XIX Geometrical Seminar

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Quasisymmetric function invariant of graphs

Chromatic Hopf algebra of graphs $\mathcal{G}$

$\Gamma = (V, E)$ simple finite graph

$$\Gamma_1 \cdot \Gamma_2 = \Gamma_1 \sqcup \Gamma_2$$
$$\Delta(\Gamma) = \sum_{I \subset V} \Gamma|_I \otimes \Gamma|_{I^c}$$

$\mathcal{G}$ is a graded commutative, cocommutative Hopf algebra

multiplicative character $\zeta : \mathcal{G} \to k$

$$\zeta_{\mathcal{G}}(\Gamma) = \begin{cases} 1, & \Gamma \text{ is discrete} \\ 0, & \text{otherwise} \end{cases}$$

CHA is the pair $(\mathcal{H}, \zeta)$ of a graded Hopf algebra $\mathcal{H}$ and a character $\zeta : \mathcal{H} \to k$.

M. Aguiar, N. Bergeron, F. Sottile, 2006.

CHA of quasisymmetric functions $QSym$, I. Gessel, R. Stanley in 1980s

Additive basis indexed by compositions of integers

$$M_{(a_1,\ldots,a_k)} = \sum_{i_1<\cdots<i_k} x_{i_1}^{a_1} \cdots x_{i_k}^{a_k}.$$

quasi-shuffle product of compositions

$$M_{(1,1)} \cdot M_{(1,1)} = 6M_{(1,1,1,1)} + 2(M_{(2,1,1)}+$$
$$+M_{(1,2,1)} + M_{(1,1,2)}) + M_{(2,2)}$$

concatenation

$$\Delta(M_\alpha) = \sum_{\alpha=\beta\gamma} M_\beta \otimes M_\gamma$$

Terminal object in the category of CHA's

$$\Psi : \mathcal{H} \to QSym$$
$$\Psi(h) = \sum_{\alpha \models n} \zeta_\alpha(h) M_\alpha$$

$$(*) \qquad \zeta_{(a_1,\ldots,a_k)}(h) = \sum \zeta_{a_1}(h_{(1)}) \cdots \zeta_{a_k}(h_{(k)})$$

$\Psi(\Gamma)$ Stanley's chromatic symmetric function of graphs

principal specialization $\mathbf{ps}(\Psi(\Gamma))(\mathrm{m}) = \chi(\Gamma, \mathrm{m})$
chromatic polynomial of a graph

Another CHA of graphs

$$\Gamma_1 \cdot \Gamma_2 = \Gamma_1 \sqcup \Gamma_2$$
$$\Delta(\Gamma) = \sum_{I \subset V} \Gamma \mid_I \otimes \Gamma / I$$

not cocommutative

A new quasisymmetric function invariant

$F(\Gamma) = \sum_{\alpha \models n} \zeta_\alpha(\Gamma) M_\alpha$

$\zeta_\alpha(\Gamma)$ = number of ordered colorings of the type α

$f(\Gamma, m) = \mathbf{ps}(\mathrm{F}(\Gamma))(\mathrm{m})$ chromatic polynomial of ordered colorings

Recurrence relation

$$F_{\emptyset} = 1$$

$$F_{\Gamma_1 \sqcup \Gamma_2} = F_{\Gamma_1} F_{\Gamma_2}$$

$$F(\Gamma) = \sum_{v \in V} (F(\Gamma \setminus v))_1$$

Stanley's inversion formula

$\chi(\Gamma,-1)=(-1)^n a(\Gamma)$ the number of acyclic orientations

Theorem. $f(\Gamma,-1)=(-1)^n c(\Gamma)$ graph-Catalan numbers

$c(\Gamma)=\sum_{v\in V} c(\Gamma\setminus v)$

Generating functions of graph-Catalan numbers

$\{\Gamma_n\}_{n\in\mathbb{N}}$

$y(t)=\sum_{n\geq 0}(-1)^n c(\Gamma_n)t^n$

Example:

1. Complete graphs K_n

$t^2y' = 1 - (1+t)y, y(0) = 1$

$y(t) = \int_0^{+\infty} \frac{e^{-x}}{1+tx}dx, t > 0$

2. Complete bipartite graphs $K_{m,n}$

$c(K_{m,n}) = mc(K_{m-1,n}) + nc(K_{m,n-1}), m, n \geq 1$

$u^2\frac{\partial z}{\partial u} + v^2\frac{\partial z}{\partial v} = 1 - (1+u+v)z - \frac{u^2}{(1+u)^2} - \frac{v^2}{(1+v)^2}, z(0,v) = \frac{1}{1+v}$

