Singular motion of a symmetric Manakov top

Božidar Jovanović

Mathematical Institute SANU

joint work with
Vladimir Dragović and Borislav Gajić
XIX GEOMETRICAL SEMINAR
Dedicated to the memory of Professor Mileva Prvanović
Zlatibor, Serbia, August 28-September 4, 2016

Euler equations na so(n)

$$\dot{M}_{ij} = \sum_{k=1}^{n} \frac{b_i - b_j}{(b_k + b_i)(b_k + b_j)} M_{ik} M_{kj}$$

$M \in so(n)$ - angular momentum.

$\Omega \in so(n)$ - angular velocity.

$M_{ij} = (b_i + b_j)\Omega_{ij}$, $B = diag(b_1, \dots, b_n)$ - mass tensor.

Kinetic energy: $H = \frac{1}{2} \sum_{i<j} M_{ij}\Omega_{ij}$.

Lie–Poisson bracket: $\{f, g\}(M) = -\langle M, [\nabla f(M), \nabla g(M)] \rangle$.

Invariants: $p_0^k = tr(M^{2k}), \qquad k = 1, 2, \dots, [n/2]$.

Mishchenko integrals

Mishchenko [1970]: The following polynomials

$$J_k = \sum_{p=1}^{k} tr(B^{p-1} M B^{k-p} \Omega), \qquad k \geq 0,$$

are independent integrals of the Euler equations. In particular, the Euler top on so(4) is completely integrable.

Proof — by calculating the Hamiltonian vector fields at the point

$$M_0 = \sum_i E_i \wedge E_{i+1},$$

Manakov integrals

Manakov [1976]: Euler equations imply LA pair

$$\dot{L}(\lambda) = [L(\lambda), N(\lambda)], \quad L(\lambda) = M + \lambda A, \quad N(\lambda) = \Omega + \lambda B, \quad A = B^2,$$

and vice verse for $b_i \neq b_j$. Manakov integrals:

$$\mathcal{L} = \{\mathrm{tr}(M + \lambda A)^k \,|\, k = 1, 2, \dots, n,\ \lambda \in \mathbb{R}\},$$

Theorem 1. [Mishchenko and Fomenko, 1978] Assume $a_i \neq a_j$. We have

$$\rho_{so(n)} = \frac{1}{2}(\dim so(n) + \mathrm{rank}\, so(n)) = \frac{1}{2}\left(\frac{n(n-1)}{2} + \left[\frac{n}{2}\right]\right)$$

independent polynomials in $\mathcal{L}$. $\mathcal{L}$ is a complete commutative set on $so(n)$.

Symmetric rigid body

If $b_i = b_j$ then $\dot{M}_{ij} = 0$, i.e., M_{ij} is the first integrals. Suppose

$$a_1 = \cdots = a_{n_1} = \alpha_1, \ \ldots, \ a_{n+1-n_r} = \cdots = a_n = \alpha_r,$$
$$n_1 + n_2 + \cdots + n_r = n, \qquad n_1 \le n_2 \le n_3 \cdots \le n_r.$$

Let $so(n)_A = \{X \in so(n) \,|\, [X, A] = 0\} = so(n_1) \times \cdots \times so(n_r)$.

$\mathcal{S}$ - linar functions on $so(n)_A$ - Noether integrals. $\{\mathcal{L}, \mathcal{S}\} = 0$, tj., Manakov integrals are $Ad_{SO(n)_A}$–invariant, where $SO(n)_A \cong SO(n_1) \times \cdots \times SO(n_r)$ is subgroup of $SO(n)$ with the Lie algebra $so(n)_A$.

Theorem 2. [Bolsinov, 1995], [Dragovic, Gajic, B.J, 2009] $\mathcal{L} + \mathcal{S}$ is a complete set on $so(n)$. Symmetric Euler top is completely integrable in a non-commutative sense.

