1^* & \xrightarrow{D_1} C_2^* & \cdots & \xrightarrow{D_{n-1}} C_{n-1}^* & \xrightarrow{D_n} C_n^* \\
\end{align*}
\]
Poincare duality for non compact manifolds

Non compact manifolds

It satisfies similar natural properties

- \( d_{k-1}d_k = 0 \),
- \( d_kD_k + (-1)^{k+1}D_{k-1}d^*_{n-k+1} = 0 \),
- \( q \circ D_k = (-1)^{k(n-k)}D^*_{n-k} \).
- \( H(D_k) : H(C^{n-k}) \to H(C_k) \) is isomorphism, that is

\[
H(D_k) : H^0_{n-k}(M) \xrightarrow{\cong} H_k(M).
\]
The case of non compact manifolds differs from the case of compact manifold: the cohomology groups $H_{0}^{n-k}(M)$ has no proper expression in terms of the homology groups $H_{k}(M)$ unlike in compact case.
Let $G \overset{\text{def}}{=} \pi_1(M)$ is fundamental group of a compact manifold $M$. The group $G$ acts freely on the universal covering $\tilde{M}$ with trivial action on $M$ such that covering

$$
\begin{array}{ccc}
\tilde{M} \\
\downarrow \\
M
\end{array}
$$

is equivariant mapping.
Free action of the fundamental group $G$

Put $C_k \overset{\text{def}}{=} C_k(\widetilde{M}, \mathcal{R})$, $C_k^\natural \overset{\text{def}}{=} C_0^k(\widetilde{M}, \mathcal{R})$. The crucial idea is the following:

**Proposition**

The spaces $C_k$ and $C_k^\natural$ are finite generated free modules over the group ring $C[G]$, and therefore the cochain group has a simple expression in the terms of the chain group:

$$C_k^\natural = \text{Hom}_0(C_k, \mathcal{R}) \approx \text{Hom}_{C[G]}(C_k, C[G]).$$
Hence the coboundary homomorphism $d_k^*: C^{k-1} \to C^k$ is adjoint to the boundary homomorphism $d_k$ over the group ring $C[G]:$

$$
\begin{align*}
C^{k-1} & \xrightarrow{d_k} C^k \\
\| & \\
\text{Hom}_{C[G]}(C_{k-1}, C[G]) & \xrightarrow{d_k^*} \text{Hom}_{C[G]}(C_k, C[G])
\end{align*}
$$
Free action of the fundamental group $G$
APC for non simply connected manifold

Again, one has similar diagram:

\[
\begin{array}{cccccccc}
C_0 & \xleftarrow{d_1} & C_1 & \xleftarrow{d_2} & C_2 & \cdots & \xleftarrow{d_{n-1}} & C_{n-1} & \xleftarrow{d_n} & C_n \\
C_0 & \xrightarrow{D_0} & C_1 & \xrightarrow{D_1} & C_2 & \cdots & \xrightarrow{D_{n-1}} & C_{n-1} & \xrightarrow{D_n} & C_n \\
C^n & \xleftarrow{d^*_n} & C^{n-1} & \xleftarrow{d^*_{n-1}} & C^{n-2} & \cdots & \xleftarrow{d^*_2} & C^1 & \xleftarrow{d^*_1} & C^0 \\
C^n & \xrightarrow{D_n} & C^{n-1} & \xrightarrow{D_{n-1}} & C^{n-2} & \cdots & \xrightarrow{D_2} & C^1 & \xrightarrow{D_1} & C^0 \\
\end{array}
\]
With similar properties:

1. \[ d_{k-1}d_k = 0 , \]
2. \[ d_kD_k + (-1)^{k+1}D_{k-1}d^*_{n-k+1} = 0 , \]
3. \[ D_k = (-1)^{k(n-k)}D^*_{n-k} . \]
4. \[ H(D_k) : H(C^{n-k}) \rightarrow H(C_k) \text{ is isomorphism.} \]
Free action of the fundamental group $G$
APC for non simply connected manifold

One can simplify notations:

$$F_k = i^k(k-1)D_k.$$  

Then

\[
\begin{array}{ccccccc}
  C_0 & \xleftarrow{d_1} & C_1 & \xleftarrow{d_2} & C_2 & \xleftarrow{d_3} & \cdots & \xleftarrow{d_{n-1}} & C_{n-1} & \xleftarrow{d_n} & C_n \\
  \uparrow F_0 & & \uparrow F_1 & & \uparrow F_{n-2} & & \cdots & & \uparrow F_{n-1} & & \uparrow F_n \\
  C^n & \xleftarrow{\delta_n} & C^{n-1} & \xleftarrow{\delta_{n-1}} & C^{n-2} & \xleftarrow{\delta_{n-2}} & \cdots & \xleftarrow{\delta_2} & C^1 & \xleftarrow{\delta_1} & C^0 
\end{array}
\]
New homomorphism $F_k$ satisfies simpler relations:

1. $d_k F_k + F_{k-1} d^*_{n-k+1} = 0$,
2. $F_k = F^*_{n-k}$.
3. $H(F_k) : H(C^{n-k}) \rightarrow H(C_k)$ also is isomorphic.
Free action of the fundamental group $G$

APC for non simply connected manifold

As a matter of fact, both module $H(C^k) = H_k(\widetilde{M})$, and module $H(C^k) = H_0^k(\widetilde{M})$ are neither free nor finite generated modules.

