Solution of the qc Yamabe equation on a 3-Sasakian manifold and extremals of the Sobolev-Folland-Stein inequality on the quaternionic Heisenberg groups

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based on joint works with Ivan Minchev, Alexander Petkov & Dimiter Vassilev

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The Folland-Stein inequality. $p^* = \frac{pQ}{Q-p}$

Theorem (G.Folland & E.Stein)

Let $\Omega \subset \mathbf{G}$ *be an open set in a group of Heisenberg type (Carnot group) of homogeneous dimension Q. For any* $1 < p < Q$ *there exists* $S_p = S_p(\mathbf{G}) > 0$ *such that for* $u \in C_o^\infty(\Omega)$

$$\left(\int_\Omega |u|^{p^*}\, dH(g)\right)^{1/p^*} \leq S_p \left(\int_\Omega |Xu|^p\, dH(g)\right)^{1/p}.$$

Here, $|Xu|^2 = \sum_{i=1}^m |X_i u|^2$ where $X_1, \ldots, X_m$ is a basis of the horizontal space.

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- When $p = 2$, -the problem reduces to the solvability of the Yamabe equation

$$\sum_{i=1}^m X_i^2 u = -u^{\frac{Q+2}{Q-2}}.$$

- Zero dimensional center - $\mathbf{G} = \mathbb{R}^n$.
  The problem can be translated to the standard sphere S^n via the stereographic map - leads to involve Riemannian geometry - the Riemannian Yamabe problem.

Let (M, g) - compact, Riemannian manifold, $2^* = \frac{2n}{n-2}$. If $\bar{g} = u^{4/(n-2)}g$, then

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- **Yamabe invariant**: $\Upsilon([g]) = \inf\{\Upsilon(u) : \int_M u^{2^*} dv_g = 1, \ u > 0\}$.

The classical Obata

The main idea is to replace the non-linear Yamabe on S^n with a geometrical system of equations which can be solved-namely conformal deformations preserving the Einstein condition -relies on the Obata theprem:

Theorem (Obata)

Let (S^n, g_{st}) be the unit sphere in $\mathbb{R}^{n+1}$. If g is a Riem. metric, $g = \phi^2 g_{st}$, and $Scal_g = S = const$, then the metric g is again Einstein.

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"Proof" Relies on a non-trivial 'divergence-type' formula (Obata).
There ia a conceptual proof found by Lee-Parker: $0 = Ric^{st}_o = Ric_o + \frac{n-2}{\phi}(\nabla^2\phi)_o$. Thus, $(\nabla^2\phi)_o \ = \ -\frac{\phi}{n-2}Ric_o$. Using $2\nabla^*(Ric_o) = \nabla S = 0$, from the contracted Bianchi and S=const, it follows

$$div\ Ric_o(\nabla\phi, .) = -\frac{\phi}{n-2}|Ric_o|^2.$$

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One-dimensional center-The CR Obata

One-dimensional center - (complex) Heisenberg group $\mathbf{G} = \mathbb{C}^n \times \mathbb{R}$.

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Theorem (D. Jerison & J. Lee: JAMS'88)

If θ is the contact form of a pseudo-Hermitian structure proportional to the standard contact form $\bar{\theta}$ on the unit sphere in $\mathbb{C}^{n+1}$, $\theta = f\bar{\theta}$ and $Scal_\theta =$const, then θ is again pseudo-Einstein.

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CR automorphism of the sphere are relatively easy to described due to

Theorem (J. Lee: Am.J.Math.'88)

If $(M, \bar{\theta})$ is pseudo-Einstein, then $\theta = e^{2u}\bar{\theta}$ is pseudo-Einstein iff u is CR-pluriharmonic on M.

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Note that this method does not give all solutions of the CR-Yamabe equation on the unit sphere/complex Heisenberg group while the geometrical approach of Jerison and Lee gives all solutions.

