# Interior estimates for Poisson type inequality and qc hyperbolic harmonic mappings

XIX Geometrical seminar

Miodrag Mateljević
(miodrag@matf.bg.ac.rs, jenamate@eunet.rs)

University of Belgrade, Faculty of Mathematics

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We study quasiconformal (qc) mappings in plane and space and in particular Lipschitz-continuity of mappings which satisfy in addition certain PDE equations (or inequalities).

Some of the obtained results can be considered as versions of Kellogg-Warshawski type theorem for qc-mappings.

Among the other things, as tool we use the interior estimates for Poisson type inequality.

# Interior estimate

The local version of Kellogg and Warschawski theorem follows from the proof of the standard version.

**Theorem (Kellogg and Warschawski, Local version)**

*Suppose that* $\Gamma$ *is a Jordan closed curve,* $D = \mathrm{Int}(\Gamma)$, $w_0 \in tr(\Gamma)$ *is interior point of an open* $C^{2,\alpha}$ *arc* $L$ *which belongs to* $tr(\Gamma)$ *and* $\omega$ *is a conformal mapping of* $\mathbb{U}$ *onto* $D$, *and that* $L_0$ *is closed subarc of* $L$. *Then* $|\omega''|$ *has a continuous extension to* $l = \omega^{-1}(L_0)$ *(which is the part of boundary of* $\mathbb{U}$*). More precisely, there is* $r > 0$ *such that* $\omega''$ *has a continuous extension to* $\overline{\mathbb{U} \cap B(z_0, r)}$, *where* $z_0 = \omega^{-1}(w_0)$. *In particular it is bounded from above on an open arc* $l'$ *on* $\mathbb{T}$, *with interior point at* $z_0$.

We study to which extent conformal theory can be extended to harmonic qc mappings. It turns out that the following result is very useful.

**Proposition (Interior estimate, Heinz-Bernstein, see [He])**

*Let $s : \overline{\mathbb{U}} \to \mathbb{R}$ be a continuous function from the closed unit disc $\overline{\mathbb{U}}$ into the real line satisfying the conditions:*

- *$s$ is $C^2$ on $\mathbb{U}$,*
- *$s_b(\theta) = s(e^{i\theta})$ is $C^2$ and*
- *$|\Delta s| \leq a_0 |\nabla s|^2 + b_0$, on $\mathbb{U}$ for some constants $a_0$ and $b_0$ (the last inequality we will call the interior estimate inequality).*

*Then the function $|\nabla s|$ is bounded on $\mathbb{U}$.*

## Theorem ([KaMa3], The main theorem)

*Let $f$ be a quasiconformal $C^2$ diffeomorphism from the plane domain $\Omega$ onto the plane domain $G$. Let $\gamma_\Omega \subset \partial\Omega$ and $\gamma_G = f(\gamma_\Omega) \subset \partial G$ be $C^{1,\alpha}$ respectively $C^{2,\alpha}$ Jordan arcs. If for some $\tau \in \gamma_\Omega$ there exist positive constants $r$, $M$ and $N$ such that*

$$|\Delta f| \leq a|\nabla f|^2 + b\,, \quad z \in \Omega \cap D(\tau, r), \tag{1}$$

*then $f$ has bounded partial derivatives in $\Omega \cap D(\tau, r_\tau)$ for some $r_\tau < r$. In particular it is a Lipschitz mapping in $\Omega \cap D(\tau, r_\tau)$.*

The idea behind the proof is to use conformal mapping to make the part of boundary of the image to be straight line and then to apply Heinz - Bernstein theorem on coordinate function which is 0 on the the part of boundary of the domain.

In [KaMa3], we used the inner estimate of Heinz-Bernstein. Using inner estimate (cf. Theorem 6.14 [GiTr]) we can extend the result to the case where $\gamma_G$ is $C^2$ Jordan arc.

We say that a bounded domain $\Omega$ in $\mathbb{R}^n$ and its boundary belong to class $C^{k,\alpha}$, $0 \leq \alpha \leq 1$ if for every point $x_0 \in \partial\Omega$ there exists a ball $B = B(x_0)$ and mapping $\psi$ from $B$ onto $D$ such that ([GiTr], p. 95)
$\psi(B \cap \Omega) \subset R^n_+$,
$\psi(B \cap \partial\Omega) \subset \partial R^n_+$,
$\psi \in C^{k,\alpha}(B)$, $\psi^{-1} \in C^{k,\alpha}(D)$.

