Geometry of loop spaces

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Introduction

- M: compact, connected orientable C^∞-manifold.
- Diff(M): the diffeomorphism group of M

Question: What can we say about the topology of Diff(M) ?

This is a difficult question!

Some known results

A few results are known:

- $\mathrm{Diff}(S^1) \sim O(2)$
- $\mathrm{Diff}(S^2) \sim O(3)$ (Smale)
- $\mathrm{Diff}(S^3) \sim O(4)$ (Hatcher)
- $\mathrm{Diff}(S^1 \times S^2) \sim O(2) \times O(3) \times \Omega SO(3)$ (Hatcher)

There are some results on the fundamental groups for diffeomorphism groups of surfaces and 3-manifolds.

Fundamental groups of diffeomorphism groups

We will consider the following question:

Question: When M has an S^1-action, is $\pi_1(\text{Diff(M)}, \text{Id})$ infinite?

Note that a circle action gives a loop of diffeomorphisms of M.

We consider the case $dim(M) = 5$

Let $(\underline{M}^4, \underline{J}, \underline{g}, \underline{\omega})$ be a compact Kaehler surface with an integral Kaehler form $\underline{\omega} \in H^2(\underline{M}^4, \mathbb{Z})$, and let $\overline{M}_k$ be the circle bundle associated to $k[\underline{\omega}] \in H^2(\underline{M}^4, \mathbb{Z})$ for $k \in \mathbb{Z}$.

Results (1)

We set

$$|\underline{R}|_\infty = max\{|\underline{R}(e_i, e_j, e_k, e_\ell)||\{e_1, e_2, e_3, e_4\} \text{ is orthonormal basis}\}$$

where $\underline{R}$ is the Riemannian curvature of $\underline{M}^4$.

Theorem A: Let $(\underline{M}^4, \underline{J}, \underline{g}, \underline{\omega})$ be as above. Assume that for $k \in \mathbb{Z} - \{0\}$,

$$|\underline{R}|_\infty < \frac{6}{7}k^2 + \frac{3\pi\sigma(\underline{M}^4)}{7vol(\underline{M}^4)k^2}$$

where $\sigma(\underline{M}^4)$ is the signature of $\underline{M}^4$. Then $\pi_1(\mathrm{Diff}(\overline{\mathrm{M}}_\mathrm{k}))$ is infinite. More precisely, the loop of diffeomorphisms of $\overline{M}_k$ given by rotation in the circle fiber give an element of infinite order in $\pi_1(\mathrm{Diff}(\overline{\mathrm{M}}_\mathrm{k}), \mathrm{id})$.

Results (2)

In particular, we have

Corollary B: Theorem A holds for the following cases:

- $k >> 0$
- $M = T^4$ and $k \neq 0$

Tools for proving Theorem A

- Geometry of Loop Space
- Psuedo-differential algebra bundles for loop spaces
- Wodzicki-Chern-Simons classes

Geometry of Loop Space

Let

- (M, g): a compact oriented smooth Riemannian manifold.
- LM: loop space of M=$\{\gamma : S^1 \to M, \gamma \in C^\infty\}$
- For s, we consider the Sobolev space $H^s(S^1; TM)$ as a model for $T_\gamma LM$; $X \in H^s(S^1; TM)$ with $X(\theta) \in \gamma(\theta)$ is a vector field along γ.

The inner product on $H^s(S^1; TM)$ is given by

$$\langle X, Y \rangle_s = \frac{1}{2\pi} \int_0^{2\pi} \langle (1 + \Delta)^s X(\theta), Y(\theta) \rangle_{\gamma(\theta)} d\theta$$

where

- $\langle . , . \rangle$ is the inner product on (M, g)
- $\Delta = D^* D$, $D = \frac{D}{d\theta}$, $\quad D^*$: formal adjoint of D.