The CR Yamabe problem

The CR-Yamabe problem is: Given a compact strongly pseudo-convex CR manifold $(M^{2n+1}, \theta) \subset \mathbb{C}^{n+1}$ find a smooth function f such that $\bar{\theta} = e^f \theta$ has constant pseudohermitian scalar curvature.
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Theorem (D. Jerison & J. Lee J. Diff. Geom. '87-'89)

a) *$\Upsilon([\theta]) \leq \Upsilon(S^{2n+1})$, where $S^{2n+1} \subset \mathbb{C}^{n+1}$ is the sphere with its standard CR structure. If $\Upsilon([\theta]) < \Upsilon(S^{2n+1})$, then the Yamabe equation has a solution. [D. Jerison & J. Lee '87]*

b) *If $n \geq 2$ and M is not locally CR equivalent to S^{2n+1}, then $\Upsilon([\theta]) < \Upsilon(S^{2n+1})$. [D. Jerison & J. Lee '89]*

$$Y(\theta_\epsilon) = \begin{cases} Y(S^{2n+1})\left(1 - c_n|S(q)|^2\epsilon^4\right) + \mathcal{O}(\epsilon^5), & n \geq 2; \\ Y(S^5)\left(1 - c_2|S(q)|^2\epsilon^4 \ln\epsilon\right) + \mathcal{O}(\epsilon^4), & n = 2. \end{cases}$$

c) *If $n = 1$ or M is locally CR equivalent to S^{2n+1}, then the Yamabe equation has a solution. [R. Yacoub '01, N. Gamara & R. Yacoub, 01]*

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S is the Chern-Moser tensor and one of the key point is the use of the Chern-Moser theorem

Theorem (Chern-Moser: Acta Math '74)

A $(2n+1), n > 1$*-dimensional CR manifold is locally CR-equivalent to the sphere exactly when the Chern-Moser tensor vanishes,* $S = 0$.

- $\mathbb{H}$-quaternions, $q = t + ix + jy + kz$, where $t, x, y, z \in \mathbb{R}$ and $i,\ j,\ k$ satisfy the multiplication rules

$$i^2 \ = \ j^2 \ = \ k^2 \ = \ -1 \text{ and } ijk \ = \ -1.$$

Three-dimensional center-Quaternionic Heisenberg Group $G(\mathbb{H})$

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$\boldsymbol{G}(\mathbb{H}) = \mathbb{H}^n \times \mathrm{Im}\mathbb{H}$, $\quad (q, \omega) \in \boldsymbol{G}(\mathbb{H})$,

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Left-invariant horizontal vector fields

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The homogeneous dimension of $\boldsymbol{G}(\mathbb{H})$ is $Q = 4n + 6, \qquad 2^* = \frac{2Q}{Q-2} = \frac{2n+3}{n+1}.$

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The 'quaternionic contact Yamabe equation' on $\boldsymbol{G}\,(\mathbb{H})$ is

$$\sum_{\alpha=1}^{n} (T_\alpha^2 + X_\alpha^2 + Y_\alpha^2 + Z_\alpha^2)\, u = -\frac{n+1}{4(n+2)} u^{2^*-1}.Const.$$

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Definition

A quaternionic contact (qc) manifold $(M, g, \mathbb{Q})$ *is a* $4n+3$*-dimensional manifold M with a codimension three distribution H locally given as the kernel of a 1-form* $\eta = (\eta_1, \eta_2, \eta_3)$ *with values in* $\mathbb{R}^3$*. In addition H has an* $Sp(n)Sp(1)$ *structure, i.e. we have*

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Example 3-Sasakian manifolds M: The cone $C = M \times \mathbb{R}$ with the metric $g_{con} = dt^2 + t^2 g$ is a hyperkähler metric.

- Given η (and H) there exists at most one triple of a.c.str. and metric g that are compatible.
- Rotating η we obtain the same qc-structure.
- Conformal transformations $\eta = (\eta_1, \eta_2, \eta_3)$, $\mu \in \mathcal{C}^\infty(M)$, $\mu > 0$, $\Psi \in \mathcal{C}^\infty(M : SO(3))$.