Let $D$ be domain in $\mathbb{R}^n$ space and $u : D \to \mathbb{R}$. We say that $u$ satisfy Poisson type inequality if

$$|\Delta u(x)| \leq a|\nabla u(x)|^2 + b, \quad x \in D. \tag{2}$$

We study those mappings.

We also point some differences between theory in the plane and the space.
The following result is very useful in space (see also inner estimate (cf. Theorem 6.14 [GiTr]))

**Theorem (Interior estimate 1, Lemma 6.18 [GiTr])**

*Let boundary of $G$ contains a part $T$ of class $C^{2,\alpha}$, $0 \le \alpha \le 1$, and let $g$ belong $C^{2,\alpha}(\overline{G})$. Suppose that $u$ belongs $C^0(\overline{G}) \cap C^2(G)$, $u = g$ on $T$, $Lu = f$ in $G$, where $f$ and coefficients of strongly elliptic operator $L$ belong $C^\alpha(\overline{G})$. Then $u \in C^{2,\alpha}(G \cup T)$.*

A corollary of this can be considered as a local version of Kellogg-Warshawski theorem. Note that this theorem is much more general result.

Interior estimate for Poisson type inequality

**Lemma**

*Let a real function* $\chi : \mathbb{B}^n \to [-1, 1]$ *be of class* $C^2$ *on* $\mathbb{B}^n$, *continuous on* $\overline{\mathbb{B}^n}$ *and satisfy the inequality (2). Suppose in addition that* $\chi$ *is* $C^2$ *(more generally* $C^{1,\alpha}$ *on* $\mathbb{S}^n$*). Then* $\nabla\chi$ *is bounded.*

# Lipschitz property of harmonic maps

Suppose that domains $D$ and $\Omega$ are bounded domains in $\mathbb{R}^n$ and its boundaries belong to class $C^{k,\alpha}$, $0 \leq \alpha \leq 1$, $k \geq 2$. Suppose further that $g$ and $g'$ are $C^1$ metric on $\overline{D}$ and $\overline{\Omega}$ respectively. Using Lemma we prove

**Theorem**

*If $u : D \to \Omega$ a qc $(g, g')$-harmonic map (or satisfies (2)), then $u$ is Lipschitz on $D$.*

The new idea behind the proof is to use local coordinates $\psi$ to make the part of boundary of the image to lay on a hyperplane whose $n-$th coordinate is 0 and then to apply inner estimate on $n-$th coordinate of function $\psi \circ u$, which is 0 on the the part of boundary of the domain.

More precisely, consider $\tilde{u} = \psi \circ u$ and $\tilde{u}^n$.

Note that $\tilde{u}^n$ is 0 on $u^{-1}(B \cap \partial\Omega)$. Define metric $g''$ on $N$, $\overline{\psi(B)} \subset N$ such that $\psi$ is isometry. Then we define the tension field $\tau(\tilde{u})$ of $\tilde{u}$ by coordinates.

In [KaMa2] it is proved the following theorem: a $K$ quasiconformal harmonic mapping of the unit ball $\mathbb{B}^n$ $(n > 2)$ onto itself is Euclidean bi-Lipschitz, providing that $u(0) = 0$ and that $K < 2^{n-1}$, where $n$ is the dimension of the space. It is an extension of a similar result for hyperbolic harmonic mappings with respect to hyperbolic metric (see Tam and Wan, (1998)). The proof makes use of Möbius transformations in the space, and of a recent result which states that, harmonic quasiconformal self-mappings of the unit ball are Lipschitz continuous.

# Some Recent Results

Among other things, in [BoMa] we prove that the above result holds if $K < 3^{n-1}$, and when codomain is only assumed to be convex. A suitable application of normal family argument allows us to take a conceptually simpler approach then in [KaMa2].