Therefore it is convenient to consider so called the cylinder of the diagram
Free action of the fundamental group $G$
APC for non simply connected manifold

Construction of the cylinder of the diagram:

\[
\begin{align*}
C_0 & \xleftarrow{d_1} C_1 \xleftarrow{d_2} C_2 \xleftarrow{d_3} \cdots \xleftarrow{d_{n-1}} C_{n-1} \xleftarrow{d_n} C_n \\
C_0 & \xrightarrow{F_0} C_1 \xrightarrow{F_1} C_2 \xrightarrow{F_{n-2}} \cdots \xrightarrow{F_{n-1}} C_{n-1} \xrightarrow{F_n} C_n \\
C^n & \xleftarrow{d^*_n} C^{n-1} \xleftarrow{d^*_{n-1}} C^{n-2} \xleftarrow{d^*_{n-2}} \cdots \xleftarrow{d^*_2} C^1 \xleftarrow{d^*_1} C^0
\end{align*}
\]
Free action of the fundamental group $G$
APC for non simply connected manifold

Construction of the cylinder of the diagram:

$$
\begin{array}{cccccccc}
C_0 & \leftarrow & d_1 & C_1 & \leftarrow & d_2 & C_2 & \leftarrow & \cdots & \leftarrow & d_{n-1} & C_{n-1} & \leftarrow & d_n & C_n \\
& \uparrow F_0 & & \uparrow F_1 & & \uparrow F_{n-2} & & & & & \uparrow F_{n-1} & & \uparrow F_n \\
C^n & \leftarrow & d^*_n & C^{n-1} & \leftarrow & d^*_{n-1} & C^{n-2} & \leftarrow & \cdots & \leftarrow & d^*_2 & C^1 & \leftarrow & d^*_1 & C^0 \\
& \downarrow d_1 & & \downarrow d_2 & & \downarrow d_{n-1} & & & & & \downarrow d_n & & \downarrow d_1 & & \downarrow d_0 \\
C_0 & \leftarrow & d_1 & C_1 & \leftarrow & d_2 & C_2 & \leftarrow & \cdots & \leftarrow & d_{n-1} & C_{n-1} & \leftarrow & d_n & C_n \\
& \uparrow F_0 & & \uparrow F_1 & & \uparrow F_{n-2} & & & & & \uparrow F_{n-1} & & \uparrow F_n \\
C^n & \leftarrow & d^*_n & C^{n-1} & \leftarrow & d^*_{n-1} & C^{n-2} & \leftarrow & \cdots & \leftarrow & d^*_2 & C^1 & \leftarrow & d^*_1 & C^0 \\
& \downarrow d_1 & & \downarrow d_2 & & \downarrow d_{n-1} & & & & & \downarrow d_n & & \downarrow d_1 & & \downarrow d_0 \\
\end{array}
$$
Free action of the fundamental group $G$
APC for non simply connected manifold

\[
\begin{array}{cccccccc}
C_0 & \overset{d_1}{\leftarrow} & C_1 & \overset{d_2}{\leftarrow} & C_2 & \overset{d_3}{\leftarrow} & \cdots & \overset{d_{n-1}}{\leftarrow} & C_{n-1} & \overset{d_n}{\leftarrow} & C_n \\
F_0 & \leftarrow & F_1 & \leftarrow & F_2 & \leftarrow & \cdots & \leftarrow & F_{n-2} & \leftarrow & F_{n-1} & \leftarrow & F_n \\
C^n & \overset{d_n^*}{\leftarrow} & C^{n-1} & \overset{d_{n-1}^*}{\leftarrow} & \cdots & \overset{d_{n-2}^*}{\leftarrow} & C^2 & \overset{d_2^*}{\leftarrow} & C^1 & \overset{d_1^*}{\leftarrow} & C^0
\end{array}
\]
Free action of the fundamental group $G$
APC for non simply connected manifold
Free action of the fundamental group $G$

APC for non simply connected manifold

\[ 0 \leftarrow C_0 \leftarrow H_1 \oplus C^1 \leftarrow H_2 \oplus C^{n-1} \leftarrow H_3 \leftarrow \cdots \]
Free action of the fundamental group $G$
APC for non simply connected manifold

\[
\begin{array}{cccc}
0 & \leftarrow & C_0 & \xleftarrow{H_1} C_1 \\
& & \oplus & \leftarrow H_2 \\
& & C^n & \oplus C^{n-1} \leftarrow H_3 \\
& & \leftarrow & \cdots \\
& & \cdots & \xleftarrow{H_{n-1}} C_{n-1} \\
& & & \oplus C^2 \leftarrow H_n \\
& & & \leftarrow C^n \oplus C^1 \\
& & & \leftarrow H_{n+1} C^0 & \leftarrow 0
\end{array}
\]
Free action of the fundamental group $G$

