

2-truncated cubes and their toric spaces

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XIX Geometrical Seminar

August 29, 2016

Zlatibor, Serbia

Theorem (M.Atiyah; V.Guillemin, S.Sternberg'82)

Let (M, ω) be a $2d$ -dimensional compact connected symplectic manifold with a hamiltonian action of a compact torus \mathbb{T}^n . Then the image of the moment map $\mu : M \rightarrow \mathbb{R}^n$ is a convex polytope P which is the convex hull of $\mu(M^{\mathbb{T}})$.

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If $d = n$ and the torus action is effective, then (M, ω) is a symplectic toric manifold.

A polytope P in \mathbb{R}^n is called Delzant if its normal fan is smooth.

Theorem (T.Delzant'88)

There is a 1-1 correspondence between compact symplectic toric manifolds (M, ω, μ) (up to equivariant symplectomorphism) and Delzant polytopes $\mu(M)$ (up to lattice isomorphism).

Definition

Let P be a combinatorial simple polytope of dimension n . A *quasitoric manifold* over P is a smooth $2n$ -dimensional manifold M with a smooth action of the torus T^n satisfying the two conditions:

- (1) the action is locally standard;
- (2) there is a continuous projection $\pi : M \rightarrow P$ whose fibers are T^n -orbits.

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Remarks

- (a) M/T is homeomorphic, as a manifold with corners, to the simple polytope P ;
- (b) The action is free over the interior of P , the vertices of P correspond to the fixed points of the torus action on M ;
- (c) A projective toric manifold is a quasitoric manifold.

A moment-angle manifold \mathcal{Z}_P

In the work of M.Davis and T.Januszkiewicz'91 the following construction appeared.

Definition

Suppose P^n is a combinatorial simple polytope with facets F_1, \dots, F_m . Denote by T^{F_i} a 1-dimensional coordinate subgroup in $T^F \cong T^m$ for each $1 \leq i \leq m$ and $T^G = \prod T^{F_i} \subset T^F$ for a face $G = \cap F_i$ of a polytope P^n . Then the *moment-angle manifold* corresponding to P is a quotient space

$$\mathcal{Z}_P = T^F \times P^n / \sim,$$

where $(t_1, p) \sim (t_2, q)$ iff $p = q \in P$ and $t_1 t_2^{-1} \in T^{G(p)}$, $G(p)$ is a minimal face of P which contains $p = q$.

Remarks

- If P_1 and P_2 are *combinatorially equivalent*, i.e. their face lattices are isomorphic, then \mathcal{Z}_{P_1} and \mathcal{Z}_{P_2} are homeomorphic. The opposite statement is **not** true (truncation polytopes);

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- For any quasitoric manifold $M^{2n} \rightarrow P$ over a simple polytope P there is a principal T^{m-n} -bundle $\mathcal{Z}_P \rightarrow M^{2n}$, s.t. the composition $\mathcal{Z}_P \rightarrow M^{2n} \rightarrow P$ is a projection onto the orbit space of the T^m -action on \mathcal{Z}_P ;

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- V.Buchstaber and T.Panov (1998) realized \mathcal{Z}_P as a nondegenerate intersection of Hermitian quadrics in \mathbb{C}^m . It follows, that \mathcal{Z}_P is a smooth closed manifold of dimension $m + n$; moreover, it is 2-connected for any simple polytope P .

Examples

- 1) If $P = \Delta^n$ then $\mathcal{Z}_P = S^{2n+1}$;
- 2) If $P = P_1 \times P_2$ then $\mathcal{Z}_P = \mathcal{Z}_{P_1} \times \mathcal{Z}_{P_2}$

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- Consider a prism $Pr_3 = vc^1(\Delta^3)$, $\mathcal{Z}_{Pr_3} = S^3 \times S^5$ and cut a vertical edge. We get a 3-cube C , for which $\mathcal{Z}_C = S^3 \times S^3 \times S^3$.