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Theorem (O. Biquard: Asterisque '99)

Under the above conditions and $n > 1$, there exists a unique supplementary distribution V of H in TM and a linear connection ∇ on M, s.t.,

1. *V and H are parallel*
2. *g and Q are parallel*
3. *torsion $T(A,B) = \nabla_A B - \nabla_B A - [A,B]$ satisfies*
   - $\forall X, Y \in H, \quad T_{X,Y} = -[X,Y]|_V \in V$
   - $\forall \xi \in V, \quad T_\xi := (X \mapsto (T_{\xi,X})_H) \in (sp(n) + sp(1))^\perp$

- Note: V is generated by the Reeb vector fields $\{\xi_1, \xi_2, \xi_3\}$ defined with the Biquard conditions

$$\eta_s(\xi_k) = \delta_{sk}, \qquad (\xi_s \lrcorner d\eta_s)_{|H} = 0, \qquad (\xi_s \lrcorner d\eta_k)_{|H} = -(\xi_k \lrcorner d\eta_s)_{|H}.$$

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- If the dimension of M is seven, $n = 1$, the above conditions do not always hold. Duchemin shows that if we assume, in addition, the existence of Reeb vector fields as above, then there is a connection as before. Henceforth, by a qc structure in dimension 7 we shall mean a qc structure satisfying the Reeb conditions

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- The Biquard connection is invariant under rotations and homotheties, i.e. the forms $\bar{\eta} = a\Psi\eta, \quad \Psi \in SO(3), a = const$. have the same Biquard connection.

Basic examples-Cayley transform

- Contact 3-form on the Quaternionic Heizenberg group $\boldsymbol{G}\,(\mathbb{H})$.
  $\tilde{\Theta} = (\tilde{\Theta}_1, \tilde{\Theta}_2, \tilde{\Theta}_3) = \frac{1}{2}\,(d\omega - q \cdot d\bar{q} + dq \cdot \bar{q})$ or

$$\tilde{\Theta}_1 = \frac{1}{2}\,dx - x^{\alpha}dt^{\alpha} + t^{\alpha}dx^{\alpha} - z^{\alpha}dy^{\alpha} + y^{\alpha}dz^{\alpha}$$
$$\tilde{\Theta}_2 = \frac{1}{2}\,dy - y^{\alpha}dt^{\alpha} + z^{\alpha}dx^{\alpha} + t^{\alpha}dy^{\alpha} - x^{\alpha}dz^{\alpha}$$
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- Left-invariant Reeb (vertical) vector fields ξ_1, ξ_2, ξ_3 are

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- $(\boldsymbol{G}\,(\mathbb{H}), \tilde{\theta})$- the flat model.

- 3-Sasakian sphere: Contact 3-form on the sphere $S^{4n+3} = \{|q|^2 + |p|^2 = 1\} \subset \mathbb{H}^n \times \mathbb{H}$,

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- Identify $\boldsymbol{G}\,(\mathbb{H})$ with the boundary Σ of a Siegel domain in $\mathbb{H}^n \times \mathbb{H}$,

$$\Sigma \;=\; \{(q', p') \in \mathbb{H}^n \times \mathbb{H} \;:\; \mathrm{Re}\, p' \;=\; |q'|^2\},$$

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- $\mathcal{C}^*\,\tilde{\Theta} \;=\; \frac{1}{2\,|1+p\,|^2}\,\lambda\,\tilde{\eta}\,\bar{\lambda}$, $\quad\lambda$-unit quaternion (eg. of *conformal quaternionic contact map*).

