Sectional Curvature in 4-dimensional Manifolds

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Abstract

This talk will review the known work on the study of the sectional curvature function on 4-dimensional manifolds for metrics with positive definite or Lorentz signature. It will then deal with the remaining case of neutral signature.

Introduction

In this talk (M, g) represents a 4−dimensional manifold with metric g of any of the three possible signatures. The associated Levi-Civita connection from g is denoted ∇, the corresponding curvature tensor is *Riem* with components $R^a{}_{bcd}$, the resulting Ricci tensor is denoted *Ricc* with components $R_{ab} \equiv R^c{}_{acb}$ and the Ricci scalar is $R \equiv R_{ab}g^{ab}$. The tangent space to M at m is T_m and the space of 2−forms (bivectors) at m is B_m. The collection of 2−dimensional subspaces (2−spaces) of T_m is the Grassmann manifold G_m at m. A member $0 \neq F \in B_m$ is skew-symmetric and has matrix rank an even integer. If this is 2, F is called *simple* and may be written symbolically as $F = p \wedge q$ for $p, q \in T_m$. The 2−space spanned by u and v is called the *blade* of F and G_m is diffeomorphic to the manifold of projective simple bivectors, the latter being more amenable to calculation.

Geometry of the tangent space I

If $u, v \in T_m$ the inner product $g(m)(u, v)$ is denoted by $u \cdot v$ and $0 \neq u \in T_m$ is called *spacelike*, respectively *timelike* or *null*, if $u \cdot u > 0$, (respectively < 0 or $= 0$). A $2-$space W of T_m is called *spacelike* if each non-zero member of W is spacelike, or each non-zero member of W is timelike, *timelike* if W contains exactly two, null $1-$dimensional subspaces (*directions*), *null* if V contains exactly one null direction and *totally null* if each non-zero member of V is null. Thus a totally null $2-$space consists, apart from the zero vector, of null vectors any two of which are orthogonal. A simple bivector is then called spacelike, timelike, null or totally null if its blade is of that type. Let $\overline{G_m}$ denote the collection of spacelike or timelike members of G_m. Then $F \in \overline{G_m}$ if and only if $F^{ab}F_{ab} \neq 0$. For positive definite metrics only spacelike $2-$spaces are possible, for Lorentz only spacelike, timelike and null $2-$spaces exist but all four are possible for neutral signature.

Sectional Curvature

At $m \in M$ define the sectional curvature function $\sigma_m : \overline{G_m} \to \mathbb{R}$ by

$$\sigma_m(F) \equiv \frac{R_{abcd}F^{ab}F^{cd}}{2F^{ab}F_{ab}} \tag{1}$$

Thus for positive definite metrics σ_m is defined on the whole of G_m whilst for Lorentz and neutral signatures it is restricted to the open submanifold $\overline{G_m}$ which in the Lorentz case excludes the null $2-$spaces and in the neutral case excludes both null and totally null $2-$spaces. The main mathematical difference is that the domain of σ_m in the positive definite case is a compact connected manifold whereas in the other two cases it is disconnected and non-compact.

The interpretation of σ_m is as follows; if F represents a member of $\overline{G_m}$ at m there exists, locally, a unique $2-$dimensional submanifold M' of M generated by geodesics of ∇ initially tangent to the blade of F at m and which has an induced metric from g. The Gauss curvature of M' at m equals $\sigma_m(F)$.

Main Idea

The main point of this talk is to show that, under certain conditions on (M, g), the specification of the function σ_m at each point of M uniquely determines the metric g on M from which it came except in very special cases. The first condition to be imposed is that σ_m is, for no $m \in M$, a constant function. [If it is constant then at m $Riem$ takes the "constant curvature" form $R_{abcd} = \frac{R}{12}(g_{ac}g_{bd} - g_{ad}g_{bc})$.] This condition is itself sufficient in the positive definite case to show that if g and g' are metrics on M with g positive definite and which have the same sectional curvature function on M then $g' = g$ [1]. In the other cases one first notes that σ_m is only defined on $\overline{G_m}$ and that the difference set $N_m \equiv G_m \setminus \overline{G_m}$ is the problem. Of course, if (M, g) takes the constant curvature form at m, σ_m may be continuously extended to the whole of G_m. Since this latter possibility is forbidden by the first condition above one asks to what extent σ_m can be continuously extended from $\overline{G_m}$.

