An Approximate Nerve Theorem

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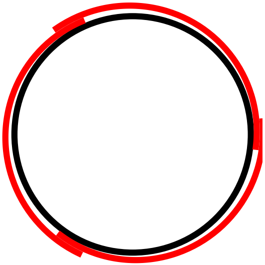
Outline

Nerve



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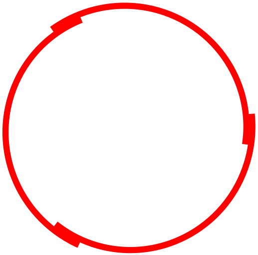

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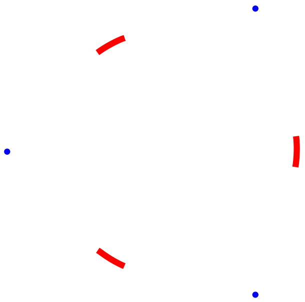

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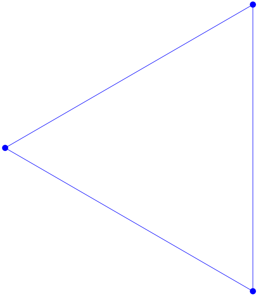

Nerve Theorem

Theorem (Borsuk, 1948)

If $\mathcal{V}$ is an open cover of a paracompact space Y such that every nonempty intersection of finitely many sets in $\mathcal{V}$ is contractible, then Y is homotopy equivalent to the nerve $\mathcal{N}(\mathcal{V})$.

Homological Nerve Theorem

Theorem (Leray, 1945)

If $\mathcal{U}$ is a cover by subcomplexes of a simplicial complex X such that every nonempty intersection of finitely many sets in $\mathcal{U}$ is acyclic, then

$$\mathrm{H}_*(X) \cong \mathrm{H}_*(\mathcal{N}(\mathcal{U})),$$

where $\mathcal{N}(\mathcal{U})$ is the nerve of $\mathcal{U}$.

Filtrations

Filtration is given by $f : X \to \mathbb{Z}$.

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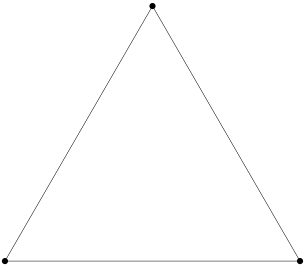

$T = 0$

Persistent homology:

$$\mathrm{H}_0(X): \quad \ldots \to \mathbb{k} \to \mathbb{k} \to \mathbb{k} \to \mathbb{k} \to \ldots$$
$$\mathrm{H}_1(X): \quad \ldots \to \mathbb{k} \to \mathbb{k}^3 \to \mathbb{k}^2 \to \mathbb{k} \to \ldots$$

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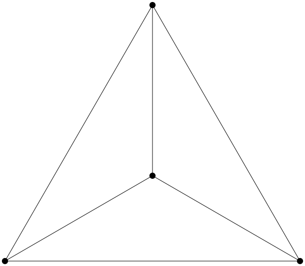

$T = 1$

Persistent homology:

$$H_0(X): \quad \ldots \to \Bbbk \to \Bbbk \to \Bbbk \to \Bbbk \to \ldots$$
$$H_1(X): \quad \ldots \to \Bbbk \to \Bbbk^3 \to \Bbbk^2 \to \Bbbk \to \ldots$$

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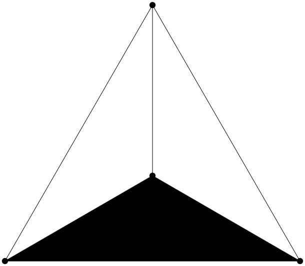

$T = 2$

Persistent homology:

$$H_0(X) : \quad \ldots \to \Bbbk \to \Bbbk \to \Bbbk \to \Bbbk \to \ldots$$
$$H_1(X) : \quad \ldots \to \Bbbk \to \Bbbk^3 \to \Bbbk^2 \to \Bbbk \to \ldots$$