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If we perform an edge truncation of C we get a 5-gonal prism Pr_5 and $\mathcal{Z}_{Pr_5} = (S^3 \times S^4)^{\#5} \times S^3$.
- Consider a 3-polytope $P = vc^1(C)$. Then \mathcal{Z}_P is **not** homotopy equivalent to a connected sum of products of spheres, though it is formal.

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- spheres;
- H-spaces;
- symmetric spaces;
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- quasitoric manifolds (T.Panov, N.Ray'08).

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- compact connected Lie groups G and their classifying spaces BG ;
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- quasitoric manifolds (T.Panov, N.Ray'08).

Moreover, formality is preserved by wedges, direct products and connected sums.

Stanley-Reisner rings

Let \mathbb{k} be a commutative ring with a unit and consider a $(n - 1)$ -dimensional simplicial complex K on the ordered set $[m] = \{1, \dots, m\}$. Let $\mathbb{k}[m] = \mathbb{k}[v_1, \dots, v_m]$ be the graded polynomial algebra on m variables, $\deg(v_i) = 2$.

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Face rings

A *face ring* (or a *Stanley-Reisner ring*) of K is the quotient ring

$$\mathbb{k}[K] := \mathbb{k}[v_1, \dots, v_m] / \mathcal{I}_K$$

where \mathcal{I}_K is the ideal generated by those square free monomials $v_{i_1} \cdots v_{i_s}$ for which $\{i_1, \dots, i_s\}$ is **not** a simplex of K . We denote $\mathbb{k}[P] = \mathbb{k}[\partial P^*]$.

Note that $\mathbb{k}[K]$ is a module over $\mathbb{k}[v_1, \dots, v_m]$ via the quotient projection.

Cohomology ring of \mathcal{Z}_P

The following result relates cohomology of \mathcal{Z}_P to combinatorics of the polytope P :

Theorem (V.Buchstaber, T.Panov'98)

If we define a differential graded algebra

$R(P) = \Lambda[u_1, \dots, u_m] \otimes \mathbb{k}[P]/(v_i^2 = u_i v_i = 0, 1 \leq i \leq m)$ with
bideg $u_i = (-1, 2)$, bideg $v_i = (0, 2)$; $du_i = v_i$, $dv_i = 0$, then:

$$H^{*,*}(\mathcal{Z}_P; \mathbb{k}) \cong H^{*,*}[R(P), d] \cong \mathrm{Tor}_{\mathbb{k}[v_1, \dots, v_m]}^{*,*}(\mathbb{k}[P], \mathbb{k}).$$

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These algebras admit $\mathbb{N} \oplus \mathbb{Z}^m$ -multigrading and we have

$$\mathrm{Tor}_{\mathbb{k}[v_1, \dots, v_m]}^{-i, 2\mathbf{a}}(\mathbb{k}[P], \mathbb{k}) \cong H^{-i, 2\mathbf{a}}(R(P), d),$$

where $\mathrm{Tor}_{\mathbb{k}[v_1, \dots, v_m]}^{-i, 2J}(\mathbb{k}[P], \mathbb{k}) \cong \tilde{H}^{|J|-i-1}(P_J; \mathbb{k})$ for $J \subset [m]$. Here we denote $P_J = \cup_{j \in J} F_j$. The multigraded component

$\mathrm{Tor}_{\mathbb{k}[v_1, \dots, v_m]}^{-i, 2\mathbf{a}}(\mathbb{k}[P], \mathbb{k}) = 0$, if \mathbf{a} is not a $(0, 1)$ -vector of length m .

Serre's conjecture

Suppose $R = \mathbb{k}[x_1, \dots, x_m]/I$ to be a quotient ring of a polynomial algebra.