APC for non simply connected manifold

\[
\begin{array}{c}
A_0 \\
A_1 \\
A_2 \\
0 \\ C_0 \\ C_1 \\
C_2 \\
\oplus \\
\oplus \\
\oplus \\
C^m \\
C^{m-1} \\
H_1 \\
H_2 \\
H_3 \\
\cdot \cdot \cdot
\end{array}
\]
Free action of the fundamental group $G$
APC for non simply connected manifold

$$
\begin{align*}
&\cdots \\ &C_{n-1} \\ &\oplus \\ &C^2 \\ &\oplus \\ &A_{n-1} \\
&H_{n-1} \\
&\cdots
\end{align*}
$$

$$
\begin{align*}
&A_0 \\
&\| \\
&\| \\
&\| \\
&A_n
\end{align*}
$$

$$
\begin{align*}
&\cdots \\ &C_n \\ &\oplus \\ &C^1 \\ &\oplus \\ &A_n \\
&H_n \\
&\cdots
\end{align*}
$$

$$
\begin{align*}
&\cdots \\ &C_{n-1} \\ &\oplus \\ &C^0 \\ &\oplus \\ &A_{n+1}
\end{align*}
$$

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Homotopy invariants of manifolds: Poincare duality and the signature formulae

XIX Geometrical seminar, Zlatibor, Serbia, August 28-September 4, 2016
Free action of the fundamental group $G$

APC for non simply connected manifold

\[
0 \leftarrow A_0 \overset{H_1}{\leftarrow} A_1 \overset{H_2}{\leftarrow} A_2 \overset{H_3}{\leftarrow} \cdots
\]
Free action of the fundamental group $G$

APC for non simply connected manifold

\[ 0 \leftarrow A_0 \xleftarrow{H_1} A_1 \xleftarrow{H_2} A_2 \xleftarrow{H_3} \cdots \]

\[ \cdots \xleftarrow{H_{n-1}} A_{n-1} \xleftarrow{H_n} A_n \xleftarrow{H_{n+1}} A_{n+1} \leftarrow 0 \]
Free action of the fundamental group $G$

APC for non simply connected manifold

\[ 0 \leftarrow A_0 \xleftarrow{H_1} A_1 \xleftarrow{H_2} A_2 \xleftarrow{H_3} \cdots \]

\[ \cdots \xleftarrow{H_{n-1}} A_{n-1} \xleftarrow{H_n} A_n \xleftarrow{H_{n+1}} A_{n+1} \leftarrow 0 \]

\[ A_k \approx (A_{n+1-k})^*, \quad H_k \approx (H_{n-k})^*. \]
Free action of the fundamental group $G$

The process of shortening of APC:

$$
0 \leftarrow A_0 \overset{H_1}{\leftarrow} A_1 \oplus A_0 \overset{H_2}{\leftarrow} A_2 \overset{H_3}{\leftarrow} \cdots
$$
Free action of the fundamental group $G$

The process of shortening of APC:

\[
\begin{array}{cccccccc}
0 & \leftarrow & A_0 & \overset{H_1}{\leftarrow} & A_1' & \leftarrow & H_2 & A_2 & \overset{H_3}{\leftarrow} & \cdots \\
\oplus & \downarrow & \downarrow & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
A_0 & \leftarrow & A_1 & \leftarrow & A_2 & \leftarrow & A_3 & \leftarrow & \cdots \\
\end{array}
\]

\[
\cdots \leftarrow A_{n-1} & \overset{H_{n-1}}{\leftarrow} & A_n' & \overset{H_n}{\leftarrow} & A_n & \leftarrow & H_{n+1} & A_{n+1} & \leftarrow & 0
\]

\[A_0 \rightarrow A_1' \leftarrow A_1 \leftarrow A_2' \leftarrow A_2 \leftarrow \cdots \leftarrow A_{n-1}' \leftarrow A_{n-1} \leftarrow A_n' \leftarrow A_n \leftarrow H_{n+1} \leftarrow A_{n+1} \leftarrow 0\]
Free action of the fundamental group $G$

The process of shortening of APC

$$
\begin{align*}
0 & \leftarrow A'_1 \xrightarrow{H_2} A_2 \xrightarrow{H_3} \cdots \\
\oplus & \\
0 & \leftarrow A_0 \xrightarrow{H_1} A_0 \leftarrow 0
\end{align*}
$$
Free action of the fundamental group $G$