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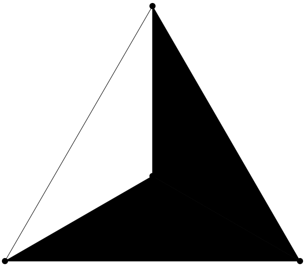

$T \geq 3$

Persistent homology:

$$H_0(X) : \quad \ldots \to \Bbbk \to \Bbbk \to \Bbbk \to \Bbbk \to \ldots$$

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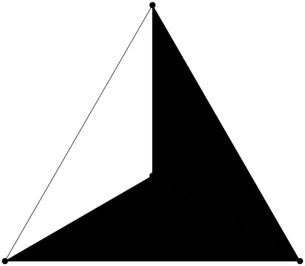

$T \geq 3$

Persistent homology:

$$\mathrm{H}_0(X) = \Bbbk[t]$$

$$\mathrm{H}_1(X) = \Bbbk[t] \oplus \frac{t\Bbbk[t]}{t^3\Bbbk[t]} \oplus \frac{t\Bbbk[t]}{t^2\Bbbk[t]}$$

Persistence Module

Persistent homology is a functor $V : (\mathbb{Z}, \leq) \to$ **Vect**

$$\mathrm{H}_0(X) : \qquad \ldots \to \mathbb{k} \to \mathbb{k} \to \mathbb{k} \to \mathbb{k} \to \ldots$$

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or a $\mathbb{k}[t]$-module

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or a $\Bbbk[t]$-module

$$\mathrm{H}_0(X) = \Bbbk[t].$$

Isomorphic categories:

$$\mathbf{Vect}^{(\mathbb{Z}, \leq)} \cong \mathbf{Mod}_{\Bbbk[t]}$$

Interleaving

Filtrations $f, g : X \to \mathbb{Z}$ with $\|f - g\|_\infty \leq \varepsilon \implies$ their homologies are ε-interleaved $\Bbbk[t]$-modules.

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Definition

$\Bbbk[t]$-modules M and N are ε-interleaved if there is a pair of ε-morphisms $f : M \xrightarrow{\varepsilon} N$ and $g : N \xrightarrow{\varepsilon} M$ such that

$$g(f(m)) = t^{2\varepsilon} m \qquad \text{and} \qquad f(g(n)) = t^{2\varepsilon} n.$$

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This yields a metric between isomorphism classes of modules:

$$d_I(M, N) = \min\{\varepsilon \in \mathbb{N}_0 \mid M \overset{\varepsilon}{\sim} N\}.$$

Mayer-Vietoris Spectral Sequence

X simplicial complex, $\mathcal{U} = (U_i)_{i \in \Lambda}$ cover $\implies$ double complex

$$E^0_{p,q} = \bigoplus_{|I|=p+1} C_q(U_I), \qquad U_I = \bigcap_{i \in I} U_i$$

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$$\partial^0_{p,q}(\sigma, I) = \sum_{k=0}^{q} (-1)^k t^{\deg(\sigma)-\deg(\sigma_k)}(\sigma_k, I)$$

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$$\partial^0_{p,q}(\sigma, I) = \sum_{k=0}^{q} (-1)^k t^{\deg(\sigma)-\deg(\sigma_k)} (\sigma_k, I)$$