Definition: face rings case

Let $b_i(K) = \dim_{\mathbb{k}} \operatorname{Tor}_{\mathbb{k}[K]}^{-i}(\mathbb{k}, \mathbb{k})$. Then **Poincaré series** of $\mathbb{k}[K]$ is

$$\operatorname{Poin}(\mathbb{k}[K]; t) = \sum_{i=0}^{\infty} b_i(K) t^i.$$

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- In 1982 D.J.Anick constructed a counterexample with a binomial ideal.
- The same year J.Backelin proved it for monomial ideals. Thus, $\operatorname{Poin}(\mathbb{k}[P]; t)$ is always a rational function.

Examples

- (1) A face ring is called **Golod** if the following identity holds:

$$\text{Poin}(\mathbb{k}[K]; t) = \frac{(1+t)^m}{1 - \sum_{i,j>0} \beta^{-i,2j}(K) t^{-i+2j-1}}.$$

We denoted $\beta^{-i,2j}(K) = \text{rk}_{\mathbb{k}} \text{Tor}_{\mathbb{k}[v_1, \dots, v_m]}^{-i,2j}(\mathbb{k}[K], \mathbb{k})$.

- (2) In 2008 T.Panov and N.Ray proved that for a flag simplicial complex K :

$$\text{Poin}(\mathbb{k}[K]; t) = \frac{(1+t)^n}{1 - h_1 t + \dots + (-1)^n h_n t^n}$$

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Question

Calculate the denominator coefficients for $\text{Poin}(\mathbb{k}[P]; t)$ effectively in terms of combinatorial commutative algebra of P .

We now turn to a discussion of flag nestohedra.

Building set

Let $S = \{1, 2, \dots, n+1\}$, $n \geq 2$. A *building set* on S is a family of subsets $B = \{B_k \subseteq S\}$, such that: 1) $\{i\} \in B$ for all $1 \leq i \leq n+1$; 2) if $B_i \cap B_j \neq \emptyset$, then $B_i \cup B_j \in B$.

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Nestohedra

Nestohedron is a simple convex n -dimensional polytope $P_B = \sum_{B_k \in B} \Delta_{B_k}$, where $\Delta_{B_k} = \text{conv}\{e_j \mid j \in B_k\} \subset \mathbb{R}^{n+1}$.

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Example: graph-associahedra

A *graphical building set* $B(\Gamma)$ for a graph Γ on the vertex set S consists of such B_k that Γ_{B_k} is a **connected subgraph** of Γ . Then $P_\Gamma = P_{B(\Gamma)}$ is called a *graph-associahedron*.

Graph-associahedra were first introduced by M.Carr and S.Devadoss (2006) in their study of Coxeter complexes.

Examples

- Γ is a complete graph on $[n + 1]$.
Then $P_\Gamma = Pe^n$ is a **permutohedron**.
- Γ is a stellar graph on $[n + 1]$.
Then $P_\Gamma = St^n$ is a **stellahedron**.
- Γ is a cycle graph on $[n + 1]$.
Then $P_\Gamma = Cy^n$ is a **cyclohedron** (or Bott-Taubes polytope).
- Γ is a chain graph on $[n + 1]$.
Then $P_\Gamma = As^n$ is an **associahedron** (or Stasheff polytope).

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Graph-associahedra are *flag* polytopes, i.e. if a number of facets has an **empty** intersection then some pair of these facets has an **empty** intersection.

In order to work with combinatorial types of graph-associahedra we should describe the structure of their face lattices.

Face poset

Facets of P_Γ are in 1-1 correspondence with **non** maximal connected subgraphs of Γ .

Moreover, a set of facets corresponding to such subgraphs $\Gamma_{i_1}, \dots, \Gamma_{i_s}$ has a **nonempty** intersection if and only if:

- (1) For any two subgraphs $\Gamma_{i_k}, \Gamma_{i_l}$, either they do not have a common vertex or one is a subgraph of another;
- (2) If any two of the subgraphs $\Gamma_{i_{k_1}}, \dots, \Gamma_{i_{k_l}}, l \geq 2$ do not have common vertices, then their union graph is disconnected.

