The proof of Blagojević-Grujić-Živaljević conjecture on symmetric products of compact Riemann surfaces with punctures

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Fact β. *For $m \geq 3$, the space $\mathrm{Sym}^2 \mathbb{R}^m$ is homeomorphic to $\mathbb{R}^m \times \mathrm{ConeInt}(\mathbb{R}P^{m-1})$. Here by $\mathrm{ConeInt}(X)$ we denote the open cone $\mathrm{Cone}(X) \backslash X$.*

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Fact γ. *There exists a canonical homeomorphism* $\mathrm{Sym}^n\mathbb{C} \cong \mathbb{C}^n$ *for all* $n \geq 2$.
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Problem 1. *Is the same true for compact Riemann surfaces with punctures? Namely, for any* $M_{g,k}^2$ *and* $N_{g,k}^2$ *we get that* $\mathrm{Sym}^n M_{g,k}^2$ *and* $\mathrm{Sym}^n N_{g,k}^2$ *are* C^∞*-diffeomorphic for all* $n \geq 2$.

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Fact δ. *The space* $\mathrm{Sym}^n S^1, n \geq 2,$ *is homeomorphic to the* D^{n-1}*-bundle over* S^1*, which is trivial for odd n, and non-oriented for even n.*

Fact ε. *The space* $\mathrm{Sym}^n \bigvee_1^m S^1, n \geq 2, m \geq 1,$ *is homotopy equivalent to* $\mathrm{Sk}^n T^m$*. Here, the cell structure on the torus* T^m *is the direct product of the minimal cell structure on* S^1 *(this structure has one 0-cell).*

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Fact ζ. *Suppose X is a connected CW-complex. Then* $\pi_1(\mathrm{Sym}^n X) = \pi_1(X)^{ab} = H_1(X; \mathbb{Z})$ *for all* $n \geq 2$.

M_g^2 and $M_{g'}^2$ are compact Riemann surfaces (g, g' - genuses).

Prop. *If $g \neq g'$ then closed manifolds $\mathrm{Sym}^n M_g^2$ and $\mathrm{Sym}^n M_{g'}^2$ are not homotopy equivalent for all $n \geq 2$. (They has different π_1)*

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$M_{g,k}^2$ and $M_{g',k'}^2$ are compact Riemann surfaces with punctures ($g, g' \geq 0$ - genuses, $k, k' \geq 1$ - number of punctures).

Prop. $M_{g,k}^2 \sim \bigvee_1^{2g+k-1} S^1$. *Open manifold $\mathrm{Sym}^n M_{g,k}^2$ is homotopy equivalent to $\mathrm{Sym}^n \bigvee_1^{2g+k-1} S^1 \sim \mathrm{Sk}^n T^s, n \geq 2, s := 2g+k-1$.*

Prop. $\mathrm{Sym}^n M_{g,k}^2 \sim \mathrm{Sym}^n M_{g',k'}^2 \;\Leftrightarrow\; 2g+k = 2g'+k'$

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Conjecture (2003). *Fix any $n \geq 2$, and two pairs (g, k) and (g', k') with the condition $2g + k = 2g' + k'$. If $g \neq g'$, then open manifolds $\mathrm{Sym}^n M^2_{g,k}$ and $\mathrm{Sym}^n M^2_{g',k'}$ are not homeomorphic.*

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Theorem (B-G-Ž, 2003).

1. *Suppose that $n \leq 2\max\{g, g'\}$. Then the n-th Betti numbers at infinity of manifolds $\mathrm{Sym}^n M^2_{g,k}$ and $\mathrm{Sym}^n M^2_{g',k'}$ are different.*
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Corollary. *If $n \leq 2\max\{g, g'\}$, then the Conjecture is true.*

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In the case $n > 2\max\{g, g'\}$ the Conjecture was open up to the author's arXiv preprint (June 2016).

