Hochschild (co)homology of exterior algebras using algebraic Morse theory

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algebraic Morse theory: formulation

digraph

Every chain complex of free R-modules of finite rank

$$C_*: \quad \ldots \longrightarrow R^{(I_{k+1})} \xrightarrow{\partial_{k+1}} R^{(I_k)} \xrightarrow{\partial_k} R^{(I_{k-1})} \to \ldots$$

induces a weighted digraph Γ_{C_*}:

- vertex set $\ldots \sqcup I_{k+1} \sqcup I_k \sqcup I_{k-1} \sqcup \ldots$;
- weighted edges correspond to nonzero entries in matrices $\ldots, \partial_{k+1}, \partial_k, \ldots$.

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(1) the edges in $\mathcal{M}$ have <u>invertible weights</u> in R;

(2) $\mathcal{M}$ is a <u>matching</u>, i.e. edges in $\mathcal{M}$ have no common endpoints;

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(2) $\mathcal{M}$ is a matching, i.e. edges in $\mathcal{M}$ have no common endpoints;

(3) in the digraph $\Gamma_{C_*}^{\mathcal{M}}$ (which is Γ_{C_*} with edges from $\mathcal{M}$ reversed and with opposite inverted weights) there are no directed cycles.

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Any Morse matching $\mathcal{M}$ on Γ_{C_*} induces a homotopy equivalence between C_* and a smaller chain complex $\mathring{C}_*$:

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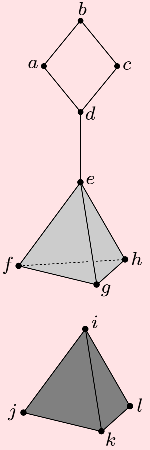
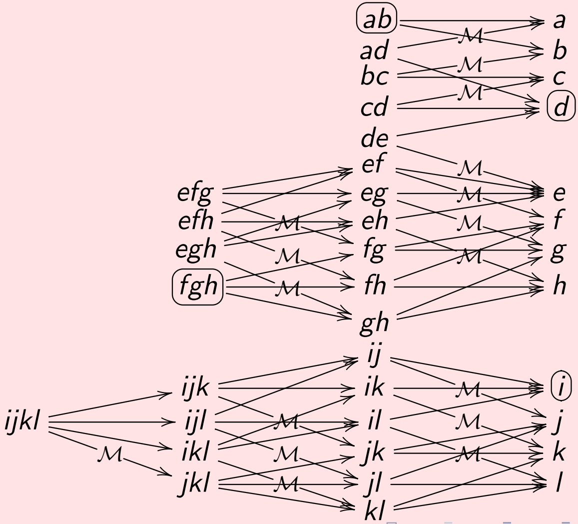
The homotopy equivalence sends an element $v \in \mathring{C}_k$ to the sum of all directed paths in $\Gamma_{C_*}^{\mathcal{M}}$ to all vertices $w \in C_k$.

algebraic Morse theory: example

Δ



$$R^{(\Delta^{[3]})} = R^1 \to R^{(\Delta^{[2]})} = R^8 \to R^{(\Delta^{[1]})} = R^{17} \to R^{(\Delta^{[0]})} = R^{12}$$



The *bar resolution* of any associative unital R-algebra A is

$$B_*: \quad \ldots \longrightarrow A^{\otimes k+2} \xrightarrow{b_k} A^{\otimes k+1} \xrightarrow{b_{k-1}} \ldots \longrightarrow A^{\otimes 2} \longrightarrow A \longrightarrow 0,$$
$$b_k : a_0 \otimes \cdots \otimes a_{k+1} \longmapsto \sum_{0 \leq i \leq k} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{k+1}.$$

This is an exact complex of A^e-modules, where A^e acts on $A^{\otimes k+2}$ by

$$(\alpha \otimes \beta)(a_0 \otimes a_1 \otimes \cdots \otimes a_k \otimes a_{k+1}) = (\alpha a_0) \otimes a_1 \otimes \cdots \otimes a_k \otimes (a_{k+1}\beta).$$