and

$$\partial^1_{p,q}(\sigma, I) = \sum_{l=0}^{p} (-1)^l (\sigma, I_l).$$

Mayer-Vietoris Spectral Sequence

$$
\begin{array}{ccccccc}
E^0_{0,3} & \xleftarrow{\partial''} & E^0_{1,3} & \xleftarrow{\partial''} & E^0_{2,3} & \xleftarrow{\partial''} & E^0_{3,3} \\
\partial' \downarrow & & \partial' \downarrow & & \partial' \downarrow & & \partial' \downarrow \\
E^0_{0,2} & \xleftarrow{\partial''} & E^0_{1,2} & \xleftarrow{\partial''} & E^0_{2,2} & \xleftarrow{\partial''} & E^0_{3,2} \\
\partial' \downarrow & & \partial' \downarrow & & \partial' \downarrow & & \partial' \downarrow \\
E^0_{0,1} & \xleftarrow{\partial''} & E^0_{1,1} & \xleftarrow{\partial''} & E^0_{2,1} & \xleftarrow{\partial''} & E^0_{3,1} \\
\partial' \downarrow & & \partial' \downarrow & & \partial' \downarrow & & \partial' \downarrow \\
E^0_{0,0} & \xleftarrow{\partial''} & E^0_{1,0} & \xleftarrow{\partial''} & E^0_{2,0} & \xleftarrow{\partial''} & E^0_{3,0}
\end{array}
$$

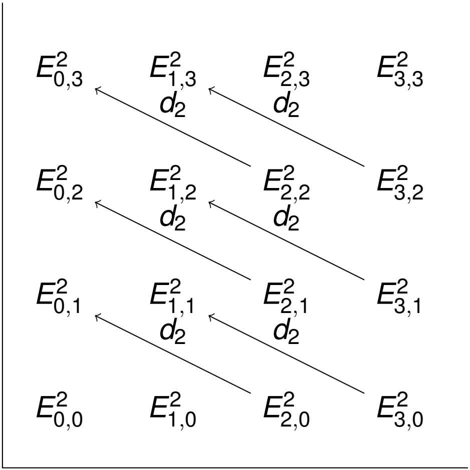
Mayer-Vietoris Spectral Sequence

$$
\begin{array}{cccc}
E^0_{0,3} & E^0_{1,3} & E^0_{2,3} & E^0_{3,3} \\
d_0 \downarrow & d_0 \downarrow & d_0 \downarrow & d_0 \downarrow \\
E^0_{0,2} & E^0_{1,2} & E^0_{2,2} & E^0_{3,2} \\
d_0 \downarrow & d_0 \downarrow & d_0 \downarrow & d_0 \downarrow \\
E^0_{0,1} & E^0_{1,1} & E^0_{2,1} & E^0_{3,1} \\
d_0 \downarrow & d_0 \downarrow & d_0 \downarrow & d_0 \downarrow \\
E^0_{0,0} & E^0_{1,0} & E^0_{2,0} & E^0_{3,0}
\end{array}
$$

Mayer-Vietoris Spectral Sequence

$$
\begin{array}{ccccccc}
E^1_{0,3} & \xleftarrow{d_1} & E^1_{1,3} & \xleftarrow{d_1} & E^1_{2,3} & \xleftarrow{d_1} & E^1_{3,3} \\
E^1_{0,2} & \xleftarrow{d_1} & E^1_{1,2} & \xleftarrow{d_1} & E^1_{2,2} & \xleftarrow{d_1} & E^1_{3,2} \\
E^1_{0,1} & \xleftarrow{d_1} & E^1_{1,1} & \xleftarrow{d_1} & E^1_{2,1} & \xleftarrow{d_1} & E^1_{3,1} \\
E^1_{0,0} & \xleftarrow{d_1} & E^1_{1,0} & \xleftarrow{d_1} & E^1_{2,0} & \xleftarrow{d_1} & E^1_{3,0}
\end{array}
$$

Mayer-Vietoris Spectral Sequence



Persistent Homology Nerve Theorem

Theorem (Chazal & Oudot, 2008)

If $\mathcal{U}$ is a cover by subcomplexes of a filtered simplicial complex X such that every nonempty intersection of finitely many sets in $\mathcal{U}$ is persistently acyclic, then

$$\mathrm{H}_*(X) \cong \mathrm{H}_*(\mathcal{N}(\mathcal{U})),$$

where $\mathcal{N}(\mathcal{U})$ is the nerve of $\mathcal{U}$.