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- If $P = Pe^n$ then its facets $F_1 \cap F_2 \neq \emptyset$ if and only if the corresponding Γ_1 and Γ_2 are subgraphs of one another;
- If $\Gamma_i, 1 \leq i \leq r$ are connected components of Γ then $P_\Gamma = P_{\Gamma_1} \times \dots \times P_{\Gamma_r}$.

Definition: special subgraphs in a graph

1) Suppose Γ is a graph. For any of its connected subgraphs γ one can compute the number $i(\gamma)$ of such connected subgraphs $\tilde{\gamma}$ in Γ that either

$$\gamma \cap \tilde{\gamma} \neq \emptyset, \gamma, \tilde{\gamma}$$

or

$$\gamma \cap \tilde{\gamma} = \emptyset,$$

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2) We denote by $i_{max} = i_{max}(\Gamma)$ the maximal value of $i(\gamma)$ over all connected subgraphs γ in Γ . A connected subgraph γ , on which i_{max} is achieved, will be called a **special subgraph**.

Using the face poset structure of P_Γ we get the following result:

Theorem

Let $P = P_\Gamma$ be a graph-associahedron of dimension $n \geq 3$ for a connected graph Γ . Then for $i > i_{max}$:

$$\beta^{-i, 2(i+1)}(P) = 0.$$

Denote the number of special subgraphs in Γ by s . Then

$$\beta^{-i_{max}, 2(i_{max}+1)}(P) = s.$$

Bigraded Betti numbers: examples

For the 4 classical series of graph-associahedra the theorem gives the following values of i_{max} and s .

Associahedron

$$\beta^{-q,2(q+1)}(P) = \begin{cases} n+3, & \text{if } n \text{ is even;} \\ \frac{n+3}{2}, & \text{if } n \text{ is odd;} \end{cases}$$

$$\beta^{-i,2(i+1)}(P) = 0 \quad \text{for } i \geq q+1,$$

where $q = q(n)$ is:

$$q = q(n) = \begin{cases} \frac{n(n+2)}{4}, & \text{if } n \text{ is even;} \\ \frac{(n+1)^2}{4}, & \text{if } n \text{ is odd.} \end{cases}$$

Cyclohedron

$$\beta^{-q,2(q+1)}(P) = \begin{cases} 2n + 2, & \text{if } n \text{ is even;} \\ n + 1, & \text{if } n \text{ is odd;} \end{cases}$$

$$\beta^{-i,2(i+1)}(P) = 0 \quad \text{for } i \geq q + 1,$$

where $q = q(n)$ is:

$$q = q(n) = \begin{cases} \frac{n(n+2)-2}{2}, & \text{if } n \text{ is even;} \\ \frac{(n+1)^2-2}{2}, & \text{if } n \text{ is odd.} \end{cases}$$

Permutohedron

$$\beta^{-q, 2(q+1)}(P) = \binom{n+1}{\lfloor \frac{n+1}{2} \rfloor}$$

$$\beta^{-i, 2(i+1)}(P) = 0 \quad \text{for } i \geq q+1,$$

where $q = q(n) = 2^{n+1} - 2^{\lfloor \frac{n+1}{2} \rfloor} - 2^{\lfloor \frac{n+2}{2} \rfloor} + 1$

Stellahedron

$$\beta^{-q, 2(q+1)}(P) = \binom{n}{\lfloor \frac{n}{2} \rfloor}$$

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where $q = q(n) = 2^n - 2^{\lfloor \frac{n}{2} \rfloor} - 2^{\lfloor \frac{n+1}{2} \rfloor} + \lfloor \frac{n+3}{2} \rfloor$.

Bigraded Betti numbers of the type $\beta^{-i,2(i+1)}(P)$ have another topological application by means of the following result.

Loop homology of moment-angle-manifolds

J.Grbić, T.Panov, S.Theriault, J.Wu (2012) proved that

$\sum_{i=1}^{m-n} \beta^{-i,2(i+1)}(P)$ equals the minimal number of multiplicative generators of the Pontryagin algebra $H_*(\Omega\mathcal{Z}_P; \mathbb{k})$ for any flag simple polytope P .

