Schwarz lemma, the Carathéodory and Kobayashi metrics and applications in complex analysis

Workshop: The perturbation of the generalized inverses, geometric structures, fixed point theory and applications

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Introduction

We study Schwarz lemma at the boundary of strongly pseudoconvex domains, and versions of now called the Carathéodory-Cartan-Kaup-Wu theorem, which generalizes the classical Schwarz lemma for holomorphic functions to higher dimensions, and iterations of holomorphic mappings.

C. Earle and R. Hamilton, *A fixed point theorem for holomorphic mappings*, Proc. Symp. Pure Math., Vol. XVI, 1968, 61-65.

The Schwarz lemma as one of the most influential results in complex analysis and it has a great impact to the development of several research fields, such as geometric function theory, hyperbolic geometry, complex dynamical systems, and theory of quasi-conformal mappings.

We plan to study Schwarz lemma at the boundary of strongly pseudoconvex domains, and versions of now called the Carathéodory-Cartan-Kaup-Wu theorem, which generalizes the classical Schwarz lemma for holomorphic functions to higher dimensions, and iterations of holomorphic mappings.

Let G be a nonempty domain in a complex Banach space and let a holomorphic function h maps G strictly inside a subset G, then h is a contraction. We proved this probably around 1980 (we found a hand written manuscript 1990 and did not pay much attention to it at that time). But we realized these days that it is a version of the Earle-Hamilton (1968) fixed point theorem, which may be viewed as a holomorphic formulation of Banach's contraction mapping theorem. The result was proved in 1968 (when I enroled Math Faculty in Belgrade) by Clifford Earle and Richard Hamilton by showing that, with respect to the Carathéodory metric on the domain, the holomorphic mapping becomes a contraction mapping to which the Banach fixed-point theorem can be applied.

Geometric interpretation

We use notation $df = pdz + qd\bar{z}$. Let f be $\mathbb{R}$-differentiable at a point $z_0 = r_0 e^{it_0}$, $r_0 = |z_0|$, $\epsilon_0 > 0$, $w_0 = f(z_0)$, $R_0 = |w_0|$ and L be a circular arc defined by $L(t) = r_0 e^{it}$, $|z_0| \leq \epsilon_0$. Since $L'(t) = ir_0 e^{it}$ and in particular $L'(t_0) = iz_0$, the tangent vector T of the curve $f \circ L$ at $f(z_0)$ equals $T = df(L'(t)) = df(iz_0)$.

If $f(L) \subset B_{R_0}$, then from obvious geometric interpretation we have $T = \lambda i f(z_0)$, $\lambda \in \mathbb{R}$. Since $T = df(iz_0)$, we find $i(pz_0 - q\overline{z_0}) = \lambda i f(z_0)$, $\lambda \in \mathbb{R}$, where $p = f_z(z_0)$ and $q = f_{\bar{z}}(z_0)$, and therefore we get

Proposition

Under the above hypothesis, $(pz_0 - q\overline{z_0}) = \lambda f(z_0)$, $\lambda \in \mathbb{R}$. *If* $|p| > |q|$, *then* $\lambda > 0$.

Lemma (Jack)

Suppose that U *is the unit disk,* f *analytic on* $\overline{\mathbb{U}}$, $\mathbb{U}$ *is invariant under the mapping* f *and* f *has a fix point* z_0 *on the boundary of* $\mathbb{U}$. *Then* $f'(z_0)$ *is positive.*

The subject related to Jack's lemma has also been discussed by Örnek in a recent paper. It seems that the following result is true. By $S^*(\alpha)$ we denote the family of starlike univalent functions of of order α. Note that $f \in S^*(\alpha)$ if $\mathrm{Re}\frac{zf'(z)}{f(z)} > \alpha$.

Theorem

If f *belongs* $S^*(\alpha)$, $0 < \alpha < 1$, *and* $1/\beta = 2(1-\alpha) = s(\alpha)$, *then*

(i) $|f(z)| \leq \dfrac{|z|}{(1-|z|)^{1/\beta}}$;

(ii) $|f''(0)| \leq 2/\beta$.

B.Örnek, *Estimates for holomorphic functions concerned with Jack's lemma* (manuscript no. 15156 Publ. Inst. Math.)

Suppose f is an analytic map of $\mathbb{U}$ into itself. If $|b| < 1$, we say b is a fixed point of f if $f(b) = b$. If $|b| = 1$, we say b is a fixed point of f if $\lim\limits_{r \to 1_-} f(rb) = b$.

Julia-Carathéodory Theorem implies that if b is a fixed point of f with $|b| = 1$, then $\lim\limits_{r \to 1_-} f'(rb)$ exists (call it $f'(b)$) and $0 < f'(b) \leq \infty$.