Algebra of $SO(n)_A$–invariant functions

Orthogonal decomposition

$$so(n) = so(n)_A \oplus \mathfrak{v} \cong so(n_1)\oplus so(n_2)\oplus\cdots\oplus so(n_r)\oplus\mathfrak{v},$$

Consider the restriction of the Lie–Poisson bracket

$$\{f,g\}_{\mathfrak{v}}(M) = -\langle M, [\nabla_{\mathfrak{v}} f(M), \nabla_{\mathfrak{v}} g(M)]\rangle, \qquad f,g\in\mathcal{F}.$$

$\mathcal{F}$: $\mathrm{Ad}_{SO(n)_A}$–invariant functions on $\mathfrak{v}$.
$\cong$ $SO(n)$–invariant functions on $T^*(SO(n)/SO(n)_A)$.
$\cong$ functions on singular Poisson variety $\mathfrak{v}/SO(n)_A$.

Theorem 3. [Dragovic, Gajic, B.J, 2009, 2014], [Mykytyuk 2014].
$\mathcal{L}_{\mathfrak{v}}$ is a complete commutative subset of $\mathcal{F}$: we have

$$\rho_{\mathfrak{v}} = \dim\mathfrak{v} - \frac{1}{2}\dim\mathcal{O}_{SO(n)}(M)\,(= \rho_{so(n)} - \dim so(n)_A \text{ for } M \text{ regular}),$$

for a generic $M\in\mathfrak{v}$, independent polynomials among restrictions $\mathcal{L}_{\mathfrak{v}}$.

Geodesic flows on homogeneous spaces G/H

Mishchenko and Fomenko [1981] stated the conjecture that noncommutative integrability implies the usual Liouville one by means of integrals that belong to the same functional class as noncommutative integrals. The conjecture is solved in C^∞–smooth case for infinite-dimensional algebras of integrals, as well as in polynomial and analytic cases for finite-dimensional algebras of integrals [Bolsinov, BJ, 2003], [Sadetov, 2004].

Let G/H be a homogeneous space of a compact connected Lie group G endowed with a normal metric ds_0^2 induced from a bi-invariant Riemannian metric on G. The geodesic flow of ds_0^2 is integrable in the noncommutative sense by means of analytic integrals polynomial in momenta $\mathcal{F} + \mathcal{G}$ [Bolsinov, BJ, 2001, 2004].

$\mathcal{F}$ - the algebra of G-invariant functions on T^*G/H, $\mathcal{G}$ - the algebra of Noether integrals with respect to the Hamiltonian G-action on T^*G/H.

Regular case: $so(n)_A = so(n-r)$, $2r \geq n$

By expanding Manakov integrals in λ, we get integrals in the form

$$p_l^k = tr(M^{2k}A^l + (\text{other permutations with } 2k \text{ M and } l \text{ A})),$$
$$k = 1, 2, \dots, [n/2], \qquad l = 0, 1, \dots, n-2k.$$

Note that the total number of polynomials p_l^k is:

$$(n-1) + (n-3) + \cdots + (n+1-2[n/2]) = \rho_{so(n)}.$$

Instead of $A = diag(a_1, a_2, \dots, a_{r-1}, a_r, a_{r+1}, \dots, a_{r+1})$, we can take $A = diag(a_{r+1}, a_1, a_{r+1}, a_2, a_{r+1}, \dots, a_r)$ and consider the vector spaces

$$W_k = span\{E_1 \wedge E_{2k}, E_2 \wedge E_{2k+1}, \dots\}, \qquad k = 1, 2, \dots, [n/2]$$
$$V_k = span\{E_1 \wedge E_{2k+1}, E_2 \wedge E_{2k+2}, \dots\},$$
$$\mathfrak{v}_k = V_k \cap \mathfrak{v}, \qquad k = 1, 2, \dots, [n/2].$$

