Transformations between Singer-Thorpe bases in 4-dimensional Einstein manifolds

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Singer, I.M. and Thorpe, J.A.: The curvature of 4-dimensional Einstein spaces, in Global Analysis, Papers in Honor of K. Kodaira, University of Tokyo Press and Princeton University Press, 1968, 355–365.

Theorem

If (M, g) is a 4-dimensional Einstein Riemannian manifold and R its curvature tensor at some fixed point p, then there is an orthonormal basis $\mathcal{B} = \{e_1, e_2, e_3, e_4\}$ in T_pM such that the complementary sectional curvatures are equal, i.e. $K_{12} = K_{34}, K_{13} = K_{24}, K_{14} = K_{23}$, and all corresponding components R_{ijkl} with exactly three distinct indices are zero.

Such a basis is called a Singer-Thorpe basis or an ST basis.

We study ST bases in a purely algebraic way.

Gilkey, P.B.: The Geometry of Curvature Homogeneous Pseudo-Riemannian Manifolds, ICP Advanced Texts in Mathematics - Vol 2, Imperial College Press, 2007.

Definition

An *algebraic curvature tensor* on a vector space $\mathbb{V}$ with a positive scalar product $\langle , \rangle$ is a tensor R of the type $(0, 4)$ on $\mathbb{V}$ which satisfies

$$\begin{aligned} R(U, V, W, Z) = -R(V, U, W, Z) = R(W, Z, U, V), \\ R(U, V, W, Z) + R(V, W, U, Z) + R(W, U, V, Z) = 0 \end{aligned}$$

for all $U, V, W, Z \in \mathbb{V}$. Further, a triplet $(\mathbb{V}, \langle , \rangle, R)$ as above is Einstein if the corresponding Ricci tensor ρ on $\mathbb{V}$ satisfies the identity $\rho = \lambda\langle , \rangle$ for some $\lambda \in \mathbb{R}$.

The algebraic version of Theorem 1:

Theorem

Let $\mathbb{V}$ be a 4-dimensional vector space provided with a positive scalar product $\langle , \rangle$. Let R be an Einstein algebraic curvature tensor on $\mathbb{V}$. Then there is an orthonormal basis $\mathcal{B} = \{e_1, e_2, e_3, e_4\}$ of $\mathbb{V}$ such that the nontrivial components of R with respect to $\mathcal{B}$ are, up to standard symmetries and antisymmetries, the following:

$$R_{1212} = R_{3434} = A, \quad R_{1313} = R_{2424} = B, \quad R_{1414} = R_{2323} = C,$$
$$R_{1234} = F, \qquad R_{1423} = G, \qquad R_{1342} = H,$$

where A, B, C, F, G are some constants satisfying $F + G + H = 0$. On the other hand, all components R_{ijkl} with exactly three distinct indices are zero.

Definition

An orthonormal basis $\mathcal{B} = \{e_1, e_2, e_3, e_4\}$ of $\mathbb{V}$ with the properties given above is called an ST basis on $\mathbb{V}$ corresponding to the curvature tensor R.

Definition

Let $(\mathbb{V}, \langle , \rangle, R)$ be an Einstein triplet. Then $\mathbb{V}$ is called 2-stein if it satisfies the following additional condition:

$$\mathcal{F}(X) = \sum_{i,j=1}^{4} (R(X, e_i, X, e_j))^2,$$

where $\mathcal{B} = \{e_1, \ldots, e_n\}$ is any orthonormal basis, is independent on the choice of the unit vector $X \in \mathbb{V}$.

Then, we have the following

Proposition

An Einstein triplet $(\mathbb{V}, \langle , \rangle, R)$ of dimension 4 is 2-stein if and only if

$$\pm F = A - \frac{\tau}{12}, \qquad \pm H = B - \frac{\tau}{12}, \qquad \pm G = C - \frac{\tau}{12}$$

hold with respect to any ST basis of $\mathbb{V}$. Here $\tau = \sum_{i=1}^{n} \rho(e_i, e_i)$.

Kowalski, O. and Vanhecke, L.: Ball-homogeneous and disk-homogeneous Riemannian manifolds, Math. Z. **80** (1982), 429–444.