Curvature of a Quaternionic Contact Structure

- curvature: $R(A,B)C = [\nabla_A, \nabla_B]C - \nabla_{[A,B]}C, \quad R(A,B,C,D) = g(R(A,B)C,D)$;
- qc-Ricci tensor: $Ric(X,Y) = tr_H\{Z \mapsto \mathcal{R}(Z,X)Y\} = R(e_a, X, Y, e_a)$ for $e_a, X, Y \in H$
- qc-Ricci 2-forms $\rho_s(X,Y) = R(X,Y,e_a,I_s e_a)$
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Theorem (I/ Vasilev: J.Math. Pure Appl.'2010)

The following tensors

- $R(X,Y,Z,V) - R(Z,V,X,Y)$
- $4R_{[-1]}(X,Y,Z,V) = 3R(X,Y,Z,V) - R(I_1X,I_1Y,Z,V) - R(I_2X,I_2Y,Z,V) - R(I_3X,I_3Y,Z,V)$
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are determined by the (horizontal!) torsion, $T_{\xi_j}, j = 1,2,3$ *and the qc scalar curvature* $Scal$.

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are determined by the (horizontal!) torsion, $T_{\xi_j}, j = 1,2,3$ *and the qc scalar curvature* $Scal$.

Corollary

A QC manifold is locally isomorphic to the quaternionic Heisenberg group exactly when the curvature of the Biquard connection restricted to H vanishes, $R_{|_H} = 0$.

Conformal transformations

$\eta = (\eta_1, \eta_2, \eta_3)$, $\mu \in \mathcal{C}^\infty(M)$, $\mu > 0$, $\Psi \in \mathcal{C}^\infty(M : SO(3))$.

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Lemma (O. Biquard: Asterisque '99)

If $\bar{\eta} = u^{4/(Q-2)} \eta$, *then*

$$4 \frac{Q+2}{Q-2} \triangle u - u\, Scal = -u^{2^*-1} \overline{Scal},$$

where $\triangle u = tr_H(\nabla du)$, $Q = 4n+6$, $2^* = 2Q/(Q-2)$.

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Set $Scal = 0$ and $\overline{Scal} = const$ this is precisely the Yamabe equation on $\boldsymbol{G}\,(\mathbb{H})$.

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The qc-Yamabe problem: Find solutions to

$$4\frac{Q+2}{Q-2}\,\triangle u - \; u\,\mathrm{Scal} \; = \; -u^{2^*-1}\,\overline{\mathrm{Scal}}, \quad \overline{Scal} = const.$$

- Yamabe functional is

$$\Upsilon(u) = \int_M (4\frac{Q+2}{Q-2}\,|\nabla_H u|^2 \; + \; \mathrm{Scal}\,u^2)\,dv_g.$$

- The Yamabe invariant is the infimum

$$\Upsilon([\eta]) = \inf_u\{\Upsilon(u) : \; \int_M u^{2^*}\,dv_g \; = \; 1, \; u > 0\}.$$

Theorem (1. I/ I. Minchev, D. Vassilev' 2015)

Let $\tilde{\eta} = \frac{1}{2h}\eta$, $\tilde{\eta}$ *standard quaternionic contact structure on the 3-Sasakian sphere of dimension* $4n+3$. *If* η *has constant qc-scalar curvature, then up to a multiplicative constant* η *is obtained from* $\tilde{\eta}$ *by a conformal quaternionic contact automorphism*

$$\phi \in \mathit{Diff}(M), \quad \eta = \phi^* \tilde{\eta} = \mu \Psi \tilde{\eta}, \quad \Psi \in \mathcal{C}^\infty(M : SO(3)),$$

The qc-Yamabe equation, the main result

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Theorem 1 shows, in particular, the following

Corollary

If Φ satisfies the qc Yamabe equation on the quaternionic Heisenberg group $\boldsymbol{G}\,(\mathbb{H})$,

$$\frac{4(Q+2)}{Q-2} \triangle_{\tilde{\Theta}} \Phi = -S_\Theta\, \Phi^{2^*-1},$$

for some constant S_Θ, then up to a left translation the function $\Phi = (2h)^{-(Q-2)/4}$ and h is given by

$$h(q,\omega) \;=\; c_0 \left[\left(\sigma \;+\; |q+q_0|^2\right)^2 \;+\; |\omega \;+\; \omega_o \;+\; 2\,\mathit{Im}\, q_o\, \bar{q}|^2 \right], \tag{1}$$

for some fixed $(q_o, \omega_o) \in \boldsymbol{G}\,(\mathbb{H})$ and constants $c_0 > 0$ and $\sigma > 0$.
Furthermore, the qc-scalar curvature of Θ is $S_\Theta = 128n(n+2)c_0\sigma$.