The process of shortening of APC

\[ 0 \leftarrow A'_1 \xleftarrow{H_2} A_2 \xleftarrow{H_3} \cdots \]

\[ 0 \leftarrow A_0 \xleftarrow{H_1} A_0 \leftarrow 0 \]

\[ \cdots \xleftarrow{H_{n-1}} A_{n-1} \xleftarrow{H_n} A'_n \leftarrow 0 \]

\[ 0 \leftarrow A_{n+1} \xleftarrow{H_{n+1}} A_{n+1} \leftarrow 0 \]
Free action of the fundamental group $G$

General algebraic formula for signature:

\[
C = \bigoplus_{k=0}^{n} C_k, \quad C^* = \bigoplus_{k=0}^{n} C_k^*,
\]

\[
B = C \oplus C^*
\]

\[
C \oplus C^* = B \xleftarrow{H} B^* = (C \oplus C^*)^* \approx C \oplus C^* = B, \quad H^2 = 0.
\]
Let \( D = \text{Im} \ H \),

\[
\begin{array}{cccccc}
0 & \leftarrow & B/D & \leftarrow & B & \leftarrow & D & \leftarrow & 0 \\
       &        \uparrow H &       &        \uparrow H &       &        \uparrow H &       &        \\
0 & \rightarrow & (B/D)^* & \rightarrow & B^* & \rightarrow & D^* & \rightarrow & 0 \\
\end{array}
\]

\[
\text{Ker} \ H
\]

\[ H : D^* \xrightarrow{\simeq} D, \quad H^* = H. \]
Free action of the fundamental group $G$
General algebraic formula for signature

Definition of non commutative signature

The module $D$ is a stable free over the group ring $\mathcal{A} = C[G]$. The pair $(D, H)$ represents an element of Hermitian $K$-theory:

$$[D, H] \in K^0_h(\mathcal{A}).$$

This element is called non commutative signature of non simply connected manifold $M$, $G = \pi_1(M)$:

$$\text{sign}(M) \overset{def}{=} [D, H] \in K^0_h(\mathcal{A}).$$
Free action of the fundamental group $G$

Remark

If $M$ is simply connected, $G = \pi_1(M) = 1$, or if we ignore the fundamental group, then $\mathcal{A} = \mathcal{R}[1] = \mathcal{R}$,

$$K^0_h(\mathcal{A}) \approx \mathbb{Z}$$

and

non commutative signature coincides with classical signature.
Free action of the fundamental group $G$
Transfer to the group $C^*$-algebra

For simplicity assume that the group ring $C[G]$ is substituted for its completion, the group $C^*$–algebra $\mathcal{A} = C^*[G]$ (with respect to the regular representation of the group $G$). The spaces of chains (cochains) also are substituted for modules over the algebra $\mathcal{A}$,

$$
\overline{C}_k \overset{\text{def}}{=} C_k \otimes_{C[G]} \mathcal{A},
$$

$$
\overline{C}^k \overset{\text{def}}{=} C^k \otimes_{C[G]} \mathcal{A},
$$
Free action of the fundamental group $G$
Transfer to the group $C^*$-algebra

Put

$$A_i = \mathcal{C}_i \oplus \mathcal{C}^{n-i+1}$$

and

$$A_{ev} = \bigoplus_{k=0}^{2l} A_{2k}, \quad A_{odd} = \bigoplus_{k=0}^{2l} A_{2k+1}.$$
Free action of the fundamental group $G$
Transfer to the group $C^*$-algebra

Then one can show that the homomorphism

$$G_{ev} = d + d^* + F \overset{\text{def}}{=} G|_{A_{ev}} : A_{ev} \to A_{ev}$$

is invertible and self-adjoint.
Consider a splitting of $G_{ev}$ into a direct sum of positive and negative summands

$$G_{ev} = \begin{pmatrix} G^+_{ev} & 0 \\ 0 & G^-_{ev} \end{pmatrix} : A^+_{ev} \oplus A^-_{ev} \longrightarrow A^+_{ev} \oplus A^-_{ev}.$$
Free action of the fundamental group $G$
Transfer to the group $C^*$-algebra

Denote

$$\text{sign} G_{ev} = \text{sign}(d + d^* + F) \overset{\text{def}}{=} [A^+_{ev}] - [A^-_{ev}] \in K(\mathcal{A}).$$

Definition

This definition coincides with classical signature in classical case:

$$\text{sign}(d + d^* + F) = \text{sign}(H(F)) = \text{sign} M.$$
The natural map $C[G] \longrightarrow C^*[G]$ induces a map

$$K^0_h(C[G]) \longrightarrow K^0_h(C^*[G]) \approx K^0(C^*[G])$$

Then algebraic signature $\text{sign}(M)$ goes to completed signature $\breve{\text{sign}}(M)$. 
The abstract Hirzebruch formula
Bordisms of non simply connected manifolds

Non commutative signature of non simply connected manifold $\text{sign}(ACP(M))$ satisfies the following properties:

- it is homotopy invariant,
- it is invariant of non simply connected bordisms.

