HYPERSURFACES IN SPACE FORMS SATISFYING SOME CURVATURE CONDITIONS

Dedicated to the memory of Professor Mileva Prvanović

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Some tensors

(M, g) - an n-dimensional, $n \geqslant 4$, semi-Riemannian connected paracompact manifold of class C^∞

∇ - the Levi-Civita connection of (M, g)

R - the Riemann-Christoffel curvature tensor of (M, g)

S - the Ricci tensor of (M, g)

κ - the scalar curvature tensor of (M, g)

C - the Weyl conformal curvature tensor (M, g)

Some subsets of semi-Riemannian manifolds

(M, g) - a semi-Riemannian manifold, $n = \dim M \geqslant 4$.

κ - the scalar curvature tensor of (M, g)

$$\mathcal{U}_R = \{x \in M \,|\, R \neq \tfrac{\kappa}{(n-1)n} G \text{ at } x\}$$

$$\mathcal{U}_S = \{x \in M \,|\, S \neq \tfrac{\kappa}{n} g \text{ at } x\}$$

$$\mathcal{U}_C = \{x \in M \,|\, C \neq 0 \text{ at } x\}$$

We note that $\mathcal{U}_S \cup \mathcal{U}_C = \mathcal{U}_R$ and $G = \frac{1}{2} g \wedge g$.

Some classes of semi-Riemannian manifolds, $n \geqslant 4$

Spaces of constant curvature, $R = \frac{\kappa}{(n-1)n} G$;
Einstein spaces, $S = \frac{\kappa}{n} g$;
conformally flat manifolds, $C = 0$;

locally symmetric spaces, $\nabla R = 0$;
Ricci-symmetric manifolds, $\nabla S = 0$;
conformally symmetric spaces, $\nabla C = 0$;

semi-symmetric manifolds, $R \cdot R = 0$;
Ricci-semi-symmetric manifolds, $R \cdot S = 0$;
Weyl-semi-symmetric manifolds, $R \cdot C = 0$;

pseudosymmetric manifolds, $R \cdot R = L_R \, Q(g, R)$;
Ricci-pseudosymmetric manifolds, $R \cdot S = L_S \, Q(g, S)$;
Weyl-pseudosymmetric manifolds, $R \cdot C = L_S \, Q(g, C)$.

The main classes of pseudosymmetry type manifolds

$R \cdot S = L_S Q(g, S)$	$\supset$	$R \cdot R = L_R Q(g, R)$	$\subset$	$R \cdot C = L_C Q(g, C)$
$\cup$		$\cup$		$\cup$
$R \cdot S = 0$	$\supset$	$R \cdot R = 0$	$\subset$	$R \cdot C = 0$
$\cup$		$\cup$		$\cup$
$\nabla S = 0$	$\supset$	$\nabla R = 0$	$\subset$	$\nabla C = 0$
$\cup$		$\cup$		$\cup$
$S = \frac{\kappa}{n} g$	$\supset$	$R = \frac{\kappa}{(n-1)n} G$	$\subset$	$C = 0$

Basic definitions

(M, g) - a hypersurface isometrically immersed in a semi-Riemannian space of constant curvature $N_s^{n+1}(c)$ with signature $(s, n+1-s)$, $n \geqslant 4$,

$c = \frac{\widetilde{\kappa}}{n(n+1)}$, $\widetilde{\kappa}$ - the scalar curvature of the ambient space

the Gauss equation:

$$R_{hijk} = \frac{1}{2}\varepsilon\,(H \wedge H)_{hijk} + \frac{\widetilde{\kappa}}{n(n+1)}\,G_{hijk}\,, \quad \varepsilon = \pm 1 \tag{1}$$

$$\Downarrow$$

$$S_{hk} = \varepsilon\left(\mathrm{tr}(H)\,H_{hk} - H^2_{hk}\right) + \frac{(n-1)\widetilde{\kappa}}{n(n+1)}\,g_{hk} \tag{2}$$

$$\Downarrow$$

$$\kappa = \varepsilon\left((\mathrm{tr}(H))^2 - \mathrm{tr}(H^2)\right) + \frac{(n-1)\widetilde{\kappa}}{n+1} \tag{3}$$