Idea of Proof

Use the Mayer-Vietoris spectral sequence.

$$
\begin{array}{cccc}
E^0_{0,3} & E^0_{1,3} & E^0_{2,3} & E^0_{3,3} \\
d_0 \downarrow & d_0 \downarrow & d_0 \downarrow & d_0 \downarrow \\
E^0_{0,2} & E^0_{1,2} & E^0_{2,2} & E^0_{3,2} \\
d_0 \downarrow & d_0 \downarrow & d_0 \downarrow & d_0 \downarrow \\
E^0_{0,1} & E^0_{1,1} & E^0_{2,1} & E^0_{3,1} \\
d_0 \downarrow & d_0 \downarrow & d_0 \downarrow & d_0 \downarrow \\
E^0_{0,0} & E^0_{1,0} & E^0_{2,0} & E^0_{3,0}
\end{array}
$$

Idea of Proof

By acyclicity, E^1 is concentrated in the bottom row.

$$
\begin{array}{ccccccc}
0 & \xleftarrow{d_1} & 0 & \xleftarrow{d_1} & 0 & \xleftarrow{d_1} & 0 \\
0 & \xleftarrow{d_1} & 0 & \xleftarrow{d_1} & 0 & \xleftarrow{d_1} & 0 \\
0 & \xleftarrow{d_1} & 0 & \xleftarrow{d_1} & 0 & \xleftarrow{d_1} & 0 \\
E^1_{0,0} & \xleftarrow{d_1} & E^1_{1,0} & \xleftarrow{d_1} & E^1_{2,0} & \xleftarrow{d_1} & E^1_{3,0}
\end{array}
$$

Idea of Proof

$(E^1_{*,0}, d^1_{*,0}) \cong (C_*(\mathcal{N}), \partial)$ as chain complexes.

$$\begin{array}{ccccccc} C_0(\mathcal{N}) & \longleftarrow & C_1(\mathcal{N}) & \longleftarrow & C_2(\mathcal{N}) & \longleftarrow & C_3(\mathcal{N}) \\ \cong \downarrow & & \cong \downarrow & & \cong \downarrow & & \cong \downarrow \\ E^1_{0,0} & \longleftarrow & E^1_{1,0} & \longleftarrow & E^1_{2,0} & \longleftarrow & E^1_{3,0} \end{array}$$

Idea of Proof

$E^2_{*,0} \cong \mathrm{H}_*(\mathcal{N})$ as graded $\Bbbk[t]$-modules.

$$\begin{array}{cccc} \mathrm{H}_0(\mathcal{N}) & \mathrm{H}_1(\mathcal{N}) & \mathrm{H}_2(\mathcal{N}) & \mathrm{H}_3(\mathcal{N}) \\ \cong\downarrow & \cong\downarrow & \cong\downarrow & \cong\downarrow \\ E^2_{0,0} & E^2_{1,0} & E^2_{2,0} & E^2_{3,0} \end{array}$$

Idea of Proof

We may conclude that $\mathrm{H}_*(\mathcal{N}) \cong E^2_{*,0} \cong E^\infty_{*,0} \cong \mathrm{H}_*(X)$. □

Approximate Nerve Theorem, Easy Version

Theorem (G & Škraba, 2016)

If $\mathcal{U}$ is an ε-acyclic cover of X and $D = \dim \mathcal{N}(\mathcal{U}) < \infty$, we have

$$\mathrm{H}_*(X) \overset{(4D+2)\varepsilon}{\sim} \mathrm{H}_*(\mathcal{N}).$$

Idea of Proof

Use Mayer-Vietoris and imitate previous proof.

$$
\begin{array}{cccc}
E^0_{0,3} & E^0_{1,3} & E^0_{2,3} & E^0_{3,3} \\
d_0 \downarrow & d_0 \downarrow & d_0 \downarrow & d_0 \downarrow \\
E^0_{0,2} & E^0_{1,2} & E^0_{2,2} & E^0_{3,2} \\
d_0 \downarrow & d_0 \downarrow & d_0 \downarrow & d_0 \downarrow \\
E^0_{0,1} & E^0_{1,1} & E^0_{2,1} & E^0_{3,1} \\
d_0 \downarrow & d_0 \downarrow & d_0 \downarrow & d_0 \downarrow \\
E^0_{0,0} & E^0_{1,0} & E^0_{2,0} & E^0_{3,0}
\end{array}
$$