Harmonic quasiconformal maps have found applications in Teichmüller theory, among other things. Recently, V. Marković [Ma] proved that a quasiconformal map of the sphere $\mathbb{S}^2$ admits a harmonic quasi-isometric extension to the hyperbolic space $\mathbb{H}^3$, thus confirming the well known Schoen Conjecture in dimension 3. Related questions of bi-Lipschitzity and bounds of Jacobian have been studied in a sequence of papers by Kalaj and Mateljević; see also a recent paper of Iwaniec-Onninen. The corresponding results for harmonic maps between surfaces were obtained previously by Jost and Jost-Karcher.

# Quasiconformal and HQC mappings between Lyapunov Jordan domains

Olli Martio observed that every quasiconformal harmonic mapping of the unit planar disk onto itself is co-Lipschitz. Later, the subject of quasiconformal harmonic mappings was intensively studied by the participants of the Belgrade Analysis Seminar[1]

[1] M. Pavlović, V. Marković, D. Kalaj, V. Božin, M. Arsenović, M. Marković, N. Lakić, D. Šarić, M. Knežević, V. Todorčević, M. Laudanović, M. Svetlik, I. Anić, etc.

As an application of Gehring-Osgood inequality[GeOs] concerning qc mappings and quasi-hyperbolic distances, in the particular case of punctured planes, we prove

### Proposition

*Let $f$ be a $K$-qc mapping of the plane such that $f(0) = 0$, $f(\infty) = \infty$ and $\alpha = K^{-1}$. If $z_1, z_2 \in \mathbb{C}^*$, $|z_1| = |z_2|$ and $\theta \in [0, \pi]$* (respectively $\theta^* \in [0, \pi]$) is the measure of convex angle between $z_1, z_2$ (respectively $f(z_1), f(z_2)$), then

$$\theta^* \le c \max\{\theta^\alpha, \theta\},$$

where $c = c(K)$. In particular, if $\theta \le 1$, then $\theta^* \le c\theta^\alpha$.

We shortly refer to this result as (GeOs-BM).

Through the paper we frequently consider the setting ($\mathbb{U}_{qc}$): Let $h : \mathbb{U} \to D$ be $K$-qc map, where $\mathbb{U}$ is the unit disk and suppose that $D$ is Lyapynov domain. Under this hypothesis, using (GeOs-BM), we prove that for every $a \in \mathbb{T} = \{|z| = 1\}$, there is a special Lyapunov domain $U_a$, of a fixed shape, in the unit disk $\mathbb{U}$ which touches $a$ and a special, convex Lyapunov domain $lyp(D)_b^-$, of a fixed shape, in $D$ such that $lyp(D)_b^- \subset h(U_a) \subset H_b$, where $H_b$ is a half-plane $H_b$, which touches $b = h(a)$. We can regard this result as "good local approximation of qc mapping $h$ by its restriction to a special Lyapunov domain so that codomain is locally convex".

In addition if $h$ is harmonic, using it, we prove that $h$ is co-Lip $\mathbb{U}$:

**Theorem**

*Suppose $h : \mathbb{U} \to D$ is a hqc homeomorphisam, where $D$ is a Lyapanov domain with $C^{1,\mu}$ boundary. Then $h$ is co-Lipschitz.*

It settles an open intriguing problem in the subject and can be regarded as a version of Kellogg - Warschawski theorem for hqc.

# Schoen Conjecture

Harmonic quasiconformal maps have found applications in Teichmüller theory, among other things. Recently, V. Marković proved that a quasiconformal map of the sphere $\mathbb{S}^2$ admits a harmonic quasi-isometric extension to the hyperbolic space $\mathbb{H}^3$, thus confirming the well known Schoen Conjecture in dimension 3 (independently we also have obtained some results of this type).

V. Marković, *Harmonic maps between 3-dimensional hyperbolic spaces*, Inventiones Mathematicae, 199 (3). pp. 921-951.

### Conjecture (Schoen, Li-Wang)

*Suppose that* $f : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ *is a quasiconformal map. Then there exists a unique harmonic and quasi-isometric map* $H : \mathbb{H}^n \to \mathbb{H}^n$ *that extends* $f$.

Let $F : X \to Y$ be a map between two metric spaces $(X, d_X)$ and $(Y, d_Y)$.