First Part of Proof

Concentrating on the Lorentz and neutral cases, σ_m is initially defined on $\overline{G_m}$. The next step is to see if σ_m can be continuously extended to any member of N_m. The answer to this is crucial and is that if such an extension to *any* member of N_m is possible then σ_m is a constant function on $\overline{G_m}$ and thus trivially continuously extendible to G_m. By the first condition insisted upon above the consequence of this is that the collection of null (in the Lorentz case) and null and totally null (in the neutral case) $2-$spaces is fixed by the metric being studied. For each signature this can be shown to reveal the null cone at each $m \in M$. Thus immediately, if g and g' have the same sectional curvature function on M they are conformally related.

Idea of proof

For either Lorentz or neutral signature if F and G are bivector representatives of two distinct members of N_m the bivector $F + \lambda G$ is, for some $\lambda \in \mathbb{R}$ (simple and) represents a member of N_m if and only if the 2−spaces represented by F and G intersect in a null direction (and then $F + \lambda G$ represents a member of N_m for each $\lambda \in \mathbb{R}$). The check on the "pencil" represented by $F + \lambda G$ is purely algebraic/geometric and independent of signature and reveals all the null members of T_m and hence fixes the null cone. It is remarked that the topology of N_m distinguishes between Lorentz and neutral signature.

Theorem I

Suppose that g and g' are metrics with the same sectional curvature which, for no $m \in M$, is a constant function. Then (using primes for the corresponding geometrical objects from g') one has for some real-valued function $\phi : M \to \mathbb{R}$ and using C to denote the Weyl conformal tensor for g with components $C^a{}_{bcd}$ (and similarly for g')

(i) $g' = \phi g$, (ii) $R'_{abcd} = \phi^2 R_{abcd}$, (iii) $R'^a{}_{bcd} = \phi R^a{}_{bcd}$,

(iv) $R'_{ab} = \phi R_{ab}$, (v) $R' = R$, (vi) $C' = \phi C$

It is noted that $C = C'$ from (i) but this does not force $\phi = 1$ in (vi) since C (and hence C') may be zero. Also *Riem* does not vanish at any $m \in M$ [2, 5]. [It is remarked that if C is nowhere zero on M, $g' = g$.]

Theorem 2

One continues with the study of the subset $U \subset M$ of those points in M at which $C(= C')$ is *not* zero and let $V \subset M$ be the subset of those points where $d\phi$ does not vanish. Thus $\phi = 1$ on U and ϕ is a non-zero constant on each component of $\text{int}(M \setminus V)$. Let $W \subset M$ be the subset of points at which $\phi = 1$ so that $U \subset W$. Finally disjointly decompose M as $M = V \cup \text{int}W \cup K$ where the closed subset K is defined by disjointness. It then follows that $\text{int}K = \emptyset$. Thus, in summary, one has $g' = g$ on W and g and g' are conformally flat on the open subset V (and quietly forget about K since $\text{int}K = \emptyset$!) All that remains is to see what happens on V.

The subset V

For each of the Lorentz and neutral signature cases one gets $R = 0$ on V and the Ricci tensor takes the Segre type $\{(211)\}$ form $R_{ab} = \psi\phi_a\phi_b$ with ϕ^a null (for both g and g'). One may choose coordinates (see, e.g. [6, 7]) x, y, u, v about any $m \in V$ to get the following metric (for the Lorentz case see [2, 5, 3, 4]).

$$ds^2 = \gamma(u)(x^2 - y^2)du^2 + 2dudv + dx^2 \pm dy^2 \qquad (2)$$

for some function γ where ϕ is a function of u only and where the plus (minus) sign corresponds to Lorentz (neutral) signature. In fact, given g, one may choose a corresponding g' with the same sectional curvature function as $g' = e^{2\sigma(u)}g$ where $\ddot{\sigma} - \dot{\sigma}^2 = \gamma(e^{2\sigma} - 1)$ for which solutions are known [5]

A special case

The following result is an interesting special case.

- *Let M be a $4-$dimensional manifold with a non-flat, Ricci-flat metric g of signature $(+,+,+,-)$ or $(+,+,-,-)$. Let g' be any other metric on M with the same sectional curvature function as g at each $m \in M$. Then $g' = g$ on M.*

- The mathematics of the map which associates g with its sectional curvature function (Lorentz case)[4].

- Applications to General Relativity Theory.

Conclusions

Thus the study of the sectional curvature function in the case of neutral signature follows similar lines to that in the Lorentz case and, the exceptional cases excluded, the determination of the metric is essentially one-to-one. The exceptional case (see (2)) in neutral signature is the analogue of the "plane waves" in Lorentz signature and which have been so useful and suggestive in describing gravitational waves in General Relativity Theory.
It is remarked that the above theorems can be technically strengthened.

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