H^s-connection of loop space (1)

We consider the Levi-Civita connenction defined by

$$
\begin{aligned}
&2\langle {\nabla^s}_X Y, Z\rangle_s \\
&\quad = X\langle Y, Z\rangle_s + Y\langle X, Z\rangle_s - Z\langle X, Y\rangle_s \\
&\qquad +\langle [X, Y], Z\rangle_s + \langle [Z, X], Y\rangle_s - \langle [Y, Z], X\rangle_s
\end{aligned}
$$

Proposition 1:

For s=0:

$$\nabla^0_X Y = D_X Y = \delta_X Y + \Gamma_X(Y)$$

where $\delta_X Y$ is the variation of Y in X direction and Γ is the Christoffel symbols of the Riemannian metric of (M, g).

H^s-connection of loop space (2)

Proposition 2:

- For $s = 1$:

$$
\begin{aligned}
\nabla^1_X Y \;=\; & D_X Y + \frac{1}{2}(1+\Delta)^{-1} \\
& [-\nabla_{\dot\gamma}(R(X,\dot\gamma))Y - R(X,\dot\gamma)\nabla_{\dot\gamma}Y - \nabla_{\dot\gamma}(R(Y,\dot\gamma)X) \\
& -R(Y,\dot\gamma)\nabla_{\dot\gamma}X + R(X,\nabla_{\dot\gamma}Y)\dot\gamma Y)\dot\gamma - R(\nabla_{\dot\gamma}X,Y)\dot\gamma]
\end{aligned}
$$

  where R is the curvature tensor of (M,g)

- We can give general formulas for any $s > \frac{1}{2}$

- The connection 1-form ω^s_X is taken values in zeroth order ΨDOs acting on Y

Remarks on ΨDOs (1)

A very brief explanation of ΨDOs.

A differential operator of order m,

$$Pf(x) = \sum_{|\alpha|=k=0}^{m} a(x)_\alpha (\partial_x)^\alpha f(x),$$

can be written using Fourier transform and Fourier inversion as

$$Pf(x) = \int\int a(x,\xi) e^{<x-y|\xi>} f(y) dy d\xi$$

where $a(x,\xi) = \sum_{k=|\alpha|=0}^{m} a(x)_\alpha (i\xi)^\alpha$.

Remarks on ΨDOs (2)

Definition

- A **ΨDO** P of order m is defined by the above integral transformation using a more general function $a(x, \xi)$.
- $a(x, \xi)$ must have the following asymptotic expansion:

$$a(x, \xi) \sim a_m(x, \xi) + a_{m-1}(x, \xi) + \cdots$$

  where $a_k(x, \xi)$ is homogeneous of degree k in ξ

- $a_k(x, \xi)$ is called the **symbol** of $a(x, \xi)$ of order k and is denoted by $\sigma_k(a)$.

H^s-curvature of Loop space

We compute the curvature form Ω^s of ∇^s:

$$\Omega^s(X,Y) = \nabla^s_X \nabla^s_Y - \nabla^s_Y \nabla^s_X - \nabla^s_{[X,Y]}$$

Proposition 3:

- s=0: $\Omega^0(X,Y) = R(X,Y)$ (the curvature on (M,g))
- $s = 1$: $\Omega^1(X,Y)$ is a ΨDO of order zero.

$$\begin{aligned} \sigma_0(\Omega^1(X,Y)) &= R(X,Y) \\ \sigma_{-1}(\Omega^s(X,Y)) &= P_{-1}(R,\nabla R) \end{aligned}$$

where $P_{-1}(R,\nabla R)$ is a polynomial in R and ∇R.