We say that $F$ a $(L, A)$-quasi-isometry if there are constants $L > 0$ and $A \geq 0$, such that

$$Ld_Y(F(x), F(y)) - A = d_X(x, y) = Ld_Y(F(x), F(y)) + A,$$

for every $x, y \in X$ (some authors call this a rough isometry but we stick to the name commonly used in hyperbolic geometry).

By $QI(X, Y)$ we denote the set of all quasi-isometries from $X$ to $Y$ and if $X = Y$ then $QI(X, X) = QI(X)$.

Each quasi-isometry $F : \mathbb{H}^3 \to \mathbb{H}^3$ extends continuously to $\mathbb{H}^3$ and the restriction of $F$ to $\mathbb{S}^2$ is a quasiconformal map. Moreover, if $F$ and $G$ are two quasi-isometries of $\mathbb{H}^3$ that have the same boundary values, then the distance $d_{\mathbb{H}^3}(F(x), G(x))$ is bounded on $\mathbb{H}^3$.

The main difficulty in working with harmonic maps between hyperbolic spaces is to control the harmonic map inside the hyperbolic space in terms of the regularity of the boundary map.

It is easy to construct a sequence of diffeomorphisms $f_n : \mathbb{S}^1 \to \mathbb{S}^1$, that converge to the identity in the $C^0$ sense, but such that the corresponding harmonic extensions degenerate on compact sets in $\mathbb{H}^2$ and we can not extract any sort of limit (this behavior is very different for the harmonic functions problem we mentioned before that was solved by Anderson and Sullivan, where the boundary map effectively controls the behavior of the harmonic function inside the disc).

The following theorem takes care of the main difficulty we described above.

## Theorem

*There exist constants* $L = L(K) > 0$ *and* $A = A(K) \geq 0$*, such that if a* $K$*-quasiconformal map* $f : \mathbb{S}^2 \to \mathbb{S}^2$ *has a harmonic quasi-isometric extension* $H(f) : \mathbb{H}^3 \to \mathbb{H}^3$*, then* $H(f)$ *is a* $(L, A)$*-quasi-isometry.*

The following theorem is a corollary of previous theorem.

## Theorem

*Every quasiconformal map of* $\mathbb{S}^2$ *admits a harmonic quasiisometric extension.*

Lemm and Marković develop new methods for studying the convergence problem for the heat flow on negatively curved spaces and prove that any quasiconformal map of the sphere $\mathbb{S}^{n-1}$, $n \geq 3$, can be extended to the $n$-dimensional hyperbolic space such that the heat flow starting with this extension converges to a quasi-isometric harmonic map. This implies the Schoen-Li-Wang conjecture that every quasiconformal map of $\mathbb{S}^{n-1}$, $n \geq 3$, can be extended to a harmonic quasi-isometry of the $n$-dimensional hyperbolic space.

M. Lemm, V. Markovic, *Heat flows on hyperbolic spaces*, arxiv.org/pdf/1506.04345

## Theorem

*For every $K > 1$ there exists a constant $C = C(K)$ with the following properties. We let $g : \mathbb{H}^2 \to \mathbb{H}^2$ be any $K$-quasiconformal homeomorphism and assume that there exists a harmonic quasiconformal homeomorphism $f : \mathbb{H}^2 \to \mathbb{H}^2$ that agrees with $g$ on the boundary $\partial\mathbb{H}^2$. Then $d(h, f) = \sup_{z\in\mathbb{H}^2} d(h(x), f(x)) \le C$, where $d(\cdot,\cdot)$ denotes the hyperbolic distance.*

For $A_K(z) = x + iKy$ we have $d(A_K, I) = \log K$.

By $\sigma(z) = 2(1-|z|^2)^{-1}$ we denote hyperbolic density. A $C^2$ map $f:\mathbb{D}\to\mathbb{D}$ is harmonic if it satisfies the equation

$$f_{z\overline{z}} + 2(\frac{\sigma_w}{\sigma}\circ f) f_z f_{\overline{z}} = 0\,.$$

For a harmonic map $f$ by $Hopf[f] = (\sigma^2\circ f) f_z \overline{f_{\overline{z}}} = (\sigma^2\circ f) f_z \overline{f}_z$ we denote the Hopf Differential. Then $Hopf[f]$ is a holomorphic function on $\mathbb{D}$.