The Folland-Stein inequality on quaternionic Heisenberg group

Theorem 1 allows the determination of **all** solutions of the qc-Yamabe problem on the sphere and on the quaternionic Heisenberg group. As a consequence of Theorem 1 we obtain that all solutions to the qc Yamabe equation are given by the functions which realize the equality case of the L^2 Folland-Stein inequality. Thus, we confirm the following result

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Theorem (I/, I. Minchev, D. Vassilev: Ann. Sc. Norm Super. Pisa'12)

a) Let $\boldsymbol{G}\,(\mathbb{H}) \;=\; \mathbb{H}^n \times Im\,\mathbb{H}$ be the quaternionic Heisenberg group. The best constant in the L^2 Folland-Stein embedding inequality is

$$S_2 \;=\; \frac{\left[2^{-2n}\,\omega_{4n+3}\right]^{-1/(4n+6)}}{2\sqrt{n(n+1)}},$$

where $\omega_{4n+3} = 2\pi^{2n+2}/(2n+1)!$ is the volume of the unit sphere $S^{4n+3} \subset \mathbb{R}^{4n+4}$. The non-negative extremals are given by the functions of the form

$$F = \; c_0 \left[\left(\sigma \;+\; |q|^2\right)^2 \;+\; |\omega|^2\right]. \tag{2}$$

Any other non-negative extremal is obtained from F by translations $\tau_{(q_0,\omega_0)}F = F(q_0 + q, \omega_0 + \omega)$ and dilations $F_\lambda = \lambda^4 F(\lambda q, \lambda^2 \omega), \lambda > 0$.
These are all solutions to the qc-Yamabe equation on the quaternionic Heisenberg group $\boldsymbol{G}\,(\mathbb{H})$.
b) The qc Yamabe constant of the standard qc structure of the sphere is

$$\lambda(S^{4n+3}, [\tilde{\eta}]) \;=\; 16\,n(n+2)\;\left[((2n)!)\,\omega_{4n+3}\right]^{1/(2n+3)}. \tag{3}$$

Main strategy: Replace the non-linear Yamabe equation on $\boldsymbol{G}(\mathbb{H})$ with a geometrical system which can be solved. Use Cayley to transform the problem to the 3-Sasakian sphere and use compactness.

The first observation is:

Theorem (I/, I. Minchev, D. Vassilev: MemAMS'14, Math.Res.Lett.15)

If M is qc-Einstein then Scal=const.

Hint: Try to replace the Yamabe equation with conformal deformations preserving the qc-Einstein condition.

The Torsion Tensor. $T_{\xi_j} = T^0_{\xi_j} + I_j U$, $U \in \Psi_{[3]}$.

$T^0_{\xi_j}$-symmetric, $I_j U$-skew-symmetric.
Biquard shows T_ξ is completely trace-free.

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Theorem (I/ I. Minchev, D. Vassilev: MemAMS'14)

Define $T^0 = T^0_{\xi_1} I_1 + T^0_{\xi_2} I_2 + T^0_{\xi_3} I_3 \in \Psi_{[-1]}$. *We have*
$Ric = (2n+2)T^0 + (4n+10)U + \frac{Scal}{4n} g$.
$(M^{4n+3}, g, \mathbb{Q})$ *is qc-Einstein manifold iff the torsion of the Biquard connection is identically zero.*

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3-Sasakian manifold: M^{4n+3} is 3-Sasaki if its cone is hyperkähler of signature (4n+3,0)-positive 3-Sasaki, or (4n,3)-negative 3-Sasaki:
$C = M^{4n+3} \times \mathbb{R}^+, \quad g_{con} = t^2 g + \epsilon dt^2, \quad Hol(g_{con}) \in Sp(n+1) \mathrm{or} Sp(n,1)$. If $J_1, J_2, J_3)$ are the three complex structures on C then $\xi_s = J_s \frac{\partial}{\partial t}$ are the Reeb vector fields of the qc-structure.
The torsion of Biquard connection vanishes, $T_\xi = 0$ and any 3-Sasakian manifold is qc-Einstein.