Chern Classes and Chern-Simons Classes

We review Chern-Weil theory and Chern-Simons theory in finite dimensions:

- G: finite dimensional Lie group with Lie algebra $\mathfrak{g}$
- $\mathfrak{g}^k = \mathfrak{g}^{\otimes k}$
- $F \to M$: a principal G-bundle over M
- θ: connection on F
- Ω_F: curvature of θ
- $I^k(G) = \{P : \mathfrak{g}^k \to \mathbb{C} | P :$ symmetric, multilinear, Ad-invariant$\}$ (degree k Ad-invariant polynomial)

Chern Classes

Chern-Weil Theory

$$\begin{aligned}(P \in I^k(G), \Omega_F \in \Lambda^2(F, \mathfrak{g})) \quad &\longrightarrow P \circ \Omega_F \in \Lambda^{2k}(F) \\ &\longrightarrow [P(\Omega_M)] \in \Lambda^{2k}(M)\end{aligned}$$

Chern-Weil theory for $U(n)$-bundles

The special case of $G{=}U(n)$:

- Consider the invariant polynomial defined by $P(A) = \text{trace}(A^k)$, where $A \in u(n)$. The corresponding cohomology class is the k^{th} Chern class.

- For a $U(n)$-connection on a $U(n)$-bundle, the curvature form Ω takes values in the Lie algebra $u(n)$.

Chern-Simons Classes (for 2 connections)

- θ_0, θ_1: connections on F
- For $P \in I^k(G)$,

$$P(\Omega_0) - P(\Omega_1) = dCS_P(\theta_0, \theta_1)$$

where

$$CS_P(\theta_0, \theta_1) = \int_0^1 P(\theta_0 - \theta_1, \Omega_t, \cdots, \Omega_t)dt$$

$\theta_t = t\theta_0 + (1-t)\theta_1$, $\Omega_t = d\theta_t + \theta_t \wedge \theta_t$.

Remark: These theories extend to associated vector bundles.

Chern-Simons Classes for ΨDO bundles

We extend Chern-Simons classes to an infinite dimensional setting (which includes the case of loop spaces).

- We consider Chern-Weil and Chern-Simons theory for the infinite dimensional Lie group ΨDO_0^*
- Main example: The Levi-Civita connection 1-forms for the metrics on loop space take values in $\Psi DO_{\leq 0}$, the Lie algebra of ΨDO_0^*
- We use the "Wodzicki residue" to produce invariant polynomials on $\Psi DO_{\leq 0}$

Wodzicki residue

- P: ΨDO of order zero acting on (sections of a bundle over) a compact manifold N with symbol $\sigma^P(x,\xi)$ and asymptotic expansion

$$\sigma^P(x,\xi) \sim \sum_{k=0} \sigma^P_{-k}(x,\xi)$$

where $\sigma^P_{-k}(x,\xi)$ is homogeneous of order $-k$.

- The Wodzicki residue of P is defined by

$$\mathrm{res}^w(P) = \frac{1}{(2\pi)^n} \int_{S^*N} \mathrm{tr}(\sigma^P_{-n}(x,\xi))\, d\xi dx$$

where S^*N is the unit cosphere bundle over N.

WCS class for loop space

Definition: The k-th WCS class for the Levi-Civita connections ∇^0, ∇^1 on the loop space LM is defined by

$$CS^W_{2k-1}(\nabla_1, \nabla_0) = \frac{1}{k!} \int_0^1 \int_{S^*S^1} \mathrm{tr}[\sigma_{-1}((\omega_1 - \omega_0) \wedge \Omega_t^{k-1})]dt$$

where $\Omega_t = d\omega_t + \omega_t \wedge \omega_t$, $\omega_t = t\omega_0 + (1-t)\omega_1$.

Application of WCS class to circle actions

Let M be a compact oriented manifold.

Let $a_0, a_1 : S^1 \times M \longrightarrow M$ be smooth actions.:

Definition:

- a_0 and a_1 are smoothly homotopic if there exists a C^∞ map $F : [0,1] \times S^1 \times M \longrightarrow M$ such that $F(0,\theta,m) = a_0(\theta,m)$ and $F(1,\theta,m) = a_1(\theta,m)$.