Sekigawa, K. and Vanhecke, L.: Volume-preserving geodesic symmetries on four-dimensional Kähler manifolds, Differential Geometry Peñiscola, (1985), 275–291.

Proposition

Let $(\mathbb{V}, \langle , \rangle, R)$ *be an Einstein triplet. Then the following two assertions are equivalent:*
(i) *For any unit vector* $X \in T_pM$*, the quadruplet* $\{X, J_1X, J_2X, J_3X\}$ *is an ST basis for* R*.*
(*For the natural quaternionic structure* $\{J_1, J_2, J_3\}$ *on* T_pM*.*)
(ii) $(\mathbb{V}, \langle , \rangle, R)$ *is 2-stein.*

Motivated also by research in so-called weakly Einstein spaces

Euh, Y., Park, J. and Sekigawa, K.: A generalization of a 4-dimensional Einstein manifold, Mathematica Slovaca, **63** (2013), 595–610.

Euh, Y., Park, J. and Sekigawa, K.: Critical metrics for quadratic functionals in the curvature on 4-dimensional manifolds, Differ. Geom. Appl. **29** (2011), 642–646.

K. Sekigawa put the following, more general question:

Let (M, g) be a 4-dimensional Einstein manifold and $\{e_1, \dots, e_4\}$ be an arbitrary fixed ST basis at any point p. Determine the relation between the fixed ST basis $\mathcal{B} = \{e_1, \dots, e_4\}$ and all ST bases $\mathcal{B}' = \{\bar{e}_1, \dots, \bar{e}_4\}$ at p.

Preliminaries

Let $P = (a^i_j) \in \mathrm{O}(4)$ be the matrix of an orthogonal transformation acting on the set of orthonormal bases of $(\mathbb{V}, \langle , \rangle)$: if $\mathcal{B} = \{e_i\}_{i=1}^4$ is an orthonormal basis, the new orthonormal basis $\mathcal{B}P = \mathcal{B}' = \{e'_j\}_{j=1}^4$ is given as $e'_j = \sum_{i=1}^4 e_i a^i_j$.

Let us denote by P^{ij}_{kl} the 2×2 submatrix of the matrix P formed by the elements in the rows i, j and in the columns k, l and $d^{ij}_{kl} = \det(P^{ij}_{kl})$.

Lemma

Let $\mathcal{B}$ be an ST basis for an Einstein algebraic curvature tensor R in which the components of R are given by the fixed A, B, C, F, G. Then the components of the tensor R in the new basis $\mathcal{B}P$ are given by the formula

$$
\begin{aligned}
R'_{ijkl} \quad = \quad & (d^{12}_{ij} \cdot d^{12}_{kl} + d^{34}_{ij} \cdot d^{34}_{kl})\, A \quad + \\
& (d^{13}_{ij} \cdot d^{13}_{kl} + d^{24}_{ij} \cdot d^{24}_{kl})\, B \quad + \\
& (d^{14}_{ij} \cdot d^{14}_{kl} + d^{23}_{ij} \cdot d^{23}_{kl})\, C \quad + \\
& (d^{12}_{ij} \cdot d^{34}_{kl} + d^{34}_{ij} \cdot d^{12}_{kl})\, F \quad + \\
& (d^{14}_{ij} \cdot d^{23}_{kl} + d^{23}_{ij} \cdot d^{14}_{kl})\, G \quad + \\
& (d^{13}_{ij} \cdot d^{42}_{kl} + d^{42}_{ij} \cdot d^{13}_{kl})\, H.
\end{aligned}
$$

Let $\mathcal{B} = \{e_1, e_2, e_3, e_4\}$ be an ST basis for an Einstein algebraic curvature tensor R on $(V, \langle , \rangle)$.

Lemma

For any matrix $P \in \mathrm{O}(4)$, the components of the tensor R in the new basis satisfy

$$R'_{1212} = R'_{3434}, \quad R'_{1313} = R'_{2424}, \quad R'_{1414} = R'_{2323}.$$

We are interested in transformations $P \in \mathrm{O}(4)$ such that in the new bases $\mathcal{B}' = \mathcal{B}P$, the tensor R have all components with just three different indices equal to zero.
Namely $R'_{ijkl} = 0$ for the following 12 choices of i, j, k, l:

$$1213, 1214, 1223, 1224, 2324, 2334,$$
$$1314, 1323, 1334, 1424, 1434, 2434.$$

The basic finite group of transformations

$\mathcal{H}_1 \subset \mathrm{O}(4)$ the group of all permutation matrices
$\mathcal{H}_2 \subset \mathrm{O}(4)$ the group of all diagonal matrices
with ± 1 on the diagonal.