Hochschild (co)homology of A with coefficients in an A^e-module M is the homology of the complex

$$(C_*, \partial_*) = M \otimes_{A^e} B_* \quad \text{and} \quad (C^*, \delta^*) = \mathrm{Hom}_{A^e}(B_*, M).$$

$k\backslash n$	1	2	3	4	...
0	$\mathbb{Z}^2$	$\mathbb{Z}^3\oplus\mathbb{Z}_2^1$	$\mathbb{Z}^5\oplus\mathbb{Z}_2^3$	$\mathbb{Z}^9\oplus\mathbb{Z}_2^7$	
1	$\mathbb{Z}\oplus\mathbb{Z}_2$	$\mathbb{Z}^4\oplus\mathbb{Z}_2^3$	$\mathbb{Z}^{12}\oplus\mathbb{Z}_2^9$	$\mathbb{Z}^{32}\oplus\mathbb{Z}_2^{25}$	
2	$\mathbb{Z}$	$\mathbb{Z}^6\oplus\mathbb{Z}_2^3$	$\mathbb{Z}^{24}\oplus\mathbb{Z}_2^{15}$	$\vdots$	
3	$\mathbb{Z}\oplus\mathbb{Z}_2$	$\mathbb{Z}^8\oplus\mathbb{Z}_2^5$	$\mathbb{Z}^{40}\oplus\mathbb{Z}_2^{25}$		
4	$\mathbb{Z}$	$\mathbb{Z}^{10}\oplus\mathbb{Z}_2^5$	$\vdots$		
5	$\mathbb{Z}\oplus\mathbb{Z}_2$	$\mathbb{Z}^{12}\oplus\mathbb{Z}_2^7$			
6	$\mathbb{Z}$	$\mathbb{Z}^{14}\oplus\mathbb{Z}_2^7$			
7	$\mathbb{Z}\oplus\mathbb{Z}_2$	$\vdots$			
8	$\mathbb{Z}$				
9	$\mathbb{Z}\oplus\mathbb{Z}_2$				
10	$\mathbb{Z}$				
	$\vdots$				

exterior algebras: associated digraph

Let $A=\Lambda_R[x_1,\ldots,x_n]$ and denote $x_\sigma=\bigwedge_{i\in\sigma}x_i$.
R-module A is free on $\{x_\sigma;\ \sigma\subseteq[n]\}$.
A^e-module $A^{\otimes k+2}$ is free on $\{1\otimes x_{\sigma_1}\otimes\ldots\otimes x_{\sigma_k}\otimes 1;\ \sigma_1,\ldots,\sigma_k\subseteq[n]\}$.

These tensors are the vertices of the digraph Γ_{B_*}. By the definition of b_*, the edges of Γ_{B_*} and their weights are of three forms:

$$1\otimes x_{\sigma_1}\otimes\ldots\otimes x_{\sigma_k}\otimes 1 \xrightarrow{\;x_{\sigma_1}\otimes 1\;} 1\otimes x_{\sigma_2}\otimes\ldots\otimes x_{\sigma_k}\otimes 1$$

$$1\otimes x_{\sigma_1}\otimes\ldots\otimes x_{\sigma_k}\otimes 1 \xrightarrow{\;(-1)^i\;} 1\otimes x_{\sigma_1}\otimes\ldots\otimes x_{\sigma_i}x_{\sigma_{i+1}}\otimes\ldots\otimes x_{\sigma_k}\otimes 1$$

$$1\otimes x_{\sigma_1}\otimes\ldots\otimes x_{\sigma_k}\otimes 1 \xrightarrow{\;(-1)^k 1\otimes x_{\sigma_k}\;} 1\otimes x_{\sigma_1}\otimes\ldots\otimes x_{\sigma_{k-1}}\otimes 1$$

Here $x_{\sigma_i}x_{\sigma_{i+1}}=\begin{cases}(-1)^j x_{\sigma_i\cup\sigma_{i+1}} & ;\ \sigma_i\cap\sigma_{i+1}=\emptyset\\ 0 & ;\ \text{otherwise}\end{cases}$.

