Angular billiard and algebraic Birkhoff conjecture

Andrey Mironov
Sobolev Institute of Mathematics, Novosibirsk

based on joint works with Michael Bialy, Tel Aviv University

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Theorem. (M. Bialy and A.M., 2015) *Let* Γ *be the dual curve to* γ*. Suppose that Birkhoff billiard admits a polynomial integral. Then, either* $\tilde{\Gamma}$ *has degree 2, or* $\tilde{\Gamma}$ *necessarily contains singular points. Moreover, all singular and inflection points of* $\tilde{\Gamma}$ *in* $\mathbb{C}P^2$ *belong to the Absolute* $\Lambda = \{x^2 + y^2 = 0\}$.

Corollary 1. *If the Birkhoff billiard inside γ is integrable with an integral which is polynomial in v, then $\tilde{\gamma}$ does not have two real algebraic ovals having a common tangent line.*

Indeed, if there were two such ovals γ and γ_1 (see Fig. 1), then the point (x, y) on Γ dual to the common tangent line τ would be a real singular point of Γ different from O, and hence $x^2 + y^2 \neq 0$. It then follows from Theorem 1 that the Birkhoff billiard inside γ does not admit polynomial integral on the energy level $|v| = 1$.

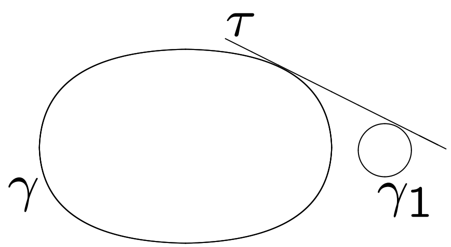

Fig. 1 Non-integrable Birkhoff billiard inside γ.

Example: Consider real algebraic curve

$$y^2 = F(x) = (x - x_1)(x - x_2)(x - x_3)(x - x_4) f(x), \quad x_j \in \mathbb{R}, \quad x_1 < x_2 < x_3 < x_4,$$

where $f(x)$ is a real polynomial such that $F(x) > 0$ for $x \in (x_1, x_2)$ and $x \in (x_3, x_4)$. Then Corollary 1 applies with

$$\gamma = \{(x, \pm\sqrt{F(x)}),\ x \in [x_1, x_2]\},\ \gamma_1 = \{(x, \pm\sqrt{F(x)}),\ x \in [x_3, x_4]\}.$$

Moreover, since the algebraic curve $\tilde{\gamma} \subset \mathbb{C}P^2$ is always singular, then Theorem by Bolotin does not apply.

Corollary 2. *Assume that* $\tilde{\Gamma}$ *is a non-singular curve (of degree* > 2*) in* $\mathbb{C}P^2$ *and has a smooth real oval* Γ *(for example,* $\tilde{\Gamma}$ *is a nonsingular cubic). Then the dual curve* γ *is also an oval and Birkhoff billiard inside* γ *is not integrable by Theorem 1. Notice, that in this case Bolotin's theorem does not apply, since* $\tilde{\gamma}$ *is necessarily singular in this case. The inflection points of* $\tilde{\Gamma}$ *correspond to singular points of* $\tilde{\gamma}$*.*

Corollary 3. *Let* γ *be the dual of Fermat oval* $\Gamma = \{x^{2n} + y^{2n} = 1,\ n > 1\}$. *Notice that* $\tilde{\Gamma}$ *is irreducible, non-singular curve and so by Theorem 1 the Birkhoff billiard inside* γ *is not integrable. One can easily compute that in this case the oval* γ *can be written as follows:*

$$\gamma = \{x^{\frac{2n}{2n-1}} + y^{\frac{2n}{2n-1}} = 1\}.$$

Therefore (the algebraic curve) γ *is a strictly convex* C^1 *curve in the plane which has 4 singular points* $(\pm 1, 0), (0, \pm 1)$ *corresponding to 4 inflection points* $(\pm 1, 0), (0, \pm 1)$ *of* Γ. *So Bolotin's theorem does not apply in this case also.*

Theorem. (M. Bialy and A.M., 2016) *If the Birkhoff billiard inside* γ *admits polynomial integral of degree 4 in* v_1, v_2*, then* γ *is an ellipse.*

We shall call the Angular billiard *integrable* if there is a function $G : U \backslash \mathcal{S} \to \mathbb{R}$ such that

$$G(A) = G(\mathcal{A}(A)), \qquad \forall A \in U \backslash \mathcal{S}.$$

Example 1 *Let* Γ *be an ellipse defined by the equation*

$$F = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0,\ G(x,y) = \frac{F(x,y)}{(x-x_0)^2 + (y-y_0)^2}.$$

Here $O(x_0, y_0)$ *is arbitrary point inside* Γ,G *is the integral.*

Suppose that the Birkhoff billiard flow admits a polynomial integral Φ. One can assume that $\Phi(q,v)$ is a *homogeneous* polynomial of a certain *even* degree n in $\sigma(v) = xv_y - yv_x,\ v_x, v_y$:

$$\Phi = \Phi(\sigma, v_x, v_y).$$

And moreover, Φ vanishes on tangent vectors to γ.

Theorem. *Let* γ *be a convex closed curve and* $\Phi(\sigma, v_x, v_y)$ *be a homogeneous polynomial integral of even degree* n, *vanishing on the tangent vectors to the boundary* γ. *Then the Angular billiard corresponding to* Γ *is also integrable with the integral of the form*

$$G_1(x,y) = \frac{F_1(x,y)}{(\sqrt{x^2+y^2})^n}, \qquad F_1(x,y) = \Phi(1,-y,x),$$

where F_1 *is a (non-homogeneous) polynomial of degree* n. *Moreover* F_1 *vanishes on* Γ.