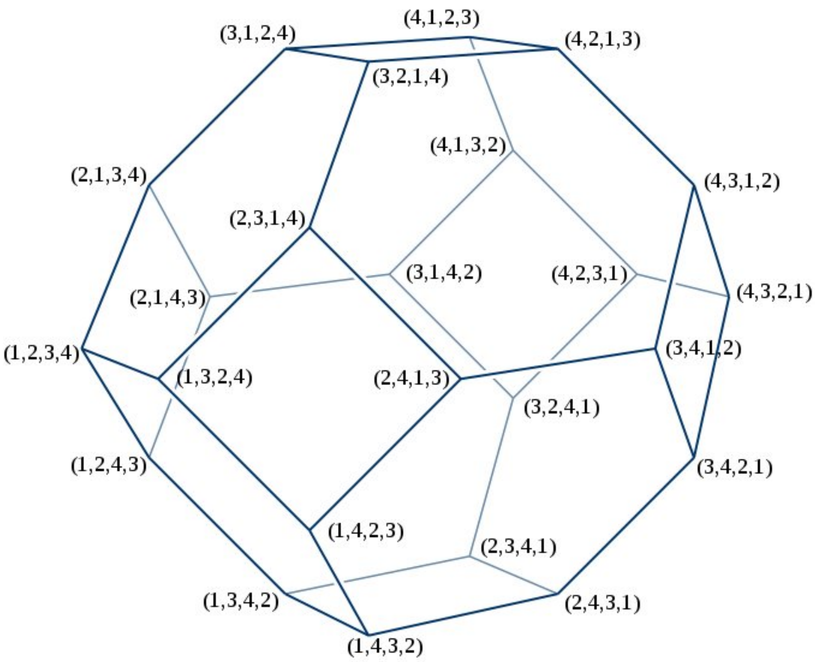
$z(u,v) = \int_0^{+\infty} \frac{e^{-x}}{(1+ux)(1+vx)}(1 - \frac{1}{(1+\frac{1}{u}+x)^2} - \frac{1}{(1+\frac{1}{v}+x)^2})dx$

Geometric interpretation

Generalized permutohedra, Postnikov, 2009.

$$Pe^{n-1} = \mathrm{Conv}\{x_\pi \mid \pi \in S_n\}$$

$$(Pe^{n-1})^* = \Delta[n]^{(1)}$$



Q convex polytope, $\Sigma_Q = \{\sigma_v \mid v \in P\}$ normal fan

Normal fan $\Sigma_{Pe^{n-1}}$ is the reduced braid arrangement

$$\mathcal{A}_{n-1} = \{x_i = x_j \mid 1 \leq i \neq j \leq n\}.$$

Normal cone in a vertex is the Weyl chamber

$$\sigma_{x_\pi} = C_\pi : x_{\pi(1)} \leq \cdots \leq x_{\pi(n)}$$

A. Postnikov (2009):
Q is a generalized permutohedron if

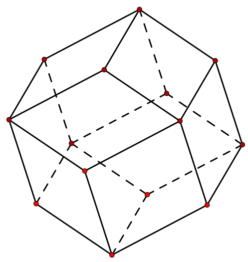
$$\Sigma_Q \subset \Sigma_{Pe^{n-1}} = \mathcal{A}_{n-1}$$

F_Q = enumerator function of integer points in normal fan

1. Grafical zonotope Z_Γ

Graphical arrangement $\mathcal{H}_\Gamma = \{x_i = x_j \mid ij \in E\}$

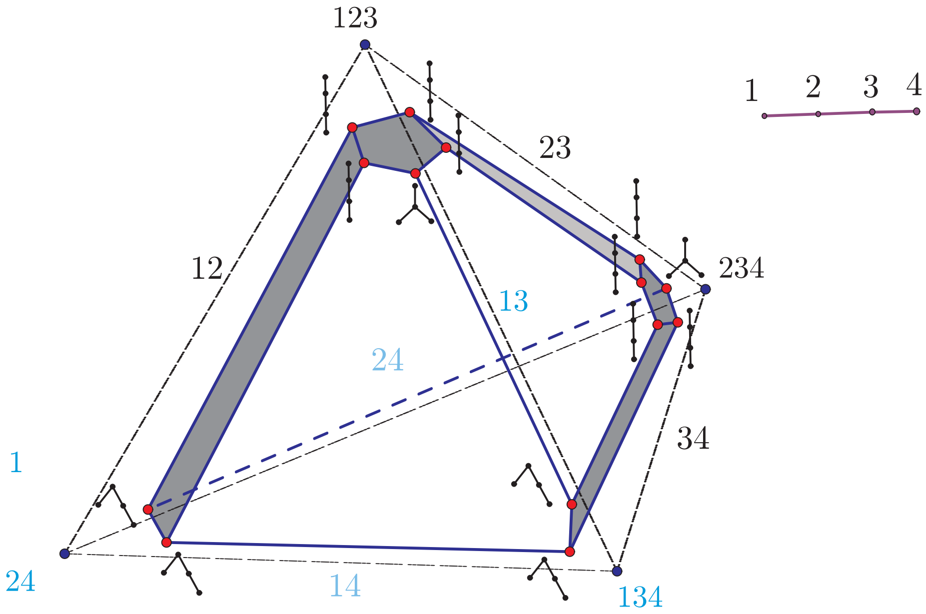
$$r(\mathcal{H}_\Gamma) = f_0(Z_\Gamma) = a(\Gamma)$$



Rhombic dodecahedron $\mathcal{Z}_{C_4}$

$F_{Z_\Gamma} = \Psi(\Gamma)$

$$F_{P_\Gamma} = F(\Gamma)$$



Problem

1. Find two non-isomorphic graphs Γ_1 and Γ_2 such that $F(\Gamma_1) = F(\Gamma_2)$.

2. Compare it with Stanley's chromatic symmetric function.