Hence one has a map

$$\text{sign} : \Omega^G_n \rightarrow K^n_h(A).$$
The abstract Hirzebruch formula
Bordisms of non simply connected manifolds

Let $BG$ be classifying space, that is Eilenberg-MacLane complex $BG = K(G, 1)$. Then each manifold $\tilde{M}$ with free cocompact action of the group $G$ induces a continuous mapping

$$\varphi_M : M = \widetilde{M}/G \longrightarrow BG,$$

such that the diagram

$$\begin{array}{ccc}
\tilde{M} & \xrightarrow{\varphi_M} & EG \\
\downarrow & & \downarrow \\
M = \tilde{M}/G & \xrightarrow{\varphi_M} & BG
\end{array}$$

is commutative.
Then one has

$$\Omega^G_n \approx \Omega^G_n(BG)$$

that is the signature induces the mapping

$$\Omega^G_n(BG) \xrightarrow{\text{sign}} K^r_h(A),$$

and can be expressed in the terms of characteristic classes that is known as the Hirzebruch formula.
The abstract Hirzebruch formula

For simply connected orientable compact manifold $M$ one has classical Hirzebruch formula

$$\text{sign} M = \langle L(M), [M] \rangle.$$

If $M$ is a non simply connected manifold with fundamental group $G = \pi_1(M)$, then one should have similar formula:

$$\text{sign} M = \langle L(M) \cdot ?, [M] \rangle \in K(A) \otimes Q,$$

for some $? \in H^*(M; K(A)) \otimes Q$, where $A = C^*[G]$ is the group $C^*$–algebra of the group $G$. 
The abstract Hirzebruch formula

The map

\[ \Omega_*(BG)^{\text{sign}} \rightarrow K_*(C[G]), \]

with natural property:

If \( N \) is a simply connected manifold and \( \tau(N) \) is its classical signature then

\[ \text{sign}(M \times N) = \text{sign}(M)\tau(N) \in K^*(C[G]). \]
General Hirzebruch formula

We shall be interested in groups modulo torsion, that is with tensor product by rational numbers $\mathbb{Q}$. Consider the map

$$\text{sign} : \Omega_*(B\pi) \otimes \mathbb{Q} \rightarrow \mathcal{K}^*(C[G]) \otimes \mathbb{Q}.$$ 

In this case the bordism group is expressed in terms of usual homologies

$$\Omega_*(B\pi) \otimes \mathbb{Q} \approx H_*(B\pi; \mathbb{Q}) \otimes \Omega_*.$$ 

Hence one has the homomorphism

$$\text{sign} : H_*(B\pi; \mathbb{Q}) \rightarrow \mathcal{K}^*(C[G]) \otimes \mathbb{Q}.$$
Thus the homomorphism \textbf{sign} is represented as a cohomology class

$$\sigma \in H^*(B\pi; K^*(C[G]) \otimes \mathbb{Q}).$$

Then for any non simply connected manifold \((M, \varphi_M)\) one has so called generalized Hirzebruch formula

$$\text{sign}(M, \varphi_M) = \langle L(M)\varphi_M^*(\sigma), [M] \rangle \in K^*(C[G]) \otimes \mathbb{Q}.$$
Let $\alpha : K^*(C[G]) \otimes \mathbb{Q} \to \mathbb{Q}$ be an additive functional and $\alpha(\sigma) = x \in H^*(B\pi; \mathbb{Q})$. Then

$$\text{sign}_x(M, f_M) = \langle L(M)f_M^*(x), [M] \rangle \in \mathbb{Q}$$

gives a homotopy invariant higher signature.
General Hirzebruch formula

We obtain the description of all homotopy invariant higher signatures. Hence for description of all homotopy invariant higher signatures one should study the universal cohomology class

$$\sigma \in H^*(B\pi; \mathcal{K}^*(C[G]) \otimes \mathbb{Q}) = H^*(B\pi; \mathbb{Q}) \otimes \mathcal{K}^*(C[G]) \otimes \mathbb{Q}.$$
General Hirzebruch formula
Comparison of the Hirzebruch formulas

Abstract Hirzebruch formula:

$$\text{sign}(M, \varphi_M) = \langle L(M) \varphi_M^*(\sigma), [M] \rangle \in \mathcal{K}^*(C[G]) \otimes \mathbb{Q}.$$ 

From the Atiyah-Singer formulas:

$$\text{sign}M = \langle L(M) \text{ch}_A(\varphi_M^*(\xi_A)), [M] \rangle \in \mathcal{K}(A) \otimes \mathbb{Q},$$

Hence

$$\sigma = \text{ch}_A(\xi_A),$$

where

$$\sigma \in H^*(B\pi; \mathcal{K}^*(C[G]) \otimes \mathbb{Q}),$$

$$\text{ch}_A(\xi_A) \in H^*(B\pi; \mathcal{K}^*(C^*[G]) \otimes \mathbb{Q}).$$
Outline