- a_0 and a_1 are smoothly homotopic through actions if $F(t,\cdot,\cdot)$ is an action for all t.

  Example: standard rotation actions on S^2 are smoothly homotopic.

Remark on circle actions

We can rewrite an action $a : S^1 \times M \longrightarrow M$ in the following two ways:

- a determines $a^D : S^1 \longrightarrow \text{Diff(M)}$ by

$$a^D(\theta)(m) = a(\theta, m).$$

  Since $a^D(0) = \text{id}$, we have $[a^D] \in \pi_1(\text{Diff(M)}, \text{id})$.

- a determines $a^L : M \longrightarrow LM$ given by

$$a^L(m)(\theta) = a(\theta, m)$$

  This determines a class $[a^L] \in H_n(LM, \mathbb{Z})$ ($n = \text{dim } M$) by setting $[a^L] = a^L_*[M]$.

Some properties

Proposition 4:

Let dim $M = 2k-1$. Let $a_0, a_1 : S^1 \times M \to M$ be two smooth actions.

(1) If $\int_{[a_0^L]} CS^W_{2k-1} \neq \int_{[a_1^L]} CS^W_{2k-1}$, then a_0, a_1 are not smoothly homotopic through actions. Moreover, $[a_0^D] \neq [a_1^D] \in \pi_1(\text{Diff(M)}, \text{id})$

(2) If $\int_{[a_1^L]} CS^W_{2k-1} \neq 0$, then $\pi_1(\text{Diff(M)}, \text{id})$ is infinite.

Proof of Proposition 4, (2)

Let a_n be the n-th iterate of a_1, i.e. $a_n(\theta, m) = a_1(n\theta, m)$.

Then,

$$CS^W_{2k-1} = \int_{S^1} \dot{\gamma}(\theta) f(\theta) d\theta$$

where $f(\theta)$ is periodic. Each loop $\gamma \in a^L_1(M)$ corresponds to the loop $\gamma(n\cdot) \in a^L_n$. Therefore, $\int_{S^1} \dot{\gamma}(\theta) f(\theta) d\theta$ is replaced by

$$\int \frac{d}{d\theta} \gamma(n\theta) f(n\theta) = n \int_0^{2\pi} \dot{\gamma}(\theta) f(\theta) d\theta$$

Thus,

$$\int_{[a^L_n]} CS^w_{2k-1} = n \int_{[a^L_1]} CS^w_{2k-1}$$

Therefore, each $[a^L_n] \in \pi_1(\mathrm{Diff(M)}, \mathrm{id})$ is distinct.

$\square$

Proof of Theorem A:

We recall the situation for Theorem A: (with slightly different notation)

We consider the case dim $M = 5$

- (M, J, g, ω) is a compact Kaehler surface with integral Kaehler form $\omega \in H^2(M, \mathbb{Z})$.
- As in geometric quantization, we construct the S^1-bundle with connection $\bar{\eta} \in \Lambda^1(L^k)$
  $$L^k \to M$$
  with curvature $\Omega_k = k\omega$.
- Let $\overline{M}_k$ be the total space of L_k.

Sasakian Structure

$\overline{M}_k$ has a Sasakian structure, i.e. there is a geometry on $\overline{M}_k$ with many special features:

- The horizontal space of the connection is $\mathcal{H} = \text{Ker}(\bar{\eta})$
- For a vertical vector $\bar{\xi}$ satisfying $\bar{\eta}(\bar{\xi}) = 1$, we have $d\bar{\eta}(\bar{\xi}, \cdot) = 0$
- $\bar{\xi}$ is the characteristic vector field of the fibration $\pi : \overline{M}_k \to M$
- $\bar{\Phi}(\bar{X}_p) = (J[\pi_* \bar{X}]_{\pi(p)})^L$ has $\bar{\Phi}^2 = -\text{id}$ on horizontal vectors; $\bar{\Phi} = 0$ on vertical vectors.
- etc.