H - the second fundamental tensor

Basic definitions

We also have

$$\begin{aligned} C_{hijk} &= \mu_1 \, G_{hijk} + \frac{1}{2} \varepsilon \, (H \wedge H)_{hijk} \\ &\quad - \frac{\varepsilon}{n-2} \, (g \wedge \left(\mathrm{tr}(H) \, H - H^2\right))_{hijk} \, , \quad \varepsilon \; = \; \pm 1 \end{aligned} \tag{4}$$

$$\mu_1 = \frac{1}{n-2} \left(\frac{\kappa}{n-1} - \frac{\widetilde{\kappa}}{n+1} \right)$$

(M, g) - a hypersurface immersed isometrically in a semi-Riemannian space of constant curvature $N_s^{n+1}(c)$ with signature $(s, n+1-s)$, $n \geqslant 4$,

$c = \frac{\widetilde{\kappa}}{n(n+1)}$, $\widetilde{\kappa}$ - the scalar curvature of the ambient space

I will consider on $\mathcal{U}_S \cap \mathcal{U}_C \subset M$ the condition

$$R \cdot C - C \cdot R = L\, Q(S, C)\,, \qquad (*)$$

where L is some function on $\mathcal{U}_S \cap \mathcal{U}_C$.

I will consider two cases:
I. $x \in (\mathcal{U}_S \cap \mathcal{U}_C) \setminus \mathcal{U}_H \subset M$
II. $x \in \mathcal{U}_H \subset (\mathcal{U}_S \cap \mathcal{U}_C) \subset M$

$$\mathcal{U}_H = \{x \in M \,:\, H^2 \neq \alpha\, H + \beta\, g \text{ at } x\,, \quad \alpha, \beta \in \mathbb{R}\}$$

I. $x \in (\mathcal{U}_S \cap \mathcal{U}_C) \setminus \mathcal{U}_H \subset M$

Theorem 1. [G, 2005]
Let M be a hypersurface in $N_s^{n+1}(c)$, $n \geqslant 4$, and let $(\mathcal{U}_S \cap \mathcal{U}_C) \setminus \mathcal{U}_H$ be a nonempty set. Then the Roter type equation

$$R = \frac{1}{2}\phi\, S \wedge S + \mu\, g \wedge S + \eta\, G$$

holds on this set.

Proposition 2. [DGJZ, 2016]
If (M, g), $n \geqslant 4$, is a semi-Riemannian manifold satisfying

$$R = \frac{1}{2}\phi\, S \wedge S + \mu\, g \wedge S + \eta\, G$$

on $\mathcal{U}_S \cap \mathcal{U}_C \subset M$ then on this set we have

$$R \cdot C - C \cdot R = -Q(S, C) + \frac{\kappa}{n-1}\, Q(g, C)\, .$$

I. $x \in (\mathcal{U}_S \cap \mathcal{U}_C) \setminus \mathcal{U}_H \subset M$

Theorem 3. [DGHZ, 2016]
If the scalar curvature κ of a hypersurface M in $N_s^{n+1}(c)$, $n \geqslant 4$, vanishes on $(\mathcal{U}_S \cap \mathcal{U}_C) \setminus \mathcal{U}_H \subset M$ then

$$R \cdot C - C \cdot R \;=\; -Q(S, C)$$

holds on this set.

Corollary 4. [DGHZ, 2016]
Let M be a hypersurface in a Riemannian space of constant curvature $N^{n+1}(c)$, $n \geqslant 4$, having at every point exactly two distinct principal curvatures. If the scalar curvature κ of M vanishes on $(\mathcal{U}_S \cap \mathcal{U}_C) \setminus \mathcal{U}_H \subset M$ then

$$R \cdot C - C \cdot R \;=\; -Q(S, C)$$

holds on this set.