Given a vertex $1\otimes x_{\sigma_1}\otimes\ldots\otimes x_{\sigma_k}\otimes 1$, going up means splitting some σ_i into σ_i' and $\sigma_i\setminus\sigma_i'$. The simplest choice is $i=1$ and $\sigma_i'=\{\max\sigma_i\}$, i.e. let $\mathcal{M}=\left\{\begin{smallmatrix}1\otimes x_i\otimes x_{\sigma_1\setminus\{i\}}\otimes\ldots\otimes x_{\sigma_k}\otimes 1\\ \downarrow\\ 1\otimes x_{\sigma_1}\otimes\ldots\otimes x_{\sigma_k}\otimes 1\end{smallmatrix};\, i=\max\sigma_1\right\}$. Then the unmatched vertices are $\mathring{\mathcal{M}}=\left\{1\otimes x_{i_1}\otimes x_{\sigma_2}\otimes\ldots\otimes x_{\sigma_k}\otimes 1;\; i_1\leq\max\sigma_2\right\}$. If we add edges $\left\{\begin{smallmatrix}1\otimes x_{i_1}\otimes x_i\otimes x_{\sigma_2\setminus\{i\}}\otimes\ldots\otimes x_{\sigma_k}\otimes 1\\ \downarrow\\ 1\otimes x_{i_1}\otimes x_{\sigma_2}\otimes\ldots\otimes x_{\sigma_k}\otimes 1\end{smallmatrix};\, i=\max\sigma_2\right\}$ to $\mathcal{M}$, then the unmatched vertices are

$\mathring{\mathcal{M}}=\left\{1\otimes x_{i_1}\otimes x_{i_2}\otimes x_{\sigma_3}\otimes\ldots\otimes x_{\sigma_k}\otimes 1;\; i_1\leq i_2\leq\max\sigma_3\right\}$. If we add edges $\left\{\begin{smallmatrix}1\otimes x_{i_1}\otimes x_{i_2}\otimes x_i\otimes x_{\sigma_3\setminus\{i\}}\otimes\ldots\otimes x_{\sigma_k}\otimes 1\\ \downarrow\\ 1\otimes x_{i_1}\otimes x_{i_2}\otimes x_{\sigma_3}\otimes\ldots\otimes x_{\sigma_k}\otimes 1\end{smallmatrix};\, i=\max\sigma_3\right\}$ to $\mathcal{M}$, then the unmatched vertices are

$\mathring{\mathcal{M}}=\left\{1\otimes x_{i_1}\otimes x_{i_2}\otimes x_{i_3}\otimes x_{\sigma_4}\otimes\ldots\otimes x_{\sigma_k}\otimes 1;\; i_1\leq i_2\leq i_3\leq\max\sigma_4\right\}$.

Seeing the emerging pattern, we collectively define

$$\mathcal{M}=\left\{\begin{smallmatrix}1\otimes x_{i_1}\otimes\ldots\otimes x_{i_{r-1}}\otimes x_i\otimes x_{\sigma_r\setminus\{i\}}\otimes\ldots\otimes x_{\sigma_k}\otimes 1\\ \downarrow\\ 1\otimes x_{i_1}\otimes\ldots\otimes x_{i_{r-1}}\otimes x_{\sigma_r}\otimes\ldots\otimes x_{\sigma_k}\otimes 1\end{smallmatrix};\, r\geq 1,\; i_1\leq\ldots\leq i_{r-1}\leq i=\max\sigma_r\right\}.$$

For a multiset $\tau = \{i_1 \le \ldots \le i_k\}$, we denote $x_{(\tau)} = 1 \otimes x_{i_1} \otimes \ldots \otimes x_{i_k} \otimes 1$, and let $\overline{\tau}$ be its corresponding set, e.g. $\tau = \{1,1,2,5,5,5\}$ implies $\overline{\tau} = \{1,2,5\}$.