Idea of Proof

By ε-acyclicity, $E^1_{p,q} \overset{\varepsilon}{\sim} 0$ holds above the bottom row.

$$
\begin{array}{ccccccc}
2\varepsilon & \xleftarrow{d_1} & 2\varepsilon & \xleftarrow{d_1} & 2\varepsilon & \xleftarrow{d_1} & 2\varepsilon \\
2\varepsilon & \xleftarrow{d_1} & 2\varepsilon & \xleftarrow{d_1} & 2\varepsilon & \xleftarrow{d_1} & 2\varepsilon \\
2\varepsilon & \xleftarrow{d_1} & 2\varepsilon & \xleftarrow{d_1} & 2\varepsilon & \xleftarrow{d_1} & 2\varepsilon \\
E^1_{0,0} & \xleftarrow{d_1} & E^1_{1,0} & \xleftarrow{d_1} & E^1_{2,0} & \xleftarrow{d_1} & E^1_{3,0}
\end{array}
$$

Idea of Proof

$(E^1_{*,0}, d^1_{*,0}) \overset{2\varepsilon}{\sim} (\mathrm{C}_*(\mathcal{N}), \partial)$ as chain complexes.

$$\begin{array}{ccccccc}
\mathrm{C}_0(\mathcal{N}) & \longleftarrow & \mathrm{C}_1(\mathcal{N}) & \longleftarrow & \mathrm{C}_2(\mathcal{N}) & \longleftarrow & \mathrm{C}_3(\mathcal{N}) \\
\wr\, 2\varepsilon & & \wr\, 2\varepsilon & & \wr\, 2\varepsilon & & \wr\, 2\varepsilon \\
E^1_{0,0} & \longleftarrow & E^1_{1,0} & \longleftarrow & E^1_{2,0} & \longleftarrow & E^1_{3,0}
\end{array}$$

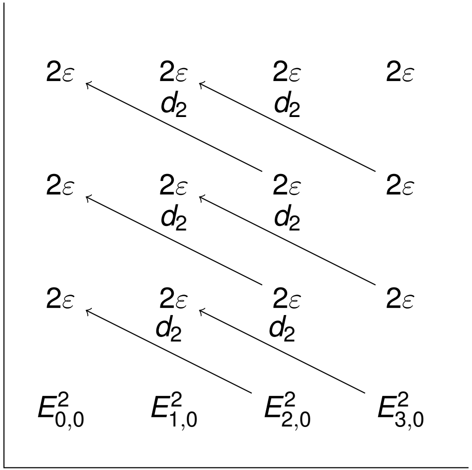
Idea of Proof

$E^2_{*,0} \overset{2\varepsilon}{\sim} \mathrm{H}_*(\mathcal{N})$ as graded $\Bbbk[t]$-modules.

$$\begin{array}{cccc} \mathrm{H}_0(\mathcal{N}) & \mathrm{H}_1(\mathcal{N}) & \mathrm{H}_2(\mathcal{N}) & \mathrm{H}_3(\mathcal{N}) \\ \wr\, 2\varepsilon & \wr\, 2\varepsilon & \wr\, 2\varepsilon & \wr\, 2\varepsilon \\ E^2_{0,0} & E^2_{1,0} & E^2_{2,0} & E^2_{3,0} \end{array}$$

Idea of Proof

At each step, we have $E^{r+1} \overset{2\varepsilon}{\sim} E^r$.



