Some systems of nonlinear PDE which are soluble in closed form

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Riemannian manifolds of conullity two

Definition

A Riemannian manifold (M, g) is said to be *of conullity two* if, for each point $p \in M$, there is an orthonormal basis $(e_1, ..., e_m)$ of the tangent space T_pM such that the curvature tensor R satisfies

$$R_{1212} = R_{2121} = -R_{1221} = -R_{2112},$$

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Our goal is not only to find many nontrivial examples but also to classify explicitly all such spaces in dimension 3, which is the first nontrivial dimension.

Lemma

Let (M^3, g) be a smooth space of conullity two. Then, in a neighborhood $\mathcal{U}$ of each point, we can introduce local coordinates (w, x, y) such that the metric g is given by the formula

$$g = (\omega^1)^2 + (\omega^2)^2 + (\omega^3)^2,$$

where ω^i are of the form

$$\begin{aligned} \omega^1 &= f(w, x, y)\mathrm{d}w, \\ \omega^2 &= A(w, x, y)\mathrm{d}x + C(w, x, y)\mathrm{d}w, \\ \omega^3 &= \mathrm{d}y + H(w, x)\mathrm{d}w. \end{aligned} \tag{1}$$

▸ (2) ▸ Theorem 2

Lemma

The local components ω^i_j $(i,j=1,2,3)$ *of the connection form* Θ *for the metric* (1) *are expressed as*

$$\begin{aligned}\omega^1_2 &= R\mathrm{d}w - A\alpha\mathrm{d}x + \beta\mathrm{d}y,\\ \omega^1_3 &= S\mathrm{d}w + A\beta\mathrm{d}x,\\ \omega^2_3 &= T\mathrm{d}w + A'_y\mathrm{d}x,\end{aligned} \tag{2}$$

where

$$\begin{aligned}\alpha &= \frac{1}{Af}(A'_w - C'_x - HA'_y),\\ \beta &= \frac{1}{2Af}(H'_x + AC'_y - CA'_y),\\ R &= \frac{f'_x}{Af} - C\alpha + H\beta,\\ S &= f'_y + C\beta, \quad T = C'_y - f\beta.\end{aligned} \tag{3}$$

One can prove easily that, for the sectional curvature $k(w, x, y) = R_{1212}$, it holds $k = \dfrac{\sigma}{Af}$, where $\sigma = \sigma(w, x)$ is some function of two variables. Recall also the standard formulas for the curvature components Ω^i_j:

$$\Omega^i_j = \mathrm{d}\omega^i_j + \sum_a \omega^i_a \wedge \omega^a_j.$$

Because our space is of conullity two, we have

$$\Omega^1_2 = k\,\omega^1 \wedge \omega^2, \quad \Omega^1_3 = \Omega^2_3 = 0.$$

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If we express these conditions explicitly, substituting from the formulas (1)–(3), we obtain the following system of partial differential equations:

$$(A\alpha)'_y + \beta'_x = 0, \tag{A1}$$

$$R'_y - \beta'_w = 0, \tag{A2}$$

$$(A\alpha)'_w + R'_x + (SA)'_y = -\sigma, \tag{A3}$$

$$(A\beta)'_y + A'_y\beta = 0, \tag{B1}$$

$$S'_y + T\beta = 0, \tag{B2}$$

$$A''_{yy} - A\beta^2 = 0, \tag{C1}$$

$$T'_y - S\beta = 0. \tag{C2}$$

(Here two of original nine conditions are not independent).

▸ Remark

Now we shall explain the important notion of asymptotic foliation. Let us express the formulas (2) through the original 1-forms ω^1, ω^2, ω^3. We obtain

$$\omega_2^1 = \frac{1}{Af} f_x' \, \omega^1 - \alpha \, \omega^2 + \beta \, \omega^3,$$
$$\omega_3^1 = a \, \omega^1 + b \, \omega^2,$$
$$\omega_3^2 = c \, \omega^1 + d \, \omega^2,$$

where the new functions a, b, c, d are given by the formulas

$$a = \frac{f_y'}{f}, \quad b = \beta,$$
$$c = \beta - \frac{Ah}{f}, \quad e = \frac{A_y'}{A}.$$

Let us remark that the function $ae - bc$ is a Riemannian invariant and the functions $a + e$, $b - c$ are Riemannian invariants up to a sign.