Theorem

The A^e-module A admits a resolution $\mathring{B}_*$, in which $\mathring{B}_k$ is free on $\{x_{(\tau)};\, \tau \in \left(\!\binom{[n]}{k}\!\right)\}$ and $\mathring{b}_k(x_{(\tau)}) = \sum_{i\in\overline{\tau}} \big(x_i \otimes 1 + (-1)^k 1 \otimes x_i\big) x_{(\tau\setminus\{i\})}$.

Since $x_i \otimes 1 \pm 1 \otimes x_i$ is a nonunit of A^e, this resolution is minimal.

The homotopy equivalence $h\colon \mathring{B}_* \to B_*$ induced by $\mathcal{M}$ sends

$$h\colon x_{(\tau)} \mapsto \sum_{\pi\in S_\tau} x_{(\pi\tau)} \quad \text{and} \quad h^{-1}\colon x_{(\pi\tau)} \mapsto \begin{cases} x_{(\tau)}; & \pi = \mathrm{id} \\ 0 \ ; & \pi \neq \mathrm{id} \end{cases}.$$

exterior algebras: additive structure of HH_*

Corollary

If $R=\mathbb{Z}$, then $HH_k(A;A) \cong \mathbb{Z}^F \oplus \mathbb{Z}_2^T$, where
$F = 2^{n-1}\left(\!\binom{n}{k}\!\right) + \begin{cases} 1; & k=0 \\ 0; & k\geq 1 \end{cases}$ and
$T = (-1)^{k+1} + 2^{n-1}\sum_{0\leq i\leq k}(-1)^{k-i}\left(\!\binom{n}{i}\!\right)$.

In particular, if $R=K$ is a field, then

$$\dim_K HH_k(A;A) = \begin{cases} 2^n\left(\!\binom{n}{k}\!\right) ; & \mathrm{char}\, K=2 \\ 2^{n-1}\left(\!\binom{n}{k}\!\right); & \mathrm{char}\, K\neq 2,\, k\geq 1 \\ 2^{n-1}+1 ; & \mathrm{char}\, K\neq 2,\, k=0 \end{cases}.$$

Corollary

Let $R = K$ be a field. If $\mathrm{char}\, K = 2$, then

$$HH^*(A;A) \cong \Lambda[x_1,\dots,x_n] \otimes_K K[x_1,\dots,x_n].$$

If $\mathrm{char}\, K \neq 2$, then $HH^*(A;A)$ is isomorphic to the subalgebra of $\Lambda[x_1,\dots,x_n] \otimes_K K[x_1,\dots,x_n]$, spanned by

$$\{x_\sigma \otimes x_\tau;\ (-1)^{|\sigma|} = (-1)^{|\tau|}\} \cup \{x_{[n]} \otimes 1\}.$$

Thus if $\mathrm{char}\, K \neq 2$, then $HH^*(A;A)$ has algebra generators

$$\{1 \otimes x_{\{i,j\}};\ i \leq j \in [n]\} \cup \{x_{\{i,j\}} \otimes 1;\ i < j \in [n]\} \cup \{x_i \otimes x_j;\ i,j \in [n]\} \cup \{x_{[n]} \otimes 1\}$$

and is subject to many relations.

Given a simplicial complex Δ on $[n]$, its Stanley-Reisner algebra is

$$\Lambda[\Delta] = \Lambda[x_1, \dots, x_n]/(x_\sigma;\, \sigma \notin \Delta).$$

What is HH^* for the sphere, path, cycle, etc.?
$\Lambda[x_1, \dots, x_n]/(x_1 \cdots x_n)$
$\Lambda[x_1, \dots, x_n]/(x_i x_j;\, |i-j| \geq 2)$
$\Lambda[x_1, \dots, x_n]/(x_i x_j;\, |i-j \mod n| \geq 2)$