Define a metric $\bar{g}$ on $\overline{M}_k$ by

$$\bar{g}(\bar{X}, \bar{Y}) = g(\pi_* \bar{X}, \pi_* \bar{Y}) + \bar{\eta}(\bar{X})\bar{\eta}(\bar{Y})$$

Then $\pi : \overline{M}_k \to M$ is a Riemannian submersion.

Formulas

Let $\bar{X} = \bar{X}^H + \bar{X}^V$ be the decomposition of $\bar{X} \in T\overline{M}_k$ into horizontal and vertical components for the Levi-Civita connection $\overline{\nabla}$ associated to the metric $\bar{g}$. For $X \in TM$, let X^L be its horizontal lift.

Lemma:

$$\overline{\nabla}_{X^L} Y^L = (\nabla_X Y)^L + kg(JX, Y)\bar{\xi}, \quad \overline{\nabla}_{X^L}\bar{\xi} = -k(JX)^L$$

Proposition:

$$\begin{aligned}
\bar{g}(\bar{R}(X^L, Y^L)Z^L, W^L) &= R(X,Y,Z,W) + k^2[-\langle JY, Z\rangle\langle JX, W\rangle \\
&\quad + \langle JX, Z\rangle\langle JY, W\rangle + 2\langle JX, Y\rangle\langle JZ, W\rangle] \\
\bar{g}(\bar{R}(X^L, Y^L, Z^L, \bar{\xi}) &= 0 \\
\bar{g}(\bar{R}(\bar{\xi}, X^L, Y^L, \bar{\xi}) &= k^2\langle X, Y\rangle
\end{aligned}$$

where $\bar{R}$ and R are the curvature tensor for $\overline{\nabla}$ and ∇, respectively.

Computation of the WCS form

We now compute CS_5^W on $\overline{M}_k$, using the formula for $\overline{R}$.

Let γ be a fiber. Set $\dot{\gamma} = \bar{\xi}$ and let $\{\bar{e}_i\}$ be an orthonormal frame of $T\bar{M}_k$ with $\bar{\xi} = \bar{e}_1$. We can take $\bar{e}_i = e_i^L$ for $i = 2, 3, 4, 5$.

We can write down the WCS formula for the case dim $M = 4$: after some calculations of $\sigma^{\overline{R}}(x, \xi)$,

$$
\begin{aligned}
&CS^W_{5,\gamma}(\bar{e}_1, \bar{e}_2, \bar{e}_3, \bar{e}_4, \bar{e}_5) \\
&\quad = \frac{1}{30} \sum_{\sigma} sgn(\sigma) \bar{R}(\bar{e}_{\sigma(1)}, \bar{e}_\ell, \bar{\xi}, \bar{e}_n) \bar{R}(\bar{e}_{\sigma(2)}, \bar{e}_{\sigma(3)}, \bar{e}_r, \bar{e}_\ell) \bar{R}(\bar{e}_{\sigma(4)}, \bar{e}_{\sigma(5)}, \bar{e}_n, \bar{e}_r)
\end{aligned}
$$

WCS formula

Now we take the orthonormal basis $\{e_2, Je_2, e_3, Je_3\}$ for M.

Plugging the curvature formula for the submersion into the WCS formula, we have

Proposition:

$$
\begin{aligned}
& CS^W_{5,\gamma}(\bar{\xi}, e_2, Je_2, e_3, Je_3) \\
& = \frac{k^2}{30}\{32\pi^2 p_1(\Omega)(e_2, Je_2, e_3, Je_3) + 32k^2[3R(e_2, Je_2, e_3, Je_3) \\
& -R(e_2, e_3, e_2, e_3) - R(e_2, Je_3, e_2, Je_3) + R(e_2, Je_2, e_2, Je_2) \\
& +R(e_3, Je_3, e_3, Je_3)] + 192k^4\}
\end{aligned}
$$

where $p_1(\Omega)$ is the first Pontrjagin form.