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Biquard shows T_ξ is completely trace-free.

Theorem (I/ I. Minchev, D. Vassilev: MemAMS'14)

Define $T^0 = T^0_{\xi_1} I_1 + T^0_{\xi_2} I_2 + T^0_{\xi_3} I_3 \in \Psi_{[-1]}$. *We have*
$Ric = (2n+2)T^0 + (4n+10)U + \frac{Scal}{4n} g.$
$(M^{4n+3}, g, \mathbb{Q})$ *is qc-Einstein manifold iff the torsion of the Biquard connection is identically zero.*

3-Sasakian manifold: M^{4n+3} is 3-Sasaki if its cone is hyperkähler of signature (4n+3,0)-positive 3-Sasaki, or (4n,3)-negative 3-Sasaki:
$C = M^{4n+3} \times \mathbb{R}^+, \quad g_{con} = t^2 g + \epsilon dt^2, \quad Hol(g_{con}) \in Sp(n+1) \mathrm{or} Sp(n,1)$. If $J_1, J_2, J_3)$ are the three complex structures on C then $\xi_s = J_s \frac{\partial}{\partial t}$ are the Reeb vector fields of the qc-structure.
The torsion of Biquard connection vanishes, $T_\xi = 0$ and any 3-Sasakian manifold is qc-Einstein.

Theorem (I/ I. Minchev, D. Vassilev: MemAMS'14, Math. Res.Lett. '15)

Let $(M^{4n+3}, g, \mathbb{Q})$ *be a QC manifold. Suppose* $Scal \neq 0$. *The next conditions are equivalent:*

i) $(M^{4n+3}, g, \mathbb{Q})$ *is qc-Einstein manifold;*
ii) *M is locally 3-Sasakian (positive if* $Scal > 0$*, negative if* $Scal < 0$*): locally there exists a matrix* $\Psi \in \mathcal{C}^\infty(M : SO(3))$*, s.t.,* $(sign(Scal)\frac{16n(n+2)}{Scal}\Psi \cdot \eta, Q)$ *is 3-Sasakian (positive or negative);*

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The proof of Theorem 2 consists of two steps.

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Theorem (I/A. Petkov, D. Vassilev: Nonlinear Anal'13, J.Geom.Anal.'14, J.Spectral Theory'16)

Let $(M, g, \mathbb{Q})$ be a compact locally 3-Sasakian manifold of dimension $4n+3$ with qc Ricci tensor $Ric = 4(n+2)g$. Then, the first positive eigenvalue λ of the sub-Laplacian $\triangle$ satisfies the equality

$$\lambda = 4n \tag{4}$$

iff the qc manifold $(M, g, \mathbb{Q})$ is qc-homothetic to the unit (4n+3)-dimensional 3-Sasakian sphere.

Recall that

$$Ric = (2n+2)T^0 + (4n+10)U + \frac{Scal}{4n} g$$

The components of the torsion tensor transform according to the following formulas: if $\bar{\eta} = \frac{1}{2h}\eta$

- $\overline{T}^0(X,Y) \; = \; T^0(X,Y) \; + \; h^{-1}\,[\nabla dh]_{[sym][-1]}$, where the symmetric part is given by

$$[\nabla dh]_{[sym]}(X,Y) = \nabla dh(X,Y) + \sum_{s=1}^{3} dh(\xi_s)\,\omega_s(X,Y).$$