Asymptotic leaves

A 2-dimensional (local) *asymptotic leaf* is surface N in (M, g) generated by geodesics and such that its tangent planes are parallel along each such geodesics with respect to the Levi-Civita connection of (M, g).

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An *asymptotic distribution* on (M,g) is a 2-dimensional distribution which is integrable and whose integral manifolds are asymptotic leaves. The integral manifolds of an asymptotic distribution determine a foliation of (M,g), which is called *asymptotic foliation*.

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Now one can deduce the following

Theorem

All asymptotic distributions are determined by the quadratic Pfaffian equation

$$c(\omega^1)^2 + (e - a)\omega^1\omega^2 - b(\omega^2)^2 = 0.$$

Corollary

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b) Two asymptotic foliations, then (M, g) is said to be hyperbolic. In particular, it is said to be orthogonally hyperbolic if the both asymptotic foliations are orthogonal, and non-orthogonally hyperbolic in the opposite case.

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c) Just one asymptotic foliation, then (M, g) is said to be parabolic.

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c) Just one asymptotic foliation, then (M, g) is said to be parabolic.

d) Infinitely many asymptotic foliations, then (M, g) is said to be planar.

Theorem

If (M, g) admits, in a given neighborhood, at least one asymptotic foliation, then the functions A, C and f from the formula (1) *are of the form*

$$A = py + q, \quad C = ry + s, \quad f = ty + u,$$

where p, q, r, s, t and u are functions of w and x only.

Remark

The system (A1)–(C2) of PDE is reduced to (A3) and the simple system of PDE

$$(A\alpha)'_y = 0, \quad R'_y = 0, \quad \beta = 0.$$

Theorem 9

Classifications

Theorem

Each locally irreducible planar 3-space (M, g) is determined, up to a local isometry, by the formulas

$$\omega^1 = t\mathrm{d}w, \quad \omega^2 = y\mathrm{d}x, \quad \omega^3 = \mathrm{d}y,$$

where $t = t(w, x)$ is an arbitrary function.

Theorem

Each locally irreducible parabolic 3-space (M, g) such that its asymptotic foliation is not totally geodesic is determined, up to a local isometry, by the formulas

$$\omega^1 = \lambda(py + q)\mathrm{d}w,$$
$$\omega^2 = (py + q)\mathrm{d}x + (ry + s)\mathrm{d}w,$$
$$\omega^3 = \mathrm{d}y + x\mathrm{d}w,$$

where $p = p(w, x)$, $q = q(w, x)$ are arbitrary functions of two variables w and x, $(p/q)'_x \neq 0$, and

$$r = \frac{p\mathcal{E} + p'_x}{\mathcal{Q}}, \quad s = \frac{q\mathcal{E}; q'_x}{\mathcal{Q}}, \quad \lambda = \sqrt{\frac{p'_w - r'_x}{p\mathcal{Q}}}$$
$$\mathcal{Q} = pq'_x - qp'_x, \quad \mathcal{E} = pq'_w - qp'_w - p^2 x.$$

Remark

There are special explicit formulas also for the singular case, when the asymptotic foliation is totally geodesic.

Theorem

Each locally irreducible orthogonally hyperbolic 3-space (M, g) not admitting any totally geodesic asymptotic foliation is determined, up to a local isometry, by the formulas

$$\omega^1 = (ty + u)\mathrm{d}w,$$
$$\omega^2 = (py + q)\mathrm{d}x,$$
$$\omega^3 = \mathrm{d}y,$$

where $p = p(w, x)$, $q = q(w, x)$ are arbitrary functions of two variables w and x, $(p/q)'_w \neq 0$, and $(p'_w/q'_w)'_x \neq 0$, and t, u are calculated from p and q as follows:

$$t = \frac{\phi p'_w}{pq'_w - qp'_w}, \quad u = \frac{\phi q'_w}{pq'_w - qp'_w},$$

where

$$|\phi| = \exp\left(\int \frac{q'_w p'_x - p'_w q'_x}{pq'_w - qp'_w}\mathrm{d}x\right).$$

