Integrable complex structures on naturally graded nilpotent Lie algebras

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Let G be the Lie group of complex matrices of the form

$$G = \begin{pmatrix} 1 & \bar{z} & v \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}, \quad z, v \in \mathbb{C}.$$

Definition

The Kodaira-Thurston nilmanifold KT is the (compact) quotient $KT = G/\Gamma$, where Γ is the subgroup of G consisting of those matrices whose entries z, v are Gaussian integers $a + ib$, $a, b \in \mathbb{Z}$.

$$\omega_1 = dz, \omega_2 = dv - \bar{z}dz,$$

is a basis for the left invariant $\Lambda^{1,0}$-forms on G and

$$d\omega_1 = 0, d\omega_2 = \omega_1 \wedge \bar{\omega}_1.$$

Left-invariant complex structures on Lie groups.

The Newlander-Nirenberg theorem implies that
a left-invariant complex structure on a real simply connected Lie group G can be defined as an almost-complex structure J on the tangent Lie algebra $\mathfrak{g}$ of G ($J^2 = -1$) satisfying the **integrability condition**=vanishing of the Nijenhuis tensor:

$$[JX, JY] = [X, Y] + J[JX, Y] + J[X, JY], \ \forall X, Y \in \mathfrak{g}.$$

Eigen-spaces $\mathfrak{g}^{\mathbb{C}}_{\pm i}$

Extending an almost complex structure J on the complexification $\mathfrak{g}^{\mathbb{C}}$ we have a splitting

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{g}^{\mathbb{C}}_{-i} \oplus \mathfrak{g}^{\mathbb{C}}_{i},$$

where $\mathfrak{g}^{\mathbb{C}}_{\pm i} = \{x - \pm iJx : x \in \mathfrak{g}\}$ are the eigen-space of the complexification of J corresponding to the eigen-values $\pm i$. J is

integrable if and only if

both $\mathfrak{g}^{\mathbb{C}}_{\pm i}$ are (complex) subalgebras of $\mathfrak{g}^{\mathbb{C}}$.

There are two special cases:

1) $\mathfrak{g}^{\mathbb{C}}_{\pm i}$ are **abelian subalgebras** of $\mathfrak{g}^{\mathbb{C}}$.
2) $\mathfrak{g}^{\mathbb{C}}_{\pm i}$ are **ideals** of $\mathfrak{g}^{\mathbb{C}}$.

A. Malcev's construction

Let $\mathfrak{g}$ be a nilpotent Lie algebra with a base $e_1, \dots, e_n$ and

$$[e_i, e_j] = \sum_k c_{ij}^k e_k, \quad c_{ij}^k \in \mathbb{Q},$$

Remark

Vector space $\mathfrak{g}$ has a group structure $\star$ (Campbell-Hausdorf formula):

$$x \star y = x + y + \frac{1}{2}[x, y] + \dots$$

such that G is a nilpotent Lie group, $\Gamma \subset G$ is a subgroup generated by $e_1, e_2, \dots, e_n$.
G/Γ is a compact nilmanifold

Descending central series and natural grading

Let $\mathfrak{g}$ be a nilpotent Lie algebra and

$$\mathfrak{g}^1{=}\mathfrak{g} \supset \mathfrak{g}^2{=}[\mathfrak{g},\mathfrak{g}] \supset \dots \supset \mathfrak{g}^k{=}[\mathfrak{g},\mathfrak{g}^{k-1}] \supset \mathfrak{g}^{s(\mathfrak{g})} \supset \{0\}$$

its descending central series.
$s(\mathfrak{g})$ is called the nil-index of the nilpotent Lie algebra $\mathfrak{g}$.
One can consider the associated graded Lie algebra
$\mathrm{gr}\mathfrak{g} = \oplus_{i=1}^{+\infty} \left(\mathfrak{g}^i/\mathfrak{g}^{i+1}\right)$ with the Lie bracket:

$$\left[x{+}\mathfrak{g}^{i+1}, y{+}\mathfrak{g}^{j+1}\right] = [x,y]{+}\mathfrak{g}^{i+j+1}, x \in \mathfrak{g}^i, y \in \mathfrak{g}^j.$$

Definition

A Lie algebra $\mathfrak{g}$ is called naturally graded if it is isomorphic to its associated graded $\mathrm{gr}\mathfrak{g}$.

Salamon's quasi-minimal model

Theorem (S.Salamon, 2001)

A real nilpotent 2n-dimensional Lie algebra $\mathfrak{g}$ admits an integrable complex structure if and only if $(\mathfrak{g}^{\mathbb{C}})^*$ has a basis $\{\omega^1, \dots \omega^n, \bar{\omega}^1, \dots \bar{\omega}^n\}$ such that

$$d\omega^{l+1} \in I(\omega^1, \dots \omega^l), \ l = 0, \dots, n-1,$$

where $I(\omega^1, \dots \omega^l)$ is an ideal in $\Lambda^*((\mathfrak{g}^{\mathbb{C}})^*)$ generated by $\omega^1, \dots \omega^l$.

It means that

$$d\omega_1 = 0, \ d\omega_2 = \omega_1 \wedge \xi_1, \ d\omega_3 = \omega_1 \wedge \eta_1 + \omega_2 \wedge \eta_2, \dots$$

Theorem (M., 2014)

Let $\mathfrak{g}$ be a nilpotent Lie algebra endowed with an integrable complex structure and $\dim \mathfrak{g} \geq 8$. Then we have the following estimate:

$$\mathrm{codim}\ \mathfrak{g}^4 \geq 5, \quad \mathrm{codim}\ \mathfrak{g}^6 \geq 8.$$

Corollary

Let nilpotent Lie algebra $\mathfrak{g}$ admit a complex structure and $\dim \mathfrak{g} \geq 8$. Then

$$s(\mathfrak{g}) \leq \dim \mathfrak{g} - 3.$$

Example

A positively graded Lie algebra $\mathfrak{D}(n) = \oplus_{l=1}^{n} \mathfrak{D}_l$:

$$\dim \mathfrak{D}_l(n) = \begin{cases} 1, & l = 2k; \\ 2, & l = 2k-1. \end{cases}$$

$\mathfrak{D}_{2k-1}(n) = \langle v_{2k-1}, u_{2k-1} \rangle$ and $\mathfrak{D}_{2k}(n) = \langle w_{2k} \rangle$. Relations:

$$\begin{aligned} [v_i, w_j] &= u_{i+j}, i+j \le n; \\ [w_j, u_l] &= v_{j+l}, j+l \le n; \\ [u_l, v_i] &= w_{l+i}, l+i \le n; \end{aligned} \qquad (1)$$

Indexes i, l (index j) take odd (even) positive integer values.

A complex structure J on $\mathfrak{D}(4m)$ and $\mathfrak{D}(4m+1)$

$$Jv_{2l+1} = u_{2l+1},\ Jw_{4k+2} = w_{4k+4}.$$

The proof consists of verifying the integrability condition for basic elements u_j, v_k, w_m.

Thank you!