II. $x \in \mathcal{U}_H \subset \mathcal{U}_S \cap \mathcal{U}_C \subset M$

Proposition 5. [DGHZ, 2016]
If M is a hypersurface in $N_s^{n+1}(c)$, $n \geqslant 4$, satisfying

$$R \cdot C - C \cdot R \;=\; L\,Q(S,C) \qquad (*)$$

on $\mathcal{U}_H \subset M$, then on this set we have

$$H^3 \;=\; \mathrm{tr}(H)\,H^2 + \psi\,H + \rho\,g \qquad (5)$$

where ψ and ρ are some functions on $\mathcal{U}_H$.
In particular, at all points of $\mathcal{U}_H$ at which $L \neq -1$ we have

$$R \cdot S \;=\; \frac{\widetilde{\kappa}}{n(n+1)}\,Q(g,S)\,, \qquad (6)$$

$$C \cdot S \;=\; 0\,, \qquad (7)$$

$$R \cdot C - C \cdot R \;=\; \frac{1}{n-2}\,Q(S,C) - \frac{\widetilde{\kappa}}{(n-2)n(n+1)}\,Q(g,R)\,. \qquad (8)$$

II. $x \in \mathcal{U}_H \subset \mathcal{U}_S \cap \mathcal{U}_C \subset M$

Theorem 6. [DGHZ, 2016]
If M is a hypersurface in $N_s^{n+1}(c)$, $n \geqslant 4$, satisfying on $\mathcal{U}_H \subset M$

$$H^3 \;=\; \mathrm{tr}(H)\, H^2 + \psi\, H \quad \text{and} \quad \mathrm{rank}(S - \alpha\, g) \;=\; 1$$

then on this set we have

$$\alpha \;=\; \frac{\kappa}{n-1} - \frac{\widetilde{\kappa}}{n(n+1)}$$

and

$$R \cdot C - C \cdot R \;=\; \frac{1}{n-2}\, Q(S, C) - \frac{\widetilde{\kappa}}{(n-2)n(n+1)}\, Q(g, R)\,.$$

In particular, if the ambient space is a semi-Euclidean space $\mathbb{R}_s^{n+1}$, $n \geqslant 4$, then

$$\alpha \;=\; \frac{\kappa}{n-1} \quad \text{and} \quad R \cdot C - C \cdot R \;=\; \frac{1}{n-2}\, Q(S, C)$$

hold on $\mathcal{U}_H \subset M$.

II. $x \in \mathcal{U}_H \subset \mathcal{U}_S \cap \mathcal{U}_C \subset M$

Theorem 7. [DGHZ, 2016]
If M is a hypersurface in $N_s^{n+1}(c)$, $n \geqslant 4$, satisfying on $\mathcal{U}_H \subset M$

$$R \cdot C - C \cdot R \;=\; -Q(S, C)$$

then on this set we have $\kappa \;=\; 0$ and

$$R \cdot C - C \cdot R \;=\; Q(g, T)\,,$$

$$\begin{aligned} T \;=\; & \left(\frac{\widetilde{\kappa}}{n(n+1)} - \frac{\varepsilon \psi}{n-1} \right) R - \frac{1}{2(n-1)(n-2)}\, S \wedge S \\ & - \frac{1}{n-1} g \wedge S^2 + \left(\frac{2\widetilde{\kappa}}{n(n+1)} - \frac{\varepsilon \psi}{n-1} \right) g \wedge S\,, \end{aligned}$$

where ε and ψ are defined by (1) and (5), respectively, i.e. by

$$\begin{aligned} R_{hijk} \;&=\; \frac{1}{2} \varepsilon\, (H \wedge H)_{hijk} + \frac{\widetilde{\kappa}}{n(n+1)}\, G_{hijk}\,, \quad \varepsilon \;=\; \pm 1\,, \\ H^3 \;&=\; \mathrm{tr}(H)\, H^2 + \psi\, H + \rho\, g\,. \end{aligned}$$

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