As graph-associahedra are flag polytopes, our previous result provides an estimate on the lower bound for the number of generators of the loop homology algebra for \mathcal{Z}_P for graph-associahedra P .

Massey k -products in $H^*[A, d]$

Defining system

Suppose (A, d) is a dga, $\alpha_i = [a_i] \in H^*[A, d]$ and $a_i \in A^{n_i}$ for $1 \leq i \leq k$. Then a *defining system* for $(\alpha_1, \dots, \alpha_k)$ is a $(k+1) \times (k+1)$ -matrix C , s.t. the following conditions hold:

- (1) $c_{i,j} = 0$, if $i \geq j$,
- (2) $c_{i,i+1} = a_i$,
- (3) $a \cdot E_{1,k+1} = dC - \bar{C} \cdot C$ for some $a = a(C) \in A$, where $\bar{c}_{i,j} = (-1)^{\deg c_{i,j}} \cdot c_{i,j}$.

This implies: $d(a) = 0$ and $a \in A^m$, $m = n_1 + \dots + n_k - k + 2$.

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Definition

A Massey k -product $\langle \alpha_1, \dots, \alpha_k \rangle$ is said to be *defined*, if there exists a defining system C for it.

If so, this Massey product consists of all $\alpha = [a(C)]$ for each defining system C . It is called *trivial*, if $[a(C)] = 0$ for some C .

Massey k -products: examples

$k=2$

If $\langle \alpha_1, \alpha_2 \rangle$ is defined, then we have:

$$a = d(c_{1,3}) - \bar{a}_1 \cdot a_2.$$

$k=3$

If $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$ is defined, then we have:

$$a = d(c_{1,4}) - \bar{a}_1 \cdot c_{2,4} - \bar{c}_{1,3} \cdot a_3,$$

$$d(c_{1,3}) = \bar{a}_1 \cdot a_2,$$

$$d(c_{2,4}) = \bar{a}_2 \cdot a_3.$$

Remarks

- (1) V.Buchstaber and V.Volodin (2011) constructed realizations of all flag nestohedra as **2-truncated cubes**, i.e. a result of a sequence of truncations of codimension 2 faces only, starting with a cube, and proved the Gal conjecture on γ -vectors for them;

Remarks

- (1) V.Buchstaber and V.Volodin (2011) constructed realizations of all flag nestohedra as **2-truncated cubes**, i.e. a result of a sequence of truncations of codimension 2 faces only, starting with a cube, and proved the Gal conjecture on γ -vectors for them;
- (2) G.Denham and A.Suciu (2005) described 5 graphs, s.t. there is a nontrivial triple Massey product of 3-dimensional classes in $H^*(\mathcal{Z}_P)$ iff one of these graphs is an induced subgraph in $sk^1(\partial P^*)$. All such products are decomposable.

Triple Massey products in $H^*(\mathcal{Z}_P)$

The following result holds for triple Massey products in the cohomology ring of \mathcal{Z}_P .

Theorem

Let P be a generalized associahedron of type A , $B(C)$, D , or $P = P_\Gamma$. Then there is a defined and nontrivial triple Massey product $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$ of some classes $\alpha_i \in H^3(\mathcal{Z}_P)$, $i = 1, 2, 3$ if and only if P is a generalized associahedron or a graph-associahedron P_Γ and in the graph Γ there is a connected component on $m \geq 4$ vertices, different from the complete graph K_m . All such Massey products are decomposable.