Consider the gradients $\nabla p_l^k|_{\mathfrak{v}} \sim P_l^k$

$$P_l^k = M_0^{2k-1}A^l + (\text{other permutations with 2k-1 } M_0 \text{ and } l\ A)$$

at the point

$$M_0 = \sum_i E_i \wedge E_{i+1} \in \mathfrak{v}.$$

Then

$$\begin{aligned} P_l^k &\in W_1 \oplus W_2 \oplus \cdots \oplus W_k \subset \mathfrak{v}, \\ Q_l^k &= [M_0, P_l^k] \in \mathfrak{v}_1 \oplus \mathfrak{v}_2 \oplus \cdots \oplus \mathfrak{v}_k, \\ U_k &= \operatorname{span}\{Q_l^k\} \subset \mathfrak{v}_1 \oplus \mathfrak{v}_2 \oplus \cdots \oplus \mathfrak{v}_k. \end{aligned} \tag{1}$$

Lemma 1. $\operatorname{pr}_{\mathfrak{v}_k}(U_k) = \mathfrak{v}_k$.

Therefore, the relation (1) becomes an equality, and we obtain the required number of integrals.

Noncompleteness of the integrals $\mathcal{L}+\mathcal{S}$ on $\mathfrak{v}$

Lemma 2. [Dragovic, Gajic, BJ, 2009] $\mathcal{L}+\mathcal{S}$ is a complete set on so(n) at the points of $\mathfrak{v}$ iff $so(n)_A$ is a commutative subalgebra.

From Theorem 3 we know that the system induces completely integrable system on $\mathfrak{v}/SO(n)_A$.

Assume $SO(n)_A = SO(n-r)$. Let:

$$v_i = (M_{i,r+1}, \dots, M_{i,n})^T \in \mathbb{R}^{n-r}, \qquad i = 1, \dots, r$$

The following polynomials are basic $Ad_{SO(n-r)}$–invariants on $\mathfrak{v}$:

$$x_{ij} = M_{ij}, \qquad 1 \le i, j \le r,$$
$$y_{kl} = (v_k, v_l), \qquad 1 \le k, l \le r,$$

and, for $n = 2r$, $z = \det(v_1, \dots, v_r)$, which is functionally dependent from the other invariants.

Reduced system

Lemma 3. The Hamiltonian flow of the reduced rigid body metric

$$H = \frac{1}{2} \sum_{1 \le i < j < r} \frac{x_{ij}^2}{b_i + b_j} + \frac{1}{2} \sum_{i=1}^{r} \frac{y_{ii}}{b_i + b_{r+1}}$$

has the following equations of motion in the coordinates x_{ij}, y_{kl}:

$$\begin{aligned} \dot{x}_{ij} &= \sum_{k=1}^{r} \frac{b_i - b_j}{(b_k + b_i)(b_k + b_j)} x_{ik} x_{kj} + \frac{b_j - b_i}{(b_{r+1} + b_i)(b_{r+1} + b_j)} y_{ij}, \\ \dot{y}_{ii} &= 2 \sum_{k=1}^{r} \frac{b_i - b_{r+1}}{(b_k + b_i)(b_k + b_{r+1})} x_{ik} y_{ik}, \\ \dot{y}_{ij} &= \sum_{k=1}^{r} \frac{b_i - b_{r+1}}{(b_k + b_i)(b_k + b_{r+1})} x_{ik} y_{kj} + \sum_{k=1}^{r} \frac{b_{r+1} - b_j}{(b_k + b_j)(b_k + b_{r+1})} x_{kj} y_{ki}. \end{aligned} \tag{2}$$