Theorem

Each locally irreducible non-orthogonally hyperbolic 3-space (M, g) not admitting any totally geodesic asymptotic foliation is determined, up to a local isometry, by the formulas

$$\omega^1 = (ty + u)\mathrm{d}w,$$
$$\omega^2 = (py + q)\mathrm{d}x + (ry + s)\mathrm{d}w,$$
$$\omega^3 = \mathrm{d}y + x\mathrm{d}w,$$

where $p = p(w, x)$, $q = q(w, x)$, $r = r(w, x)$ and $s = s(w, x)$ are arbitrary functions of w and x such that $ps - qr = 1$, and $t = t(w, x)$, $u = u(w, x)$ are calculated from p, q, r and s as follows:

(to be continued)

Theorem (continuation)

$$t = \frac{uD}{E},$$

$$u = \exp\left(\frac{1}{2}\int P\mathrm{d}x\right)\left(\int\left(Q\exp\left(-\int P\mathrm{d}x\right)\mathrm{d}x\right)\right)^{1/2},$$

where

$$P = \frac{2q(ED'_x - DE'_x)}{E(pE - qD)}, \quad Q = \frac{2E^2}{pE - qD},$$

$$D = p'_w - r'_x, \quad E = q'_w - s'_x - px,$$

$$E(pE - qD) \neq 0.$$

Elliptic case. Only partial classes of solutions are known. The most interesting one is the following:

Theorem (E. Boeckx)

Let $F(z)$ be a holomorphic function of one complex variable z defined on the whole complex plane and such that $F(z) \neq 0$ everywhere. Let $P(x, y)$, $Q(x, y)$ be its real part and imaginary part. I.e., real functions of the two real variables x, y defined by $F(x + iy) = P(x, y) + iQ(x, y)$. Define the function $f(X, Y, Z)$ in $\mathbb{R}^3$ by

$$f(X,Y,Z) = P\left(\frac{X+YZ}{1+Z^2}, \frac{Y-XZ}{1+Z^2}\right) - ZQ\left(\frac{X+YZ}{1+Z^2}, \frac{Y-XZ}{1+Z^2}\right).$$

Then the hypersurface $W = f(X, Y, Z)$ of $\mathbb{R}^4$ with the induced metric determines a complete and irreducible elliptic 3-space of conullity two.

General dimension

Theorem

Let (M^{n+2}, g) *be an* $(n+2)$*-dimensional Riemannian manifold of conullity two,* $n > 0$*. Then, in a neighborhood* $\mathcal{U}$ *of each generic point* $p \in M$*, there is a system of local coordinates* $(w, x, y^1, \dots, y^n)$ *such that*

$$g = \sum_{i=1}^{n+2} (\omega^i)^2,$$

where

$$\begin{aligned} \omega^1 &= f(w, x, y^1, \dots, y^n)\mathrm{d}w, \\ \omega^2 &= A(w, x, y^1, \dots, y^n)\mathrm{d}x + C(w, x, y^1, \dots, y^n)\mathrm{d}w, \\ \omega^{i+2} &= \mathrm{d}y^i + H^i(w, x, y^1, \dots, y^n)\mathrm{d}x \quad (i = 1, \dots, n). \end{aligned}$$

In the dimension $n > 3$ one has to consider so-called algebraic rank, which is a number $r \in \{0, 1, 2, 3, 4\}$. It can be shown that all spaces of rank 0 are locally direct products $M_2 \times \mathbb{R}^n$. Of special interest are spaces of algebraic rank one. Here one can define analogous asymptotic foliations as in the 3-dimensional case, and the analogous geometric classification follows.

An analogue of ▸ Theorem 2 also holds:

Theorem

If (M^{n+2}, g) *admits an asymptotic foliation, then the functions* A, f *and* C *are linear functions of* $y^1, \dots, y^n$ *with coefficients which are functions of* w *and* x *only.*

Due to this simplification, E. Boeckx was able to express all Riemannian manifolds of conullity two and with algebraic rank 1, admitting at least one asymptotic foliation, in a closed form.

He proved also the following

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Every complete $(n+2)$-dimensional Riemannian manifold of conullity two has algebraic rank either zero or one.

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Boeckx also constructed (necessarily incomplete) explicit examples of spaces with algebraic rank 2, 3 and 4 (where asymptotic foliations never exist). Such examples exist only in dimensions > 3.