Torsion Under Conformal Transformation

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- $\bar{U}(X,Y) \;=\; U(X,Y) \;+\; (2h)^{-1}[\,\nabla dh - 2h^{-1}dh\otimes dh]_{[3][0]}$
  or if $f = \frac{1}{2h}$, $\bar{\eta} = f\eta$, then

$$\bar{U}(X,Y) \;=\; U(X,Y) \;-\; (2f)^{-1}[\,\nabla df]_{[3][0]}.$$

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Let $\Theta = \frac{1}{2h}\tilde{\Theta}$ be a conformal deformation of the standard qc-structure $\tilde{\Theta}$ on the quaternionic Heisenberg group $\boldsymbol{G}\,(\mathbb{H})$. If Θ is also qc-Einstein, then up to a left translation the function h is given by

$$h = c\left[\left(1 + \nu\,|q|^2\right)^2 + \nu^2\,(x^2 + y^2 + z^2)\right],$$

where c and ν are positive constants. All functions h of this form have this property.

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The problem is to show that these are all solutions to the Yamabe equation on $G\left(\mathbb{H}\right)$ - we need Obata type theorem Using Cayley we translate the problem to the sphere which is compact and can apply the horisontal divergence formula:

Proposition

Let $\left(M^{4n+3},\ \eta,\ g_H\right)$ be a compact closed manifold with a contact quaternionic structure and σ a horizontal 1-form, $\sigma \in \Lambda^1\left(H\right)$. Then we have

$$\int_M \left(\nabla^*\sigma\right)\eta_1 \wedge \eta_2 \wedge \eta_3 \wedge \omega_1^{2n} = 0,$$

where $\nabla^*\sigma = -(\nabla\sigma)(e_\alpha; e_\alpha)$ and $\{e_\alpha\}_\alpha$ is an ONB frame on H, $\alpha = 1, \ldots, 4n$.

The divergence formula.

Theorem (I/ I. Minchev, D. Vassilev, 2015)

Suppose (M^{4n+3}, η) *is a quaternionic contact structure conformal to a 3-Sasakian structure* $(M^{4n+3}, \bar{\eta})$, $\tilde{\eta} = \frac{1}{2h}\eta$. *If* $Scal_\eta = Scal_{\tilde{\eta}} = 16n(n+2)$, *then with f given by*

$$f = \frac{1}{2} + h + \frac{1}{4}h^{-1}|\nabla h|^2, \quad \text{we} \quad \text{have} \tag{5}$$

$$\nabla^*\Big(f(D+E) + \sum_{s=1}^{3} dh(\xi_s)I_sE + \sum_{s=1}^{3} dh(\xi_s)\,F_s + 4\sum_{s=1}^{3} dh(\xi_s)I_sA_s - \frac{10}{3}\sum_{s=1}^{3} dh(\xi_s)\,I_sA\Big)$$
$$= \Big(\frac{1}{2}+h\Big)\Big(|T^0|^2 + |\boldsymbol{E}|^2\Big) + 2h|\mathbb{D}+\mathbb{E}|^2 + h\,\langle QV,\ V\rangle. \tag{6}$$

Here, Q is a 7 by 7 positive definite matrix, $V = (E, D_1, D_2, D_3, A_1, A_2, A_3)$,

$$A_i = I_i[\xi_j, \xi_k], \quad A = A_1 + A_2 + A_3$$

$$D_s(X) = -\frac{1}{2h}\Big[T^0(X, \nabla h) + T^0(I_s X, I_s \nabla h)\Big], \quad F_s(X) \ = \ -h^{-1}\ T^0(X, I_s \nabla h)$$

$$E(X) = \frac{h^{-2}}{4}\Big[\nabla^2 h(X, \nabla h) + \sum_{s=1}^{3} \nabla^2 h(I_s X, I_s \nabla h) + \Big(-2 + 4h - 3h^{-1}|\nabla h|^2\Big) dh(X)\Big].$$

$$\mathbb{D}(X, Y, Z) = -\frac{h^{-1}}{8}\Big[dh(X) T^0(Y, Z) + dh(Y) T^0(X, Z) + \sum_{s=1}^{3} dh(I_s X) T^0(I_s Y, Z) + \sum_{s=1}^{3} dh(I_s Y) T^0(I_s X, Z)\Big] \quad (7)$$

