Integrable complex structures on naturally graded nilpotent Lie algebras

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Let $G$ be the Lie group of complex matrices of the form

$$G = \begin{pmatrix} 1 & \bar{z} & v \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}, \quad z, v \in \mathbb{C}.$$  

**Definition**

The Kodaira-Thurston nilmanifold $KT$ is the (compact) quotient $KT = G/\Gamma$, where $\Gamma$ is the subgroup of $G$ consisting of those matrices whose entries $z, v$ are Gaussian integers $a + ib, a, b \in \mathbb{Z}$.

$$\omega_1 = dz, \omega_2 = dv - \bar{z}dz,$$

is a basis for the left invariant $\Lambda^{1,0}$-forms on $G$ and

$$d\omega_1 = 0, d\omega_2 = \omega_1 \wedge \bar{\omega}_1.$$
The Newlander-Nirenberg theorem implies that a **left-invariant complex structure** on a real simply connected Lie group $G$ can be defined as an almost-complex structure $J$ on the tangent Lie algebra $\mathfrak{g}$ of $G$ ($J^2 = -1$) satisfying the **integrability condition**—vanishing of the Nijenhuis tensor:

$$[JX, JY] = [X, Y] + J[JX, Y] + J[X, JY], \quad \forall X, Y \in \mathfrak{g}.$$
Eigen-spaces $\mathfrak{g}^\mathbb{C}_{\pm i}$

Extending an almost complex structure $J$ on the complexification $\mathfrak{g}^\mathbb{C}$ we have a splitting

$$\mathfrak{g}^\mathbb{C} = \mathfrak{g}^\mathbb{C}_{-i} \oplus \mathfrak{g}^\mathbb{C}_{i},$$

where $\mathfrak{g}^\mathbb{C}_{\pm i} = \{x - \pm iJx : x \in \mathfrak{g}\}$ are the eigen-space of the complexification of $J$ corresponding to the eigen-values $\pm i$. $J$ is integrable if and only if both $\mathfrak{g}^\mathbb{C}_{\pm i}$ are (complex) subalgebras of $\mathfrak{g}^\mathbb{C}$.

There are two special cases:

1) $\mathfrak{g}^\mathbb{C}_{\pm i}$ are abelian subalgebras of $\mathfrak{g}^\mathbb{C}$.
2) $\mathfrak{g}^\mathbb{C}_{\pm i}$ are ideals of $\mathfrak{g}^\mathbb{C}$.
A. Malcev’s construction

Let $\mathfrak{g}$ be a nilpotent Lie algebra with a base $e_1, \ldots, e_n$ and

$$[e_i, e_j] = \sum_k c_{ij}^k e_k, \quad c_{ij}^k \in \mathbb{Q},$$

**Remark**

Vector space $\mathfrak{g}$ has a group structure $\ast$ (Campbell-Hausdorff formula):

$$x \ast y = x + y + \frac{1}{2}[x, y] + \ldots$$

such that $G$ is a nilpotent Lie group, $\Gamma \subset G$ is a subgroup generated by $e_1, e_2, \ldots, e_n$. $G/\Gamma$ is a compact nilmanifold
Let $g$ be a nilpotent Lie algebra and

$$g^1 = g \supset g^2 = [g, g] \supset \cdots \supset g^k = [g, g^{k-1}] \supset g^{s(g)} \supset \{0\}$$

its descending central series.

$s(g)$ is called the nil-index of the nilpotent Lie algebra $g$. One can consider the associated graded Lie algebra $\text{gr} g = \bigoplus_{i=1}^{+\infty} (g^i / g^{i+1})$ with the Lie bracket:

$$[x + g^{i+1}, y + g^{j+1}] = [x, y] + g^{i+j+1}, \quad x \in g^i, \quad y \in g^j.$$

**Definition**

A Lie algebra $g$ is called naturally graded if it is isomorphic to its associated graded $\text{gr} g$.  

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Salamon’s quasi-minimal model

Theorem (S. Salamon, 2001)

A real nilpotent $2n$-dimensional Lie algebra $\mathfrak{g}$ admits an integrable complex structure if and only if $(\mathfrak{g}^\mathbb{C})^*$ has a basis
\[
\{\omega^1, \ldots \omega^n, \bar{\omega}^1, \ldots \bar{\omega}^n\}
\]
such that
\[
d\omega^{l+1} \in I(\omega^1, \ldots \omega^l), \quad l = 0, \ldots, n-1,
\]
where $I(\omega^1, \ldots \omega^l)$ is an ideal in $\Lambda^*((\mathfrak{g}^\mathbb{C})^*)$ generated by $\omega^1, \ldots \omega^l$.

It means that
\[
d\omega_1 = 0, \quad d\omega_2 = \omega_1 \wedge \xi_1, \quad d\omega_3 = \omega_1 \wedge \eta_1 + \omega_2 \wedge \eta_2, \ldots
\]
Theorem (M., 2014)

Let \( \mathfrak{g} \) be a nilpotent Lie algebra endowed with an integrable complex structure and \( \dim \mathfrak{g} \geq 8 \). Then we have the following estimate:

\[
\text{codim} \mathfrak{g}^4 \geq 5, \quad \text{codim} \mathfrak{g}^6 \geq 8.
\]

Corollary

Let nilpotent Lie algebra \( \mathfrak{g} \) admit a complex structure and \( \dim \mathfrak{g} \geq 8 \). Then

\[
s(\mathfrak{g}) \leq \dim \mathfrak{g} - 3.
\]
A positively graded Lie algebra $\mathcal{D}(n) = \bigoplus_{l=1}^{n} \mathcal{D}_l$:

$$\dim \mathcal{D}_l(n) = \begin{cases} 1, & l = 2k; \\ 2, & l = 2k-1. \end{cases}$$

$\mathcal{D}_{2k-1}(n) = \langle v_{2k-1}, u_{2k-1} \rangle$ and $\mathcal{D}_{2k}(n) = \langle w_{2k} \rangle$. Relations:

$$[v_i, w_j] = u_{i+j}, i+j \leq n;$$
$$[w_j, u_l] = v_{j+l}, j+l \leq n;$$
$$[u_l, v_i] = w_{l+i}, l+i \leq n;$$

(1)

Indexes $i, l$ (index $j$) take odd (even) positive integer values.
A complex structure $J$ on $\mathcal{D}(4m)$ and $\mathcal{D}(4m+1)$

$$Jv_{2l+1} = u_{2l+1}, \quad Jw_{4k+2} = w_{4k+4}.$$  

The proof consists of verifying the integrability condition for basic elements $u_j, v_k, w_m$. 

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Thank you!