1. Classical Hirzebruch formula
2. Finite dimensional representation
3. Underlying algebraic Poincare complex. Abstract generalization
4. Novikov’s conjecture
5. Topological invariance of rational Pontryagin classes
6. Manifolds with proper actions
Novikov’s conjecture

The Novikov conjecture says that the higher signature

$$\text{sign}_x M = 2^{2k} \langle L(M) \cdot f_M^*(x), [M] \rangle,$$

is a homotopy invariant, where $x$ is an arbitrary cohomology class of the classifying space $BG$, that is $x = \text{ch}\xi$ for arbitrary vector bundle over $BG$. The left side part is called higher signature of $M$. 
If $\rho$ is a representation then

$$\text{sign}_\rho M = 2^{2k} \langle L(M) \cdot f_M^*(\text{ch}\rho), [M] \rangle,$$

is a homotopy invariant.
Novikov’s conjecture

Example

If $\rho_t \in T$ is a continuous family of representations then

$$K(T) \ni \text{sign}_\rho M = 2^{2k} \langle L(M) \cdot f^*_M(\text{ch}\rho_t), [M] \rangle \in H^*(T; \mathbb{Q}),$$

is a homotopy invariant.
Novikov’s conjecture
Example: Free abelian group

If $\pi \approx \mathbb{Z}^n$ is free abelian group, $\rho_t : \pi \to U(1) \approx S^1$ be characters of the group $\pi$, $t \in T^n$, then

$$\text{ch} \rho_t = \sum_{i=1}^{n} x^i y_i \in H^2(T^n \times B\pi,)$$

$x^i \in H^1(T^n); \quad y_i \in H^1(B\pi); \quad (B\pi \approx T^n).$ Hence

$$\text{sign}_{x^i}M = 2^{2k} \langle L(M) \cdot f_M^*(x^i), [M] \rangle$$

is a homotopy invariant (Lusztig, 1972).

G. Lusztig, Novikov’s higher signature and families of elliptic operators, J.Diff. Geometry, (1972), vol.7, p.229–256,
Novikov’s conjecture
The state of Novikov’s conjecture

Novikov’s conjecture
The state of Novikov’s conjecture

- a-T-menable groups
- amenable groups
- elementary amenable groups
- virtually poly-cyclic
- virtually solvable subgroups of $GL(n, C)$
- discrete subgroups of Lie groups with finitely many path components
- subgroups of groups which are discrete cocompact subgroups of Lie groups with finitely many path components
Novikov’s conjecture
The state of Novikov’s conjecture

- countable groups admitting a uniform embedding into Hilbert space
- $G$ admits an amenable action on some compact space
- linear groups
- torsion free discrete subgroups of $GL(n, R)$
- Groups with finite $BG$ and finite asymptotic dimension
- $G$ acts properly and isometrically on a complete Riemannian manifold $M$ with non-positive sectional curvature
- $\pi_1(M)$ for a complete Riemannian manifold $M$ with non-positive sectional curvature
Novikov’s conjecture
The state of Novikov’s conjecture

- $\pi_1(M)$ for a complete Riemannian manifold $M$ with non-positive sectional curvature which is $A$-regular
- $\pi_1(M)$ for a complete Riemannian manifold $M$ with pinched negative sectional curvature
- $\pi_1(M)$ for a closed Riemannian manifold $M$ with non-positive sectional curvature
- $\pi_1(M)$ for a closed Riemannian manifold $M$ with negative sectional curvature
- word hyperbolic groups
- one-relator groups
- torsion free one-relator groups
Novikov’s conjecture
The state of Novikov’s conjecture

- Haken 3-manifold groups (in particular knot groups)
- $SL(n, \mathbb{Z}), n \geq 3$
- Artin’s braid group $B_n$
- pure braid group $C_n$
- Thompson’s group $F$
Outline

1. Classical Hirzebruch formula
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6. Manifolds with proper actions
The right hand side of the Hirzebruch formula

\[ \text{sign}_\rho X = 2^k \langle L(X) \text{ch}_\rho \xi_X, [X] \rangle. \]

is an invariant of bordisms of the classifying space \( BG \) whereas left hand side is homotopy invariant.

This means that each representation \( \rho \) defines a higher signature that automatically is homotopy invariant.
But for higher signatures defined by representations one can say more: they are invariants with respect to topological bordisms of the classifying space $BG$!
This observation allowed M. Gromov to simplify the proof of the topological invariance of the rational Pontryagin classes of a smooth manifold.