The reduced bracket

$$
\begin{aligned}
&\{x_{ij}, x_{jk}\}_{\mathfrak{v}} = -x_{ik},\\
&\{x_{ij}, x_{kl}\}_{\mathfrak{v}} = 0 \quad \text{for} \quad \{i,j\}\cap\{k,l\} = \emptyset,\\
&\{x_{ij}, y_{jj}\}_{\mathfrak{v}} = -2y_{ij},\\
&\{x_{ij}, y_{ji}\}_{\mathfrak{v}} = y_{jj} - y_{ii},\\
&\{x_{ij}, y_{jk}\}_{\mathfrak{v}} = -y_{ik},\\
&\{x_{ij}, y_{kl}\}_{\mathfrak{v}} = 0 \quad \text{for} \quad \{i,j\}\cap\{k,l\} = \emptyset,\\
&\{y_{ii}, y_{jj}\}_{\mathfrak{v}} = 4x_{ij}y_{ij},\\
&\{y_{ii}, y_{ij}\}_{\mathfrak{v}} = 2x_{ij}y_{ii},\\
&\{y_{ij}, y_{jk}\}_{\mathfrak{v}} = x_{ij}y_{jk} + y_{ij}x_{jk} + x_{ik}y_{jj},\\
&\{y_{ii}, y_{jk}\}_{\mathfrak{v}} = 2x_{ij}y_{ik} + 2x_{ik}y_{ij},\\
&\{y_{ij}, y_{kl}\}_{\mathfrak{v}} = x_{il}y_{jk} + x_{jl}y_{ik} + x_{ik}y_{jl} + x_{jk}y_{il} \quad \text{for} \quad \{i,j\}\cap\{k,l\} = \emptyset.
\end{aligned} \tag{3}
$$

Theorem 4. The reduced system (2) is completely integrable. The dimension of the invariant tori is

$$\Delta_r = r^2 - \frac{(r+1)r}{2}$$

and a complete set of independent commuting SO(n − r)–invariant integrals on $\mathfrak{v}$ is given by

$$\begin{aligned}
&p_0^1|_{\mathfrak{v}}, \quad p_1^1|_{\mathfrak{v}}, \quad \dots, \quad p_{r-2}^1|_{\mathfrak{v}}, \quad p_{r-1}^1|_{\mathfrak{v}},\\
&p_0^2|_{\mathfrak{v}}, \quad p_1^2|_{\mathfrak{v}}, \quad \dots, \quad p_{r-2}^2|_{\mathfrak{v}},\\
&\vdots\\
&p_0^{r-1}|_{\mathfrak{v}}, \, p_1^{r-1}|_{\mathfrak{v}},\\
&p_0^r|_{\mathfrak{v}}.
\end{aligned}$$

The reconstruction problem

Lemma 4. Consider the r–dimensional subspaces

$$\mathfrak{w}_k = \mathrm{span}\{E_1 \wedge E_k, E_2 \wedge E_k, \dots, E_r \wedge E_k\} \subset \mathfrak{v}, \qquad k = r+1, \dots, n$$

The Euler equations on $\mathfrak{w}_k$

$$\dot{M}_{ik} = \sum_{j=1}^{r} \frac{b_i - b_{r+1}}{(b_j + b_i)(b_j + b_{r+1})} x_{ij} M_{jk}, \qquad 1 \le i \le r < k \le n,$$

have the integrals

$$F_k = \sum_{i=1}^{r} \frac{M_{ik}^2}{b_{r+1}^2 - b_i^2} = \sum_{i=1}^{r} \frac{M_{ik}^2}{a_{r+1} - a_i}, \qquad k = r+1, \dots, n.$$

Lemma 5. The r–plane π in $\mathbb{R}^{n-r}$ spanned by the vectors $v_1, \dots, v_r$ is invariant under the Euler equations restricted to $\mathfrak{v}$. Equivalently, the Plucker coordinates of π

$$S_{i_1 \dots i_r} = (i_1 \dots i_r) \text{ minor of } [v_1, \dots, v_r], \qquad 1 \le i_1 < i_2 < \cdots < i_r \le n-r,$$

are integrals of the motion.

Among the Plücker coordinates, there are $(n-2r)r+1$ independent ones. The sum of their squares is an SO(n − r)–invariant function, a Casimir function of the reduced bracket (3). Geometrically, by means of the integrals (15) we reduce the problem to the case $n = 2r$.