Main Theorem (G., 2016). *Fix any* $n \geq 2$, *and two pairs* (g, k) *and* (g', k') *with the condition* $2g + k = 2g' + k'$. *If* $g \neq g'$, *then open manifolds* $\mathrm{Sym}^n M^2_{g,k} \times \mathbb{R}^N$ *and* $\mathrm{Sym}^n M^2_{g',k'} \times \mathbb{R}^N$ *are not homeomorphic for all* $N \geq 0$.

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Our method.

1. Topological invariance of *Stiefel* − *Whitney* classes of open smooth manifolds.
2. Calculations and the fact: (coefficients $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$)
   Suppose $2g + k = 2g' + k'$ and $g \neq g'$.
   $A^* := H^*(\mathrm{Sym}^n M^2_{g,k})$ and $B^* := H^*(\mathrm{Sym}^n M^2_{g',k'})$.
   $A^* \cong B^*$, but for any isomorphism $\varphi\colon A^* \to B^*$ we get
   $\varphi(w_2(\mathrm{Sym}^n M^2_{g,k})) \neq w_2(\mathrm{Sym}^n M^2_{g',k'})$.

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Step 1. $\mathrm{Sym}^n M^2_{g,k} \sim \mathrm{Sk}^n T^s$ has torsion-free integral homology.
$\Rightarrow H^*(\mathrm{Sym}^n M^2_{g,k}; \mathbb{Z}_2) \cong H^*(\mathrm{Sym}^n M^2_{g,k}; \mathbb{Z}) \otimes \mathbb{Z}_2$.

Step 2. $H^*(\mathrm{Sym}^n M^2_{g,k}; \mathbb{Z})$ is equal to the $(n+1)$-cutted exterior algebra $\Lambda^{\leq n}_{\mathbb{Z}}(\alpha_1, \alpha_2, \dots, \alpha_s)$ for some $\mathbb{Z}$-basis $\alpha_1, \alpha_2, \dots, \alpha_s$ of $H^1(\mathrm{Sym}^n M^2_{g,k}; \mathbb{Z})$.
$\Rightarrow H^*(\mathrm{Sym}^n M^2_{g,k}; \mathbb{Z}_2)$ is equal to $\Lambda^{\leq n}_{\mathbb{Z}_2}(\overline{\alpha}_1, \overline{\alpha}_2, \dots, \overline{\alpha}_s)$ for some (any) $\mathbb{Z}_2$-basis $\overline{\alpha}_1, \overline{\alpha}_2, \dots, \overline{\alpha}_s$ of $H^1(\mathrm{Sym}^n M^2_{g,k}; \mathbb{Z}_2)$.

$\mathrm{Sym}^n M^2_{g,k} \sim \mathrm{Sk}^n T^s$ and the cell structure of T^s is standard (minimal)

Step 3.

$$c_1(\mathrm{Sym}^n M^2_{g,k}) = -(\alpha_1 \smile \alpha_2 + \alpha_3 \smile \alpha_4 + \ldots + \alpha_{2g-1} \smile \alpha_{2g})$$

for some $\mathbb{Z}$-basis $\alpha_1, \alpha_2, \ldots, \alpha_s$ of $H^1(\mathrm{Sym}^n M^2_{g,k}; \mathbb{Z})$.

Derived from Macdonald's calculation of the total Chern class of $\mathrm{Sym}^n M^2_g$.

Step 4.

The Stiefel-Whitney classes w_k of the realization $\xi_{\mathbb{R}}\colon E_{\mathbb{R}} \to B$ can be computed from the Chern classes c_l of the initial complex vector bundle $\xi\colon E \to B$ as follows:

$$w_{2k+1}(\xi_{\mathbb{R}}) = 0 \ \ \forall k \geq 0; \quad w_{2k}(\xi_{\mathbb{R}}) = \rho_2(c_k(\xi)) \ \ \forall k \geq 1.$$

Here, $\rho_2\colon H^*(B; \mathbb{Z}) \to H^*(B; \mathbb{Z}_2)$ is the reduction homomorphism.