We let $BQD(\mathbb{D})$ the space of holomorphic function $\phi$ on $\mathbb{D}$ (more generally denote the space of bounded holomorphic quadratic functions on $\mathbb{D}$) for which $|\phi|_\infty = \sup \sigma^{-2}(z)|\phi(z)| < \infty$.

# Boundary regularity of gradient and elliptic PDE and Dirichlet Eigenfunctions

In communication with Sinai (April 2016, Princeton) the following question appeared: what can we say about the boundary regularity of Dirichlet Eigenfunctions on bounded domains which are $C^2$ except at a finite number of corners (we address this question as Y. Sinai's question). We have discussed the subject with Lamberti who informed about huge literature related to this subject and in particular about items 1) - 3).

1) the eigenfunctions of the Dirichlet Laplacian are always bounded, not matter what the boundary regularity is.

2) the gradient of the eigenfunctions may not be bounded. The typical situation in the plane is as follows. If you have a corner with angle $\beta$, then the gradient is bounded around it if $\beta \leq \pi$ and unbounded if $\beta > \pi$.

3) An example: if $\Omega$ is a circular sector in the plane with central angle $\beta$, then for all $n \in \mathbb{N}$, $\nabla \varphi_n \in L^\infty(\Omega)$ if $0 < \beta \le \pi$; if $\pi < \beta < 2\pi$ then for all $n \in \mathbb{N}$, $\nabla \varphi_n \in L^p(\Omega)$ for all $1 \le p < \frac{2\beta}{\beta - \pi}$ and there exists an infinite number of eigenfunctions $\varphi_n$ such that $\nabla \varphi_n \notin L^p(\Omega)$ if $p \ge \frac{2\beta}{\beta-\pi}$. This example is discussed in Example 6.2.5 in E.B. Davies, Spectral theory and differential operators, Cambridge University Press, Cambridge, 1995. Note that typically the first eigenfunction is bad, while higher eigenfunctions could be better, depending of the Bessel function involved.

4) Lamberti tends to believe that in a bounded convex domain eigenfunctions have bounded gradients. Indeed, the issue of the boundedness of the gradients is somewhat close to the issue of the $H^2$ regularity, and the classical paper

J. KADLEC, *The regularity of the solution of the Poisson problem in a domain whose boundary is similar to that of a convex domain (Russian)*, Czechoslovak Math. J. 14 (89) 1964, 386-393,

guarantees that in a convex domain the solutions to a Dirichlet problem are in $H^2$.

By the item 2), it seems reasonable that in the case of a triangle the gradient is bounded around it.

It seems to me if we suppose that the Dirichlet eigenfunction solution $w \in W_0^{1,2}(\Omega)$ and $\Omega$ is bounded plane convex domain, then we have that gradient of $w$ is bounded.

Yes, by applying Kozlov and Mazya. Otherwise, without their result (or others'), it is not straightforward.

By $w$ we denote a unique solution to the Dirichlet problem $L(\partial_x)w = f$, $w \in W_0^{m,2}(\Omega)$, where $f \in W^{1-m,q}(\Omega)$ with $q \in (2, \infty)$. Here $W_0^{l,p}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ in the Sobolev space $W^{l,p}(\Omega)$, $1 < p < \infty$, and $W^{-l,p'}(\Omega)$ with $p' = p/(p-1)$ is the dual of $W_0^{l,p}(\Omega)$. The operator $L(\partial_x)$ is strongly elliptic and given by $L(\partial_x) = \sum_{0 \leq k \leq 2m} a_k \partial_1^k \partial_2^{2m-k}$.

We can apply Kozlov-Mazya result:

**Theorem (Theorem 2, [KozMa])**

*Let $u$ be a solution of the Dirichlet problem for elliptic equations of order $2m$ with constant coefficients in an arbitrary bounded plane convex domain $G$. Then $m$-th order derivatives of $u$ are bounded if the coefficients of the equation are real.*

Thus, by application Kozlov-Mazya result we have

**Theorem**

*Suppose that $\Omega$ is bounded plane convex domain. If $w \in W_0^{1,2}(\Omega)$ the Dirichlet eigenfunction solution in $\Omega$, then that gradient of $w$ is bounded.*

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Thank you for attention!