Topological invariance of rational Pontryagin classes

Similar to S.P. Novikov the crucial idea consists in reduction of verification of topological invariance of the Pontryagin classes \( \{p_k(M)\} \) to the Hirzebruch (non homogeneous) class \( L(M) \in H^*(M; \mathbb{Q}) \),

\[
L(M) = \sum_{k=0}^{\infty} L_k(M),
\]

precisely for its value on an arbitrary rational homology cycle on \( M \),

\[
\{\langle L(M), x \rangle : x \in H_*(M, \mathbb{Q}) \}.
\]
So the problem can be formulated as following:

Let $f : M \to M'$ be a topological homeomorphism of smooth (orientable closed compact) manifolds. We should check that

$$f^*(p_k(M')) = p_k(M) \in H^{4k}(M; \mathbb{Q})$$

where

$$H^{4k}(M; \mathbb{Q}) \leftarrow H^{4k}(M'; \mathbb{Q}) : f^*$$

is the homomorphism in cohomology. It is equivalent to the condition

$$f^*(L_k(M')) = L_k(M) \in H^{4k}(M; \mathbb{Q}).$$
The last condition is equivalent to following: let \( x \in H_{4k}(M; \mathbb{Z}) \) be a cycle. Then one should check the equality

\[
\langle L(M), x \rangle = \langle f^*(L(M')), x \rangle
\]

or

\[
\langle L(M), x \rangle = \langle L(M'), f_*(x) \rangle.
\]
Notice that both cycle $x$ and $f_*(x)$ can be realized with smooth compact orientable framed (that is with trivial normal bundle) submanifold:

$$N \subset M, \quad N' \subset M', \quad [N] = x, \quad [N'] = f_*(x).$$

So the condition can be expressed in the geometrical form:

$$\langle L(M), [N] \rangle = \langle L(M'), [N'] \rangle.$$
Topological invariance of rational Pontryagin classes

\[ f[N] = [N'] \]
Then the value of the Hirzebruch class $L(M)$ on the element $x$ coincides with the signature of the submanifold $N$, and similar the value of the Hirzebruch class $L(M')$ on the element $f_*(x)$ coincides with the signature of the submanifold $N'$. So it is sufficient to establish that one can choose manifolds $N$ and $N'$ with identical signatures:

$$\text{sign}(N) = \langle L(M), [N] \rangle = \langle L(M'), [N'] \rangle = \text{sign}(N').$$
In this point the proofs by Novikov and Gromov become different. S.P. Novikov has proven that the manifold $N'$ can be chosen homotopy equivalent to $N$, and hence with the same signatures.
M. Gromov suggested calculation of signatures of the manifolds $N$ and $N'$ as higher signatures of new submanifolds, saying $V$ and $V'$, non-simply connected, with respect to a representation of the fundamental group, and in turn submanifolds $V$ and $V'$ are topologically bordant with preserving fundamental group.
Namely let $N \subset U$ be a tubular neighborhood of the submanifold $N$. Since the normal bundle of the submanifold $N$ is trivial a neighborhood $U$ is diffeomorphic to Cartesian product

$$N \subset U = N \times \mathbb{R}^{n-4k}.$$
Topological invariance of rational Pontryagin classes

$N \subset U = N \times \mathbb{R}^{n-4k}$

$f[N] = [N']$
Consider a compact hypersurface \( W \subset \mathbb{R}^{n-4k} \) of codimension one with trivial normal bundle. For example let take the torus or products of two-dimensional surfaces. It is known that all higher signatures of the surface \( W \) are generated by some representations of the fundamental group. Then the signature of the manifold \( N \) coincides with a higher signature of the manifold \( V = N \times W \). Namely, one should take the fundamental cocycle \( w \in H^{n-4k-1}(W; \mathbb{Z}) \) of the surface \( W \).

\[
\text{sign}(N) = \text{sign}_w(N \times W).
\]
Topological invariance of rational Pontryagin classes

\[ \begin{array}{ccc}
N \times R^{n-4k} & \overset{\varnothing}{\longrightarrow} & U \\
\downarrow & & \downarrow \\
N \times W & \overset{\varnothing}{\longrightarrow} & V \\
\end{array} \]

\[ \text{sign} N = \text{sign}_w (V). \]
Topological invariance of rational Pontryagin classes

\[ f[N] = [N'] \]

\[ N \subset X \cong N \times W \times \mathbb{R}^1 C U = N \times \mathbb{R}^{n-4k} \]

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Homotopy invariants of manifolds: Poincare duality and the signature formulae

XIX Geometrical seminar, Zlatibor, Serbia, August 28-September 4, 2016
Thus one has submanifolds

\[ N \subset V = N \times W \subset N \times W \times R^1 = U \subset M. \]

The topological homeomorphism \( f \) maps the open set \( U \subset M \) on an open set \( U' \subset M' \) which can be split by a compact hypersurface \( V' \subset U' \).
Topological invariance of rational Pontryagin classes

\[ N \cap (N \times W) \subset U = N \times W \times R^1 \]

\[ V' \subset U' = f(U') \]
Topological invariance of rational Pontryagin classes
The submanifold $V'$ can be mapped onto the manifold $V$, and then onto the surface $W$ by means of a continuous mapping

$$\varphi' : V' \xrightarrow{\psi} V = N \times W \xrightarrow{\chi} W.$$ 

Substituting the mapping $\varphi'$ for homotopic smooth mapping $\varphi''$, one has that inverse image of the regular point $\varphi''^{-1}(q_0)$ is a submanifold that realizes the cycle $f_*(x) \in H_{4k}(V')$. 