Step 5. Combining two previous steps

$$w_2(\mathrm{Sym}^n M^2_{g,k}) = \overline{\alpha}_1 \smile \overline{\alpha}_2 + \overline{\alpha}_3 \smile \overline{\alpha}_4 + \ldots + \overline{\alpha}_{2g-1} \smile \overline{\alpha}_{2g}$$

for some $\mathbb{Z}_2$-basis $\overline{\alpha}_1, \overline{\alpha}_2, \ldots, \overline{\alpha}_s$ of $H^1(\mathrm{Sym}^n M^2_{g,k}; \mathbb{Z}_2)$.

$$H^2(\mathrm{Sym}^n M^2_{g,k}; \mathbb{Z}_2) \cong \Lambda^2(H^1(\mathrm{Sym}^n M^2_{g,k}; \mathbb{Z}_2)),$$

we get

$$w_2(\mathrm{Sym}^n M^2_{g,k}) = \overline{\alpha}_1 \wedge \overline{\alpha}_2 + \overline{\alpha}_3 \wedge \overline{\alpha}_4 + \ldots + \overline{\alpha}_{2g-1} \wedge \overline{\alpha}_{2g}.$$

Step 6. (Topological invariance of Stiefel-Whitney classes for open smooth manifolds)

Suppose we have closed smooth connected manifolds M^n and N^n. By celebrated Wu formula, if $f\colon M^n \to N^n$ is a homotopy equivalence, then $f^*(w_k(N^n)) = w_k(M^n)$ for all $k \geq 1$. It is the famous *Homotopy invariance* of Stiefel-Whitney classes for closed manifolds.

$M^2 = S^1 \times \mathbb{R}^1$ and $N^2 =$ (open Möbius strip) $\Rightarrow$ even w_1 is not a *homotopy* invariant for open manifolds.

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Theorem (Thom, 1952). *Suppose we have a homeomorphism $f\colon M^n \to N^n$ of two open connected smooth (paracompact) manifolds. Then $f^*(w_k(N^n)) = w_k(M^n)$ for all $1 \leq k \leq n$.*

Step 7. Combining steps 5 and 6 $\Rightarrow$ topological type of $\mathrm{Sym}^n M^2_{g,k}$ determines genus g.

$$w_2(\mathrm{Sym}^n M^2_{g,k}) = \overline{\alpha}_1 \wedge \overline{\alpha}_2 + \overline{\alpha}_3 \wedge \overline{\alpha}_4 + \ldots + \overline{\alpha}_{2g-1} \wedge \overline{\alpha}_{2g}$$

$$V := H^1(\mathrm{Sym}^n M^2_{g,k}; \mathbb{Z}_2) \;\Rightarrow\; \Lambda^2(V) = H^2(\mathrm{Sym}^n M^2_{g,k}; \mathbb{Z}_2)$$

$$V = \langle \alpha_1, \alpha_2, \ldots, \alpha_s \rangle = \langle \beta_1, \beta_2, \ldots, \beta_s \rangle$$

$$w = \sum_{i<j} \varepsilon_{ij} \alpha_i \wedge \alpha_j = \sum_{k<l} \zeta_{kl} \beta_k \wedge \beta_l$$

$$\text{Define } \; \bar{\varepsilon} \in \mathrm{Mat}_{s\times s}(\mathbb{Z}_2):$$

$$\bar{\varepsilon}_{ij} = \varepsilon_{ij} \;\text{ if }\; i<j, \;\text{ and }\; \bar{\varepsilon}_{ii} = 0 \text{ for any } i, \;\text{ and }\; \bar{\varepsilon}^{\mathrm{T}} = \bar{\varepsilon}$$

$$\text{If } C := C_{\alpha\to\beta} \in \mathrm{GL}(s, \mathbb{Z}_2) \;\Rightarrow\; \bar{\zeta} = C^{-1} \bar{\varepsilon} C^{-\mathrm{T}}$$

$$\Rightarrow\; \mathrm{rank}\bar{\varepsilon} = \mathrm{rank}\bar{\zeta} =: \mathrm{rank}(w)$$

$$w_2(\mathrm{Sym}^n M^2_{g,k}) = \overline{\alpha}_1 \wedge \overline{\alpha}_2 + \overline{\alpha}_3 \wedge \overline{\alpha}_4 + \ldots + \overline{\alpha}_{2g-1} \wedge \overline{\alpha}_{2g}$$