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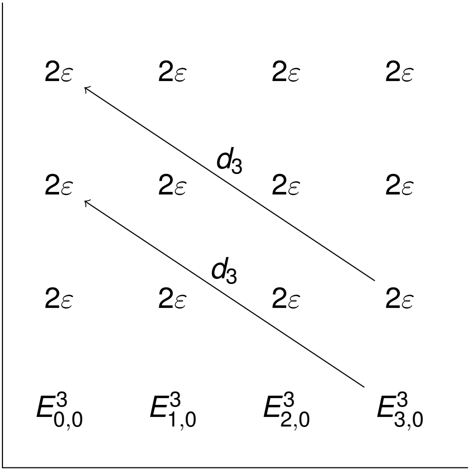

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- We conclude $H_*(\mathcal{N}) \overset{2\varepsilon}{\sim} E^2_{*,0} \overset{2\varepsilon}{\sim} \dots \overset{2\varepsilon}{\sim} E^{D+1}_{*,0} = E^{\infty}_{*,0}$.

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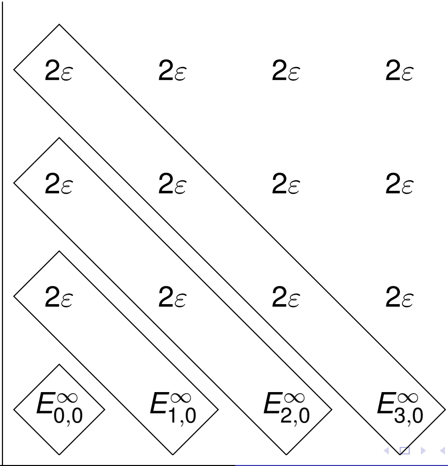
- After D steps, the spectral sequence collapses.
- We conclude $H_*(\mathcal{N}) \overset{2\varepsilon}{\sim} E^2_{*,0} \overset{2\varepsilon}{\sim} \dots \overset{2\varepsilon}{\sim} E^{D+1}_{*,0} = E^\infty_{*,0}$.
- Therefore, $E^\infty_{*,0} \overset{2D\varepsilon}{\sim} H_*(\mathcal{N})$.

Idea of Proof

To obtain $H_*(X)$ from E^∞, solve a series of extension problems.



Idea of Proof

For $0 \leq n \leq D$, we obtain:

$$\mathrm{H}_n(X) \cong \frac{\mathrm{H}_n(X)^n}{\mathrm{H}_n(X)^{-1}} \overset{2\varepsilon}{\sim} \dots \overset{2\varepsilon}{\sim} \frac{\mathrm{H}_n(X)^n}{\mathrm{H}_n(X)^{n-2}} \overset{2\varepsilon}{\sim} \frac{\mathrm{H}_n(X)^n}{\mathrm{H}_n(X)^{n-1}} \cong E^{\infty}_{n,0},$$

while $n > D$ is treated separately by a similar argument.

Idea of Proof

We finally conclude that

$$\mathrm{H}_*(\mathcal{N}) \overset{2D\varepsilon}{\sim} E^\infty \overset{2(D+1)\varepsilon}{\sim} \mathrm{H}_*(X),$$

as desired. □

Approximate Nerve Theorem

Theorem (G & Škraba, 2016)

Let $D = \dim \mathcal{N}(\mathcal{U})$, $\Delta = \dim X$ *and* $Q = \min(D, \Delta)$.
If $\mathcal{U}$ *is an* ε*-acyclic cover of* X *and* $D < \infty$, *we have*

$$\mathrm{H}_*(X) \overset{2(Q+1)\varepsilon}{\sim} \mathrm{H}_*(\mathcal{N}).$$

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To prove this, we introduce left- and right-interleavings.

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To prove this, we introduce left- and right-interleavings.

This bound is sharp, as we show by explicit examples.

Summary

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For more details, see arXiv:1608.06956.

Summary

- Given an approximately acyclic cover $\mathcal{U}$ of a filtered simplicial complex X, the persistent homology of the nerve $\mathcal{N}(\mathcal{U})$ is approximately isomorphic to the persistent homology of X.
- The error can be estimated precisely.

For more details, see arXiv:1608.06956.

Thank you for your attention!