SO(n − 2)–symmetry

In the case $r = 2$, we have $\Delta_2 = 1$, and a generic trajectory on $\mathfrak{v}/SO(n-r)$ is periodic. The integrals (14), (15) can be written in the form

$$F_k = (b_3^2 - b_2^2)M_{1k}^2 + (b_3^2 - b_1^2)M_{2k}^2 = (a_3 - a_2)M_{1k}^2 + (a_3 - a_1)M_{2k}^2,$$
$$S_{ij} = M_{1i}M_{2j} - M_{1j}M_{2i}, \qquad k = 3, \dots, n, \qquad 3 \le i < j \le n.$$

Together with the independent Manakov first integrals $p_0^1|_{\mathfrak{v}}, p_0^2|_{\mathfrak{v}}, p_1^1|_{\mathfrak{v}}$, they imply the following statement

Theorem 5. In the case of SO(n − 2)–symmetry, the generic trajectories on $\mathfrak{v}$ are periodic. The trajectories can be expressed by means of elliptic functions.

We can consider $\mathfrak{v}/SO(n-2)$ in the coordinates (x_1, x_2, x_3, x_4), $x_1, x_4 \in \mathbb{R}$, $x_2, x_3 \geq 0$, where

$$x_1 = x_{12} = M_{12}, \quad x_4 = y_{12} = M_{13}M_{23} + \cdots + M_{1n}M_{2n},$$
$$x_2 = y_{11} = M_{13}^2 + \cdots + M_{1n}^2, \quad x_3 = y_{22} = M_{23}^2 + \cdots + M_{2n}^2.$$

Note that in the case $n = 4$, there is an additional basic $SO(2)$–invariant polynomial $x_5 = M_{13}M_{24} - M_{14}M_{23}$ and the relation $x_2x_3 - x_4^2 = x_5^2$. The reduced bracket gets the form:

$$\{x_1, x_2\}_{\mathfrak{v}} = 2x_4, \qquad \{x_1, x_3\}_{\mathfrak{v}} = -2x_4,$$
$$\{x_1, x_4\}_{\mathfrak{v}} = x_3 - x_2, \quad \{x_2, x_3\}_{\mathfrak{v}} = 4x_1x_4,$$
$$\{x_2, x_4\}_{\mathfrak{v}} = 2x_1x_2, \qquad \{x_3, x_4\}_{\mathfrak{v}} = -2x_1x_2,$$

having two independent Casimir functions, given by $\mathrm{Ad}_{SO(n)}$–invariants

$$p_0^1 = -2x_1^2 - 2x_2 - 2x_3,$$
$$p_0^2 = 2x_1^4 + 4x_1^2(x_2 + x_3) + 2x_2^2 + 2x_3^2 + 4x_4^2.$$

We can write down the above invariants in terms of the quadratic Casimir functions:

$$I_1 = x_1^2 + x_2 + x_3,$$
$$I_2 = x_2 x_3 - x_4^2 \qquad (= x_5^2 \quad \text{for} \quad n = 4).$$

The reduced flow

$$\dot{x}_1 = (b_2^2 - b_1^2)x_4, \qquad \dot{x}_2 = 2(b_1^2 - b_3^2)x_1 x_4,$$
$$\dot{x}_3 = 2(b_3^2 - b_2^2)x_1 x_4, \qquad \dot{x}_4 = (b_3^2 - b_2^2)x_1 x_2 + (b_1^2 - b_3^2)x_1 x_3.$$

Here we supposed $(b_1 + b_2)(b_2 + b_3)(b_3 + b_1) = 1$.
The equations transform to

$$\dot{x}_1^2 = P(x_1; I_1, I_2, F),$$

where the polynomial $P(x; I_1, I_2, F)$ is given by

$$P = (b_1^2 - b_3^2)(b_3^2 - b_2^2)(I_1 - x^2)^2 + F(b_1^2 - 2b_3^2 + b_2^2)(I_1 - x^2) - F^2 - (b_2^2 - b_1^2)^2 I_2.$$

and F is linear first integral $F = (b_2^2 - b_3^2)x_2 + (b_1^2 - b_3^2)x_3$.