$|\mathcal{H}_1| = 24$ and $|\mathcal{H}_2| = 16$.
$\mathcal{H}_3 = \mathcal{H}_1 \cdot \mathcal{H}_2 = \mathcal{H}_2 \cdot \mathcal{H}_1$, $|\mathcal{H}_3| = 16 \cdot 24 = 384$.

For all $\underline{P \in \mathcal{H}_3, \mathcal{B}P \text{ are ST bases}}$ for R.

Special matrices

$$P_4 = \frac{1}{2} \begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix}, P_5 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

The components of R in the basis $\mathcal{B}P_4$ are

$$\begin{array}{lcl} A' = & R'_{1212} = R'_{3434} & = 1/2(B + C - F - 2G), \\ B' = & R'_{1313} = R'_{2424} & = 1/2(A + C - F + G), \\ C' = & R'_{1414} = R'_{2323} & = 1/2(A + B + 2F + G), \\ F' = & R'_{1234} & = 1/2(-B + C + F), \\ G' = & R'_{1423} & = 1/2(-A + B + G). \end{array}$$

and $R'_{ijkl} = 0$ for all 12 choices of $ijkl$.
The components of R in the basis $\mathcal{B}P_5$ are

$$\begin{array}{lcl} A' = & R'_{1212} = R'_{3434} & = A, \\ B' = & R'_{1313} = R'_{2424} & = 1/2(B + C + F + 2G), \\ C' = & R'_{1414} = R'_{2323} & = 1/2(B + C - F - 2G), \\ F' = & R'_{1234} & = F, \\ G' = & R'_{1423} & = 1/2(B - C - F) \end{array}$$

and $R'_{ijkl} = 0$ for all 12 choices of $ijkl$.

We see that both $\mathcal{B}P_4$ and $\mathcal{B}P_5$ are ST bases for R.

Lemma

The group $\mathcal{H}_4$ generated by $\mathcal{H}_3$ and P_4 is the union of cosets
$\mathcal{H}_4 = \mathcal{H}_3 \cup \mathcal{H}_3 P_4 \cup \mathcal{H}_3 P_4'$.
It holds $|\mathcal{H}_4| = 3 \cdot 384 = 1152$.

Lemma

The group $\mathcal{H}_5$ generated by $\mathcal{H}_4$ and P_5 is the union of cosets
$\mathcal{H}_5 = \mathcal{H}_4 \cup \mathcal{H}_4 P_5$.
It holds $|\mathcal{H}_5| = 2 \cdot 1152 = 2304$.

For all $P \in \mathcal{H}_5$, the bases $\mathcal{B}P$ are ST bases for R.
(We denote this set of bases by $\mathcal{B}\mathcal{H}_5$.)

Observations

We recall the formula for the transformation of the tensor R:

$$
\begin{aligned}
R'_{ijkl} = \; & (d_{ij}^{12} \cdot d_{kl}^{12} + d_{ij}^{34} \cdot d_{kl}^{34})\, A \; + \\
& (d_{ij}^{13} \cdot d_{kl}^{13} + d_{ij}^{24} \cdot d_{kl}^{24})\, B \; + \\
& (d_{ij}^{14} \cdot d_{kl}^{14} + d_{ij}^{23} \cdot d_{kl}^{23})\, C \; + \\
& (d_{ij}^{12} \cdot d_{kl}^{34} + d_{ij}^{34} \cdot d_{kl}^{12})\, F \; + \\
& (d_{ij}^{14} \cdot d_{kl}^{23} + d_{ij}^{23} \cdot d_{kl}^{14})\, G \; + \\
& (d_{ij}^{13} \cdot d_{kl}^{42} + d_{ij}^{42} \cdot d_{kl}^{13})\, H.
\end{aligned}
$$

- For $P \in \mathcal{H}_5$, all coefficients are zero for the 12 choices of indices
- For *special* R one gets weaker conditions

The universal Singer-Thorpe group

Let us now fix an orthonormal basis $\mathcal{B}$ of $(\mathbb{V}, \langle , \rangle)$ and consider the set of *all* tensors R for which $\mathcal{B}$ is an ST basis.