Triple Massey products in $H^*(\mathcal{Z}_P)$

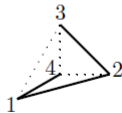
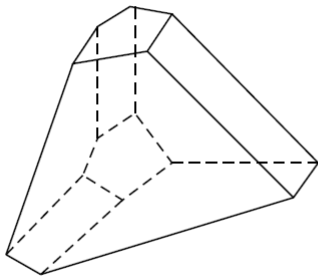
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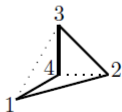
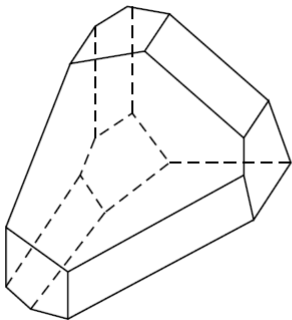
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The face lattice allows us to reduce the general case to the case of $n = 3$ and apply the result of Denham and Suciu.

Example: generalized associahedron of type A or Stasheff polytope, $n = 3$



Example: generalized associahedron of type B (C) or Bott-Taubes polytope, $n = 3$

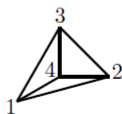
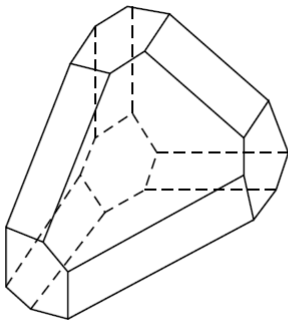


Any nestohedron on a connected building set can be obtained from a simplex as a result of a truncation sequence of the simplex's faces only.

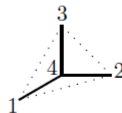
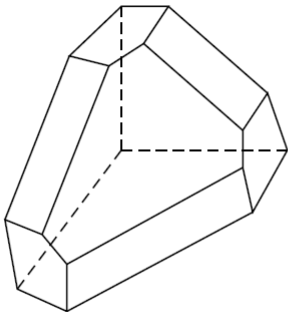
Theorem

- If $P = Pe^n$, $n \geq 2$ and the classes $\alpha_i \in H^3(\mathcal{Z}_P)$, $1 \leq i \leq n+1$ are represented by $(n+1)$ pairs of the opposite permutohedra facets, then $\langle \alpha_1, \dots, \alpha_{n+1} \rangle$ is defined and trivial;
- If $P = St^n$, $n \geq 2$ and the classes $\alpha_i \in H^3(\mathcal{Z}_P)$, $1 \leq i \leq n$ are represented by n pairs of the opposite stellahedra facets, then $\langle \alpha_1, \dots, \alpha_n \rangle$ is defined and trivial.

Example: 3-dimensional permutohedron



Example: 3-dimensional stellatedhedron



Example: $n = 2$

- $P = As^2$ is a 5-gon, $\mathcal{Z}_P = (S^3 \times S^4)^{\#5}$ and the vanishing cup product corresponds to 2 pairs of non-adjacent edges in a 5-gon.
- $P = Pe^2$ is a 6-gon, $\mathcal{Z}_P = (S^3 \times S^5)^{\#6} \# (S^4 \times S^4)^{\#8} \# (S^5 \times S^3)^{\#3}$ and the vanishing triple Massey product corresponds to 3 pairs of parallel edges in a regular 6-gon.

Massey operations and 2-truncated cubes

We next consider a particular family of 2-truncated n -cubes \mathcal{P} , one for each dimension n , for which $\mathcal{Z}_{\mathcal{P}}$ has a nontrivial Massey product of order n .

Definition

Suppose I^n is an n -dimensional cube with facets F_1, \dots, F_{2n} , such that F_i and F_{n+i} , $1 \leq i \leq n$ are parallel (do not intersect). Then we define \mathcal{P} as a result of a consecutive cut of faces of codimension 2 from I^n , having the following Stanley-Reisner ideal:

$$I = (v_1 v_{n+1}, \dots, v_n v_{2n}, v_1 v_{n+2}, \dots, v_{n-1} v_{2n}, \dots, v_1 v_{2n-1}, v_2 v_{2n}, \dots),$$

where v_i correspond to F_i , $1 \leq i \leq 2n$ and in the last dots are the monomials corresponding to the new facets. This determines uniquely the combinatorial type of \mathcal{P} .