Spectral curve

The spectral curve Γ : $P(\lambda,\mu)=\det(M+\lambda A-\mu I)=0$, has the form

$$\Gamma: \quad P(\lambda,\mu)=(\lambda a_1-\mu)(\lambda a_2-\mu)(\lambda a_3-\mu)^2+P_{20}\mu^2+P_{11}\mu\lambda+P_{02}\lambda^2+P_{00}=0$$

The coefficients P_{ij} are first integrals and they are given explicitly:

$$\begin{aligned}
P_{00} &= (M_{13}M_{24}-M_{14}M_{23})^2 = x_5^2 = I_2,\\
P_{20} &= M_{12}^2+M_{13}^2+M_{14}^2+M_{23}^2+M_{24}^2 = x_1^2+x_2+x_3 = I_1,\\
P_{11} &= -2a_3M_{12}^2-(a_3+a_1)(M_{24}^2+M_{23}^2)-(a_3+a_2)(M_{14}^2+M_{13}^2)\\
&= -2a_3x_1^2-(a_2+a_3)x_2-(a_1+a_3)x_3,\\
P_{02} &= a_3(a_3M_{12}^2+a_2(M_{14}^2+M_{13}^2)+a_1(M_{24}^2+M_{23}^2))\\
&= a_3(a_3x_1^2+a_2x_2+a_1x_3).
\end{aligned}$$

There is a relation among these first integrals: $a_3^2P_{20}+a_3P_{11}+P_{02}=0$.

The curve Γ is regular in the affine part. It is a 4–fold covering $\pi : \Gamma \to \mathbb{P}^1(\lambda)$. The intersection of the curve with the line at infinity is $\pi^{-1}(\infty) = \mathrm{P}_1 + \mathrm{P}_2 + \mathrm{P}_3$, where P_3 is a singular point.

P_3 is not an ordinary double point. The multiplicity of P_3 is 2, the number of local brunches is 2 (with a common tangent), and its δ–invariant is equal to 2. Therefore, the normalization $\tilde{\Gamma}$ of Γ is an elliptic curve:

$$\mathrm{genus}(\tilde{\Gamma}) = \frac{3 \cdot 3}{2} - 2 = 1.$$

In the Manakov nonsymmetric case ($\mathrm{a}_3 \neq \mathrm{a}_4$) the spectral curve is nonsingular and of genus 3.

In the symmetric case $\mathrm{a}_3 = \mathrm{a}_4$, with the additional condition $\mathrm{M}_{34} \neq 0$, the spectral curve is singular. But, opposite to the case studied above, the point P_3 is an ordinary double point, and thegenus of the normalization of the spectral curve is 2.

SO(n − 3)–symmetry

For r = 3 there are 6 independent Manakov integrals $\{p_0^1|_{\mathfrak{v}}, p_0^2|_{\mathfrak{v}}, p_0^3|_{\mathfrak{v}}, p_1^1|_{\mathfrak{v}}, p_1^2|_{\mathfrak{v}}, p_2^1|_{\mathfrak{v}}\}$ and the dimension of the invariant tori on $\mathfrak{v}/SO(n-3)$ is $\Delta_3 = 3$.

We can reduce the problem to the case $n = 6$ when $\dim \mathfrak{v} = 12$.

Then apart from the Manakov first integrals, there are two independent quadratic first integrals among F_4, F_5, F_6. As a result, the invariant manifolds are 4-dimensional.