Proof of Theorem A (1)

Set

$$|R|_\infty = \max\{|R(e_i, e_j, e_k, e_\ell)| i, j, k, \ell \in \{2, 3, 4, 5\}\}$$

Here, we use the "old notation" for the orthonormal frame $\{e_2, e_3, e_4, e_5\}$.

Thus, we have

Proposition: Let $[a_k^L] \in H_5(L\overline{M}_k, \mathbb{R})$ be the class associated to the rotation action in the fiber of $\pi : \overline{M}_k \to M$. Then $\int_{[a_k^L]} CS_5^W > 0$ if

$$k^2(32\pi p_1(\Omega)(e_2, Je_2, e_3, Je_3) - 224k^2|R|_\infty + 192k^4) > 0$$

pointwise on M.

Proof of Theorem A (2)

We first recall:

Proposition If $\int_{[a_k^L]} CS_5^W > 0$, then the loop of diffeomorphisms of $\overline{M}_k$ given by the rotation in the circle fiber gives an element of infinite order in $\pi_1(\mathrm{Diff}(\overline{\mathrm{M}}_\mathrm{k}))$

By this Proposition, we have

Corollary A:

The loop of diffeomorphisms of $\overline{M}_k$ given by the rotation in the circle fiber gives an element of infinite order in $\pi_1(\mathrm{Diff}(\overline{\mathrm{M}}_\mathrm{k}))$ provided

$$|R|_\infty < \frac{6}{7}k^2 + \frac{3\pi\sigma(M)}{7vol(M)k^2}, \quad k \neq 0$$

where $\sigma(M)$ is the signature of M.

Results (3)

In particular, we have

Corollary B: Theorem A holds in the following cases:

- $k >> 0$
- $M = T^4$ and $k \neq 0$

The case of complex projective space

These arguments can be refined for $M = P_2(\mathbb{C})$ by computing the curvature explicitly for the Fubini-Study metric. We get:

Theorem:

Let $|k| > 1$. Then the loop of diffeomorphisms of $\overline{P_2(\mathbb{C})_k}$ given by the rotation in the circle fiber gives an element of infinite order in $\pi_1(\mathrm{Diff}(\overline{\mathrm{P}_2(\mathbb{C})}_\mathrm{k}))$.

Proof: A calculation gives $\int_{P_2(\mathbb{C})} CS_5^W > 0$ provided $k^2(k^2-1)^2 > 0$.

$\square$

Remark 1

For $k = 1$, we note that $\overline{P_2(\mathbb{C})_1} = S^5$. Then $\pi_1(\text{Diff}(\overline{\text{P}_2(\mathbb{C})_1})) = \pi_1(\text{Diff}(\text{S}^5))$ contains the image of $\pi_1(\text{Isom}(S^5)) = \pi_1(SO(6)) = \mathbb{Z}_2$ as a subgroup. Rotating the fiber is an action by isometries, so it has order at most 2. Thus the Theorem must fail for $k = 1$.

For $k = 0$, $\overline{P_2(\mathbb{C})_0} = P_2(\mathbb{C}) \times S^1$. This should be the easiest case, but we get no information.

Another example

These calculations are also explicit for $M = S^2 \times S^2$ with the product metric:

Theorem:

The loop of diffeomorphisms of $\overline{(S^2 \times S^2)_k}$ given by the rotation of the circle fiber gives an element of infinite order in $\pi_1(\text{Diff}(\overline{\text{S}^2 \times \text{S}^2{}_\text{k}}))$ for $k \neq 0$.

Remark 2

Why dim(M)=2k-1

Our construction of CS^W is valid only for the "odd dimentional" case

Why dim (M) = 5

For the case $2k-1=3$, we get

Theorem:

$CS_3^W = 0$

So, our argument does not work for $dim(\mathrm{M}) = 3$.

Thank you