Remarks

- For $n = 2$ we get a 2-dimensional cube (the square) and for $n = 3$ we get a simple 3-polytope P with 8 facets giving a nontrivial triple Massey product due to I.Baskakov result (2003);

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- For $n = 2$ we get a 2-dimensional cube (the square) and for $n = 3$ we get a simple 3-polytope \mathcal{P} with 8 facets giving a nontrivial triple Massey product due to I.Baskakov result (2003);
- \mathcal{P} is a flag nestohedron: we can easily construct the building set B for \mathcal{P} on the vertex set $S = [n + 1]$ by identifying F_i with $\{1, \dots, i\}$ for $1 \leq i \leq n$ and identifying F_i with $\{i - n + 1\}$ for $n + 1 \leq i \leq 2n$. Then we consecutively cut the following faces:

$$\{1\} \sqcup \{3\}, \{1, 2\} \sqcup \{4\}, \dots, \{1, \dots, n - 1\} \sqcup \{n + 1\}$$

...

$$\{1\} \sqcup \{n\}, \{1, 2\} \sqcup \{n + 1\}.$$

Remarks

- Thus, $\mathcal{P} = P_B$ for the building set B consisting of the building set

$$B_0 = \{\{i\}_1^{n+1}, \{1, 2\}, \{1, 2, 3\}, \dots, [n+1]\}$$

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- \mathcal{P} is not a graph-associahedron: its number of facets $f_0(\mathcal{P}) = \frac{n(n+3)}{2} - 1 < f_0(As^n) = \frac{n(n+3)}{2}$, thus we can apply the lower and upper bounds for f -vectors of graph-associahedra proved by Buchstaber and Volodin.

Massey operations and 2-truncated cubes

Our main result on nontrivial higher Massey products for moment-angle manifolds is the following.

Theorem

Suppose $\alpha_i \in H^3(\mathcal{Z}_{\mathcal{P}})$ is represented by a 3-dimensional cocycle $v_i u_{n+i} \in R^{-1,4}(\mathcal{P})$ for $1 \leq i \leq n$ and $n \geq 2$. Then the n -fold Massey product $\langle \alpha_1, \dots, \alpha_n \rangle$ is defined and nontrivial.

Massey operations and 2-truncated cubes

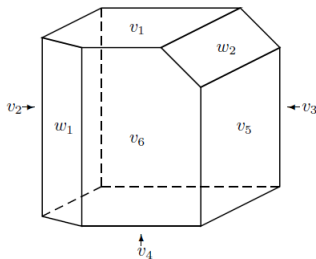
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- Any element $\alpha = [a] \in \langle \alpha_1, \dots, \alpha_n \rangle$ is s.t. $a \in R^*(\mathcal{P})$ is a sum of its multigraded components and $d : R^{-i,2J}(\mathcal{P}) \rightarrow R^{-(i-1),2J}(\mathcal{P})$. Thus, a is a coboundary iff each of its multigraded components is a coboundary;
- For any such $a \in R^*(\mathcal{P})$ its component in $R^{-(2n-2),2(1,\dots,1,0,\dots,0)}(\mathcal{P})$ with $2n$ "1"s is always represented by the cocycle $v_1 v_{2n} u_2 u_3 \dots u_{2n-1}$ (up to sign), which is not a coboundary.

Example: $n = 3$



$$I_P = (v_1 v_4, v_2 v_5, v_3 v_6, v_1 v_5, v_2 v_6, \dots).$$

Then for $a_i = v_i u_{n+i}$ we have (up to sign):

$$c_{1,3} = v_1 u_2 u_4 u_5, c_{2,4} = v_2 u_3 u_5 u_6, a = v_1 v_6 u_2 u_3 u_4 u_5.$$

Thus, $\alpha = [a] = -[v_1 u_4 u_5] \cdot [v_6 u_2 u_3]$.

THANK YOU FOR YOUR ATTENTION!

♥*Serbia*♥