Theorem (Denjoy-Wolff (1926))

(a) *If f is an analytic map of $\mathbb{U}$ into itself, not the identity map, there is a unique fixed point, a, of f in $\overline{\mathbb{U}}$ such that $|f'(a)| \leq 1$.*

(b) *If f is not an automorphism of $\mathbb{U}$ (i.e. a Möbius transformation) with fixed point in $\mathbb{U}$, iterates of f tend to a uniformly on compact subsets of $\mathbb{U}$.*

This distinguished fixed point will be called the Denjoy-Wolff point of f.

The Schwarz-Pick Lemma implies f has at most one fixed point in $\mathbb{U}$ and if f has a fixed point in $\mathbb{U}$, it must be the Denjoy-Wolff point.
C. C. Cowen, *Iteration and the Solution of Functional Equations for Functions Analytic in the Unit Disk* , Trans. Amer. Math. Soc. 265 (1981) 69-95.

Question

Is there a version of this result for quasi-regular mappings?

We prove Cartan uniqueness theorem (strongly convex pluriharmonic version)

Theorem (Cartan uniqueness theorem)

Let D be a bounded domain in C^n and given $a \in D$. If $f \in Aut_a(D)$ satisfies $f'(a) = 1$, then $f(z) = z$ for all $z \in D$.

If $f \in Aut(B)$ fixes a point of B, then fix point set of f is affine. Conversely, we have

Theorem (Hayden-Suffridge)

If $f \in Aut(B)$ fixes three point of $\mathbb{S}$, then f fixes a point of $\mathbb{B}$.

Theorem

Suppose that G *is bounded connected open subset of complex Banach space and* $f : G \to G_*$ *is holomorphic,* $G_* \subset G$*,* $s_0 = dist(G_*, G^c)$*,* $d_0 = diam(G)$ *and* $q_0 = \frac{d_0}{d_0+s_0}$*. Then* $k_{G_*}(fz, fz_1) \leq q_0 k_{G_*}(z, z_1)$ *for* $z, z_1 \in G_*$*.*

Hence we can derive Earle-Hamilton theorem.

Schwarz lemma at the boundary of strongly pseudoconvex domain

Example

Recall in complex dimension 2 we use notation
$\varphi_a(z,w) = (T_a(z), \lambda(z)w)$, *where* $\lambda(z) = \frac{s_a}{1-\overline{a}z}$. *If* $(a,0) \in B^2$,
$J_{\varphi_a}(z,w) = ((1-|a|^2)(1-\overline{a}z)^{-2}, 0; \lambda'(z)w, \lambda(z))$.
Set $v = (1,0)$ *and* $A = J_{\varphi_a}(1,0)$. *Then* $A = (\lambda, 0; 0, \mu)$, *where*
$\lambda = (1-|a|^2)(1-\overline{a})^{-2}$ *and* $\mu = s_a(1-\overline{a})^{-1}$; *so that* $\lambda = |\mu|^2$. *If*
$-1 < a < 1$, φ_a *has two fp* $\pm(1,0)$, $\lambda = \frac{1+a}{1-a}$ *and therefore* $\lambda > 1$ *if*
$0 < a < 1$ *and* $\lambda < 1$ *if* $-1 < a < 0$.

The next result is a generalization of previous example.

Theorem (Theorem 1.1, Wang and Ren)

Let $G \subset C^n$ be a bounded strongly pseudoconvex domain with C^2 boundary and $f : G \to G$ a holomorphic mapping. Suppose that f extends smoothly past some point $p \in \partial G$ and $f(p) = p$. Then for the eigenvalues $\lambda, \mu_2, \dots, \mu_n$ (counted with multiplicities) of $J_f(p)$, the following statements hold:

(i) *λ is positive and is also an eigenvalue of $\overline{J_f(p)}^t$ such that $\overline{J_f(p)}^t \upsilon_p = \lambda \upsilon_p$, where υ_p is the unit outward normal vector to ∂G at the point p;*

(ii) *$\mu_j \in C$ and $|\mu_j| \leq \sqrt{\lambda}$ for $j = 2, 3, \dots, j_n$;*

(iii) *For $j = 2, 3, \dots, j_n$, there exists $\tau_j \in T_p^{(1,0)}$ such that $J_f(p)\tau_j = \mu_j \tau_j$;*

(iv) *$|det J_f(p)| \leq \lambda^{(n+1)/2}$, $|tr J_f(p)| \leq \lambda + (n-1)\sqrt{\lambda}$;*

Moreover, if in addition f has an interior fixed point $z_0 \in G$, then $\lambda \geq 1$.