Let Γ be defined by the equation $f = 0$, where f is an irreducible polynomial in $\mathbb{C}[x,y]$ of degree d. Since $F_1 = 0$ on Γ one can write F_1 in the form: $F_1(x,y) = f^k(x,y)g_1(x,y), k \in \mathbb{Z}_+$. It is important, that f, g_1 can be assumed to be *real* polynomials. Next we replace G_1 by $G := G_1^{\frac{1}{k}}$:

$$G(x,y) = \frac{(F_1(x,y))^{\frac{1}{k}}}{(x^2+y^2)^{\frac{n}{2k}}} := \frac{F}{(x^2+y^2)^m}, F := (F_1(x,y))^{\frac{1}{k}} = fg, \qquad m = \frac{n}{2k}.$$

Then G is also an integral, which also vanishes on Γ, but F, g are not necessarily polynomials anymore.

Lemma. For the integral F of Angular billiard for Γ, for all small ε, and $(x, y) \in \Gamma$ we have:

$$F(x + \varepsilon F_y, y - \varepsilon F_x) \left(-\frac{\mu}{\varepsilon}\right)^{2m} = F(x + \mu F_y, y - \mu F_x), \quad (1)$$

$$\mu = -\frac{(x^2 + y^2)\varepsilon}{x^2 + y^2 + 2\varepsilon(xF_y - yF_x)}.$$

Remarkable Identity For any function f we define affine Hessian:

$$H(f) := f_y(f_{xx}f_y - f_{xy}f_x) + f_x(f_{yy}f_x - f_{xy}f_y) f_{xx}f_y^2 - 2f_{xy}f_xf_y + f_{yy}f_x^2.$$

Theorem. *The following formula holds true*

$$g^3(x,y)H(f(x,y)) = c_1(x^2+y^2)^{3m-3}, \tag{2}$$

where c_1 is a constant.

For the proof we extract terms of order ε^3 in power series of the equation (1) of Lemma. Then it turns out that these terms form a complete derivative

$$L_v\left(H(g(x,y)f(x,y))(x^2+y^2)^{-3m+3}\right) = 0$$

Notation: For any polynomial $p(x,y)$, we denote by $\tilde{p}(x,y,z)$ the corresponding homogeneous polynomial of the same degree as $p(x,y)$.

Lemma. *The identity*

$$\tilde{g}_1^3(x,y,z)(\mathrm{Hess}(\tilde{f}(x,y,z)))^k + c(x^2+y^2)^{k(3m-3)} = \tilde{f}(x,y,z)\tilde{h}(x,y,z), \tag{3}$$

holds true for all $(x,y,z) \in \mathbb{C}^3$, *where* c *is a constant,* $\tilde{h}$ *is a homogeneous polynomial, and*

$$\mathrm{Hess}(\tilde{f}(x,y,z)) := \det \begin{pmatrix} \tilde{f}_{xx} & \tilde{f}_{xy} & \tilde{f}_{xz} \\ \tilde{f}_{xy} & \tilde{f}_{yy} & \tilde{f}_{yz} \\ \tilde{f}_{xz} & \tilde{f}_{yz} & \tilde{f}_{zz} \end{pmatrix}.$$

Reminder Given an algebraic curve C of degree d, $p \in C$ a regular point, let T be the tangent line at p.

The *order* r of the inflection point: $I_p(C, T) = r + 2$,

1) $r \geq 1$, and

2) $r + 2 \leq d$, by Bezout theorem.

Theorem.

$I_p(C, Hess(C)) = r$, and hence $\leq d - 2$.

Proof of the Main Theorem Consider the situation in $\mathbb{C}P^2$. Any intersection point in $\mathbb{C}P^2$ between Hessian curve of Hess($\tilde{\Gamma}$) with $\tilde{\Gamma}$ is either singular or inflection point of $\tilde{\Gamma}$. So, if there is a singular or inflection point $(x_0 : y_0 : z_0) \in \tilde{\Gamma}$ such that $x_0^2 + y_0^2 \neq 0$, it then follows from (3) that $c = 0$. Therefore, $\text{Hess}(\tilde{f}) \equiv 0$ since $\tilde{g}_1 \neq 0$ identically on $\tilde{\Gamma}$. This implies that $\tilde{\Gamma}$ is a line, but this is impossible.

Let us prove now that $\tilde{\Gamma}$ must have singular points. If on the contrary $\tilde{\Gamma}$ is a smooth curve, then it follows from (3) that all inflection points must belong to two lines L_1 and L_2 defined by the equations

$$L_1 = \{x + iy = 0\}, \qquad L_2 = \{x - iy = 0\}.$$

Recall, d is the degree of $\tilde{\Gamma}$. Then the Hessian curve intersects $\tilde{\Gamma}$ exactly in inflection points, and moreover, it is remarkable fact that the intersection multiplicity of such a point of intersection equals exactly the order of inflection point, and hence does not exceed $(d-2)$. Furthermore, the lines L_1 and L_2 intersect $\tilde{\Gamma}$ maximum in $2d$ points together . Hence, we have altogether counted with multiplicities not more than $2d(d-2)$, but on the other hand the Hessian curve has degree $3(d-2)$ and thus by Bezout theorem the number of intersection points with multiplicities is $3d(d-2)$. This contradiction shows that $\tilde{\Gamma}$ can not be a smooth curve unless $d=2$.