If $w = w_2(\mathrm{Sym}^n M^2_{g,k}) \Rightarrow \mathrm{rank}(w) = 2g$

$w_k(M \times \mathbb{R}^N) = w_k(M), k = 1, 2, \ldots.$ $\Rightarrow$ Main Theorem

Pontryagin classes of $\mathrm{Sym}^n M^2_{g,k}$.

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Prop. *All integral Pontryagin classes of* $\mathrm{Sym}^n M^2_{g,k}$ *are zero.*

Idea of the proof:

1. Macdonald's formula for total Chern class of $\mathrm{Sym}^n M^2_g$
2. Calculation of total Chern class of $\mathrm{Sym}^n M^2_{g,k}$
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Prop. *If $n \geq 4$ and $g \geq 2$, then $p_1(\mathrm{Sym}^n M^2_g) \neq 0$.*

Attaching the boundary to open manifolds $\mathrm{Sym}^n M^2_{g,k}$.

Fact η. *$\mathrm{Sym}^n D^2$ is homeomorphic to D^{2n}. But, there is no natural smoothing of $\mathrm{Sym}^n D^2$.*

Corollary. *If $\overline{M}^2$ is a 2-manifold with a boundary, then $\mathrm{Sym}^n \overline{M}^2$ is a TOP $2n$-manifold with the boundary for all $n \geq 2$.*

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Suppose $\overline{M}^2_{g,k}$ is a compact Riemann surface of genus g with k small open disks removed (locally $|z| < \varepsilon$ removed).

$\Rightarrow$ $\mathrm{Sym}^n \overline{M}^2_{g,k}$ is a compact TOP $2n$-manifold with a boundary, and

$$\mathrm{int}(\mathrm{Sym}^n \overline{M}^2_{g,k}) = \mathrm{Sym}^n \mathrm{int}(\overline{M}^2_{g,k})$$

$\mathrm{int}(\overline{M}^2_{g,k})$ is a 1-dimensional complex manifold

$\Rightarrow$ $\mathrm{int}(\mathrm{Sym}^n \overline{M}^2_{g,k})$ has a natural structure of a complex n-dimensional manifold $\Rightarrow$ it has a C^∞-structure

There is no natural C^∞-structure on the ∂-manifold $\mathrm{Sym}^n \overline{M}^2_{g,k}$!!!

There is no natural C^∞-structure on the ∂-manifold $\mathrm{Sym}^n \overline{M}^2_{g,k}$!!!

Question. Is $\mathrm{Sym}^n \overline{M}^2_{g,k}$ smoothable?

Answer. Yes!

Idea of the proof.

1. $n = 2$. The 4-manifold $\mathrm{Sym}^2 \overline{M}^2_{g,k}$ with the boundary is a compact polyhedron. $\Rightarrow$ it is a triangulated manifold of dim 4 $\Rightarrow$ it is a PL-manifold of dim 4 $\Rightarrow$ it is smoothable (and C^∞-structure up to a Diffeomorphism is determined by (g, k))
2. $n \geq 3$.
   Prop. *$\pi_1(\partial \mathrm{Sym}^n \overline{M}^2_{g,k})$ is a free abelian group ($= \mathbb{Z}^{2g+k-1}$).*

Product Structure Theorem (Kirby-Siebenmann1977). *Suppose M^m is a connected TOP manifold (without boundary), and $m \geq 5$. If Σ is a smooth structure on $M^m \times \mathbb{R}$, then there exists a smooth structure Σ_0 on M^m such that*

$$(M^m \times \mathbb{R})_\Sigma \quad \text{is diffeomorphic to} \quad M^m_{\Sigma_0} \times \mathbb{R}.$$

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Thank you for your attention!