Reconstruction

For $b_1, b_2, b_3 < b_4$, in the new coordinates

$$\omega_1 = -\frac{x_{23}}{b_2+b_3}\sqrt{\frac{(b_4-b_2)(b_4-b_3)}{(b_4+b_2)(b_4+b_3)}}, \qquad \gamma_{1k} = \frac{M_{1k}}{\sqrt{b_4^2-b_1^2}},$$

$$\omega_2 = \frac{x_{13}}{b_3+b_1}\sqrt{\frac{(b_4-b_1)(b_4-b_3)}{(b_4+b_1)(b_4+b_3)}}, \qquad \gamma_{2k} = \frac{M_{2k}}{\sqrt{b_4^2-b_2^2}},$$

$$\omega_3 = -\frac{x_{12}}{b_1+b_2}\sqrt{\frac{(b_4-b_1)(b_4-b_2)}{(b_4+b_1)(b_4+b_2)}}, \qquad \gamma_{3k} = \frac{M_{3k}}{\sqrt{b_4^2-b_3^2}},$$

we get the reconstruction equations in the form of the Poisson equations

$$\dot{\gamma}_k = \gamma_k \times \omega(t), \qquad k = 4, \dots, n,$$

where $\gamma_k = (\gamma_{1k}, \gamma_{2k}, \gamma_{3k}), \omega = (\omega_1, \omega_2, \omega_3)$.

Integrability

In general, the Poisson equation are not necessarily solvable by quadratures, e.g., see [Fedorov, Maciejewski, Przybylska, 2009]. To solve by quadratures, one needs one particular solution [Kozlov, 2014].

It can be shown that the commuting Hamiltonian vector fields of the Manakov integrals p_1^1, p_1^2, p_2^1 are tangent to $\mathcal{M}_c$. Since we have an invariant volume form on $\mathcal{M}_c$, from the Euler-Jacobi-Lie theorem [Kozlov, 2013], we get:

Theorem 6. $\mathfrak{v}$ is almost everywhere foliated on 4 dimensional manifolds $\mathcal{M}_c$. The motion on $\mathcal{M}_c$ is solvable by quadratures.

Spectral curve

For simplicity, we denote

$x_1 = x_{32}, x_2 = x_{13}, x_3 = x_{21}, y_1 = y_{11}, y_2 = y_{22}, y_3 = y_{33}$.

The polynomial $P(\mu, \lambda)$, determining the spectral curve $\Gamma: \det(M + \lambda A - \mu I) = P(\mu, \lambda) = 0$, reads

$$P(\mu, \lambda) = \prod_{i=1}^{6}(\lambda a_i - \mu) + \sum_{k=0}^{2} \sum_{i+j=2k} P_{ij}\mu^i \lambda^j.$$

The coefficients

$$\begin{aligned} P_{00} &= \det(v_1, v_2, v_2)^2 = \det(y_{ij}), \\ P_{20} &= 2(x_1x_2y_{12} + x_3x_1y_{31} + x_2x_3y_{23}) - (y_{12}^2 + y_{23}^2 + y_{31}^2) \\ &\quad + x_1^2y_1 + x_2^2y_2 + x_3^2y_3 + y_1y_2 + y_2y_3 + y_3y_1, \\ P_{40} &= x_1^2 + x_2^2 + x_3^2 + y_1 + y_2 + y_3, \end{aligned}$$

are the restriction to $\mathfrak{v}$ of $\mathrm{Ad}_{SO(6)}$–invariant polynomials (Casimir functions)

The other $\mathrm{Ad}_{SO(3)}$–invariant integrals:

$$
\begin{aligned}
P_{11} &= -2a_4(x_1^2y_1 + x_2^2y_2 + x_3^2y_3) + (a_1 + a_4)(y_{23}^2 - y_2y_3) + (a_2 + a_4)(y_1 \\
&\quad +(a_3 + a_4)(y_{12}^2 - y_1y_2) - 4a_4(x_1x_2y_{12} + x_3x_1y_{31} + x_2x_3y_{23}),\\
P_{0,2} &= a_4^2(x_1^2y_1 + x_2^2y_2 + x_3^2y_3) + a_1a_4(y_2y_3 - y_{23}^2) + a_2a_4(y_3y_1 - y_{13}^2)\\
&\quad +a_3a_4(y_1y_2 - y_{12}^2) + 2a_4^2(x_1x_3y_{13} + x_2x_3y_{23} + x_1x_2y_{12}),\\
P_{31} &= -(a_1 + a_2 + 2a_4)y_3 - (a_1 + a_3 + 2a_4)y_2 - (a_2 + a_3 + 2a_4)y_1\\
&\quad -(a_1 + 3a_4)x_1^2 - (a_2 + 3a_4)x_2^2 - (a_3 + 3a_4)x_3^2,\\
P_{22} &= (a_2a_3 + 2a_2a_4 + 2a_3a_4 + a_4^2)y_1 + (a_1a_3 + 2a_3a_4 + 2a_1a_4 + a_4^2)y_2\\
&\quad +(2a_1a_4 + 2a_2a_4 + a_1a_2 + a_4^2)y_3\\
&\quad +3(a_4^2 + a_1a_4)x_1^2 + 3(a_4^2 + a_2a_4)x_2^2 + 3(a_4^2 + a_3a_4)x_3^2,\\
P_{13} &= -a_4^2\left((3a_1 + a_4)x_1^2 + (3a_2 + a_4)x_2^2 + (3a_3 + a_4)x_3^2\right)\\
&\quad -a_4\left((2a_2a_3 + a_2a_4 + a_3a_4)y_1 + (2a_1a_3 + a_3a_4 + a_1a_4)y_2 + (2a_1a_2\right.\\
P_{04} &= a_4^2\left(a_4(a_1x_1^2 + a_2x_2^2 + a_3x_3^2) + a_2a_3y_1 + a_3a_1y_2 + a_1a_2y_3\right).
\end{aligned}
$$

The description of Γ is a modification of a description of a spectral curve in the regular case considered by Haine [1984], where $a_i \neq a_j$. For a generic initial conditions x_{ij}, y_{ij}, the curve is regular in the affine part.

Γ is a 6–fold covering $\pi : \Gamma \to \mathbb{P}^1(\lambda)$ and $\pi^{-1}(\infty) = P_1 + P_2 + P_3 + P_4$, where the points P_1, P_2, P_3 are regular, while the point P_4 is a singular point with multiplicity 3 and δ–invariant equal to 6.

Let $\tilde{\Gamma}$ be the normalization of Γ. Its genus is

$$\mathrm{genus}(\tilde{\Gamma}) = \frac{5 \cdot 4}{2} - 6 = 4.$$

Note that there is an involution $\sigma : (\lambda, \mu) \mapsto (-\lambda, -\mu)$. It can be proved that the curve $\tilde{\Gamma}/\sigma$ is an elliptic curve, thus the dimension of the Prym variety $\mathrm{Prym}(\tilde{\Gamma}, \sigma)$ is 3 and it is equal to the dimension of the invariant tori of the reduced flow.

- Dragović, V., Gajić B. and Jovanović, B.: Singular Manakov Flows and Geodesic Flows of Homogeneous Spaces of SO(n), Transfomation Groups, (2009), arXiv:0901.2444
- Dragović, V., Gajić B. and Jovanović, B.: Systems of Hess–Appelrot type and Zhukovskii property , International Journal of Geometric Methods in Modern Physics, (2009), arXiv:0912.1875.
- Dragović, V., Gajić B. and Jovanović, B.: On the completeness of the Manakov integrals, Fundametalnaya i prikldnaya matematika, (2015) arXiv:1504.07221 (Dedicated to Academician Anatoly Timofeevich Fomenko on the occasion of his 70th birthday)
- Dragović, V., Gajić B. and Jovanović, B.: Note on Free Symmetric Rigid Body Motion, Regular and Chaotic Dynamics, (2015). (Dedicated to Academician Valery Vasil'evich Kozlov on the occasion of his 65th anniversary)
- Mykytyuk, I. V.: Integrability of geodesic flows for metrics on suborbits of the adjoint orbits of compact groups, Transfomation Groups, (2016), arXiv:1402.6526.

Thank you for your attention!