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Homotopy invariants of manifolds: Poincare duality and the signature formulae XIX Geometrical seminar, Zlatibor, Serbia, August 28-September 4, 2016
Topological invariance of rational Pontryagin classes
Hence we should only to compare two higher signatures:

\[ \text{sign}_w(V'') = \text{sign}_w(V') = \text{sign}_w(V). \]

Crucial point here is that two compact (topological) submanifolds \( V \) and \( V'' = f^{-1}(V') \) in the neighborhood \( U_1 = V \times \mathbb{R}^1 \) are topological bordant!

Thus their higher signatures coincide.
Topological invariance of rational Pontryagin classes
Outline

1. Classical Hirzebruch formula
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6. Manifolds with proper actions
Manifolds with proper actions

There are many different definitions of proper action of the discrete (countable) group $G$ on the manifold $M$. The essential properties of the proper action are the following:

- The action is co-compact that is the quotient space $M/G$ is compact space.
- The quotient space $M/G$ is Hausdorff.
- For any point $x \in M$ the stationary subgroup $G_x \subset G$, $G_x \overset{def}{=} \{ g \in G : gx = x \}$ is finite.
Manifolds with proper actions

We will follow the definition by S.Illman (see for example)


A map $f : X \to Y$ is said to be proper if the inverse image of each compact set is compact.

An action $\varphi : G \times X \to X$, $(g, x) \mapsto gx$, of the group $G$ on $X$ is said to be proper if for every compact subset $A$ of $X$ we have that

$$G[A] = \{ g \in G : gA \cap A \neq \emptyset \}$$

is a finite subset of $G$. 
Manifolds with proper actions

The main idea lies in the theorem by S.Illman and T.Korppi


Theorem.
Let $M$ be a smooth proper $G$-manifold, where $G$ is a discrete group.
Then:
(i) There exists a distinguished equivariant triangulation of $M$.
(ii) Any two distinguished equivariant triangulations of $M$ have equivariant subdivisions that are $G$-equivariantly isomorphic, by a $G$-equivariant isomorphism which is $G$-homotopic to $\text{Id}_M$. 
For the manifold with proper action we have similar diagram:
The difference from the non simply connected manifolds is that unlike for free action all modules $C_0 \overset{\text{def}}{=} C_0(M, \mathcal{R})$ and $C^0 \overset{\text{def}}{=} \text{Hom}_0(C_0(M, \mathcal{R}), \mathcal{R})$ are neither free nor projective modules but they are finitely generated over the group ring $\mathcal{R}[G]$ in the case $\mathcal{R} \approx \mathbb{Z}$.
The crucial points is that in the case when $\mathcal{R}$ is a field of the characteristic zero (for example $\mathcal{R} \approx \mathbb{Q}, \mathbb{R}, \mathbb{C}$ ) one has

**Theorem**

All modules $C_k$ over the field $\mathcal{R}$ of the characteristic zero are finitely generated projective modules.

$$C^k \approx \text{Hom}_\mathcal{R}(C_k, \mathcal{R}).$$
Let $K_p(A)$ be an analogue of the groups of Hermitian $K$–theory that is based on projective finitely generated modules. Clearly one has the natural map

$$K(A) \longrightarrow K_p(A).$$

**Theorem**

If $A$ is a $C^*$–algebra then this natural map is isomorphic:

$$K(A) \approx K_p(A).$$
Therefore one can define a noncommutative signature for manifolds with proper actions has similar value like the classical one:

\[ \text{sign}_p M \overset{\text{def}}{=} \text{sign}_p (ACP(M, G)) \in K_p(\mathcal{A}). \]
Definitions

Definition

Let $X$ be an oriented manifold with proper action of the group $G$.

We say that $X$ is bordant to zero if there exist an oriented manifold $W$ with proper action of the group $G$ such that

$$\partial W = X.$$ 

Denote by $p\Omega^G_\ast$ the set of all bordism classes of oriented manifolds with proper action of the group $G$. 

The non commutative signature $\text{sign}_p(ACP(M, G))$ has the properties:

- it is homotopy invariant,
- it is bordism invariant.

$$\text{sign}_p: \Omega_n^G \to K_p(A).$$
Let $f\Omega^G_n$ be similar bordisms for free action of the group $G$. Then one has the commutative diagram

$$
\begin{array}{ccc}
_p\Omega^G_n & \xrightarrow{\text{sign}_p} & K_p(\mathcal{A}) \\
\cup & & \cup \\
_f\Omega^G_n & \xrightarrow{\text{sign}} & K(\mathcal{A})
\end{array}
$$
Open problems

- Compare the index of the signature operator on the manifolds with proper action with the signature of the manifold.
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- Calculate the signature of the manifold with proper action in the terms of the fixed points.
- Give the description of the equivariant vector bundles for almost free actions on the base.
Thank you for attention!