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$$Q := \begin{bmatrix} \frac{5}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -2 & -2 & -2 \\ -\frac{1}{2} & \frac{5}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{10}{3} & -\frac{2}{3} & -\frac{2}{3} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{5}{2} & -\frac{1}{2} & -\frac{2}{3} & \frac{10}{3} & -\frac{2}{3} \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{5}{2} & -\frac{2}{3} & -\frac{2}{3} & \frac{10}{3} \\ -2 & \frac{10}{3} & -\frac{2}{3} & -\frac{2}{3} & \frac{22}{3} & -\frac{2}{3} & -\frac{2}{3} \\ -2 & -\frac{2}{3} & \frac{10}{3} & -\frac{2}{3} & -\frac{2}{3} & \frac{22}{3} & -\frac{2}{3} \\ -2 & -\frac{2}{3} & -\frac{2}{3} & \frac{10}{3} & -\frac{2}{3} & -\frac{2}{3} & \frac{22}{3} \end{bmatrix}$$

with eigenvalues 1, $\frac{9}{2} \pm \frac{\sqrt{73}}{2}$ and $\frac{11}{2} \pm \frac{\sqrt{89}}{2}$.

For the first step of the proof of Theorem 2, integrate the divergence formula and then use the divergence theorem to see that the integral of the left-hand side is zero. Thus, the right-hand side vanishes as well, which shows that the quaternionic contact structure $\bar{\eta}$ has vanishing torsion, i.e., it is also qc-Einstein.

Sketch of Proof of the second part of Theorem 2

Recall that a vector field Q on a qc manifold (M, η) is a *qc vector field* if its flow preserves the horizontal distribution $H = \ker \eta$.

$$\mathcal{L}_Q \eta = (\nu I + O) \cdot \eta,$$

where ν is a smooth function and $O \in so(3)$ is a matrix valued function with smooth entries.

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Lemma (I/ I. Minchev, D. Vassilev, 2015)

For any qc-vector field on (M,η) we have the formula

$$\Delta(\nabla^* Q_H) \;=\; -\frac{n}{2(n+2)}Q(Scal) \;-\; \frac{Scal}{4(n+2)}\nabla^* Q_H.$$

Let (M,η) and $(M,\bar\eta)$ be qc-Einsten manifolds with equal qc-scalar curvatures $16n(n+2)$. If η and $\bar\eta$ are qc conformal to each other, $\bar\eta = \frac{1}{2h}\eta$ for some smooth positive function h, then

$$Q = \frac{1}{2}\nabla f + \sum_{s=1}^{3} dh(\xi_s)\xi_s \tag{9}$$

is a qc vector field on M, where the function f appears in the divergence formula.

Proof of the second part of Theorem 2

Consider the qc vector field Q. The function $\phi = \frac{1}{2}\triangle f$ is either an eigenfunction of the sub-Laplacian with eigenvalue $-4n$, $\triangle\phi = -4n\phi$, or it vanishes identically and $f = const$. In the first case, using the quaternionic contact version of the Lichnerowicz-Obata eigenfunction sphere theorem we conclude that (M, η) is the 3-Sasakain sphere.

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These considerations provide also a certain geometric insight for the mysterious function f in the divergence formula. In fact, up to an additive constant, the misterious function f is the unique function on M for which $Q_H = \frac{1}{2}\nabla f$ is the horizontal part of a qc vector field Q with vertical part $Q_V = dh(\xi_s)\xi_s$, $Q = Q_H + Q_V$.

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Remarkably, the same situation occur in the CR case: the mysterious function in the Jerison-Lee divergence formula is the unique function on M for which $Q_H = \frac{1}{2}\nabla f$ is the horizontal part of a CR-vector field Q with vertical part $Q_V = dh(\xi)\xi$, $Q = Q_H + Q_V$. This provides geometric insight for the mysterious function f in the Jerison-Lee divergence formula from 1988.