Denote by $\mathcal{S}$ the set of bases which are ST bases *for all these tensors* and denote by $\mathcal{G}$ the set of orthogonal matrices corresponding to all transformations between the bases from $\mathcal{S}$.

$\mathcal{G} \subset \mathrm{O}(4)$ is a group, independent of the initial basis $\mathcal{B}$.

Theorem

The group $\mathcal{G}$ is just the group $\mathcal{H}_5$ of 2304 elements.

Dušek, Z and Kowalski, O.: Transformations between Singer Thorpe bases for 4-dimensional Einstein manifolds, Hokkaido Math. J., **44** (2015), 441–458.

Special Lie groups and corresponding invariant tensors

We consider the following matrix

$$X = \begin{pmatrix} 0 & s_1 & s_2 & s_3 \\ -s_1 & 0 & s_4 & s_5 \\ -s_2 & -s_4 & 0 & s_6 \\ -s_3 & -s_5 & -s_6 & 0 \end{pmatrix} \in \mathfrak{so}(4)$$

and the 1-parameter group $P(t) = \exp(tX) = E + tX + o(t^2)$.

We consider again a fixed ST basis $\mathcal{B}$ for R.
We want to determine the necessary conditions
for the new bases $\mathcal{B}P(t)$ to be ST bases.

The components of the tensor R in the new bases $\mathcal{B}P(t)$ are

$$\begin{aligned}
R'_{1214} &= \big[s_2(G-F)+s_5(A-C)\big]t+o(t^2),\\
R'_{1223} &= \big[s_2(C-A)+s_5(F-G)\big]t+o(t^2),\\
R'_{1224} &= \big[s_3(B-A)+s_4(H-F)\big]t+o(t^2),\\
R'_{1312} &= \big[s_3(F-H)+s_4(A-B)\big]t+o(t^2),\\
R'_{1314} &= \big[s_1(H-G)+s_6(B-C)\big]t+o(t^2),\\
R'_{1323} &= \big[s_1(B-C)+s_6(H-G)\big]t+o(t^2).
\end{aligned}$$

We rewrite the necessary conditions under which a 1-parameter group transforms the ST basis $\mathcal{B}$ into new ST bases in the form

$$\begin{aligned}
(s_5-s_2)(A-C+F-G) &= 0,\\
(s_5+s_2)(A-C-F+G) &= 0,\\
(s_4-s_3)(A-B+H-F) &= 0,\\
(s_4+s_3)(A-B-H+F) &= 0,\\
(s_6+s_1)(B-C+H-G) &= 0,\\
(s_6-s_1)(B-C-H+G) &= 0.
\end{aligned}$$

Definition

Let us denote by $G_1, G_2, G_3, H_1, H_2, H_3$ the 1-parameter subgroups of the matrix group $\mathrm{SO}(4)$, each of them defined as $\exp(tX)$ for $X \in \mathfrak{so}(4)$ with just two nonzero parameters s_i satisfying the corresponding condition in the second column of the table below and with other four parameters s_i equal to zero.

Proposition

Let $\mathcal{B}$ be an ST basis for an Einstein algebraic curvature tensor R and let $\mathcal{G}$ be some of the 1-parameter matrix groups $G_1, \ldots, H_3$. If all the new bases $\mathcal{B}\mathcal{G}$ are ST bases, the tensor R must satisfy the corresponding condition in the following table.

$$\begin{array}{lll}
G_1: & s_1 + s_6 = 0, & B - C - H + G = 0. \\
G_2: & s_3 + s_4 = 0, & A - B + H - F = 0, \\
G_3: & s_2 - s_5 = 0, & A - C - F + G = 0, \\
H_1: & s_1 - s_6 = 0, & B - C + H - G = 0, \\
H_2: & s_3 - s_4 = 0, & A - B - H + F = 0, \\
H_3: & s_2 + s_5 = 0, & A - C + F - G = 0.
\end{array}$$

Proposition

Let R be an Einstein curvature tensor in an ST basis $\mathcal{B}$ and let $\mathcal{G} = H_1$. All the new bases $\mathcal{B}\mathcal{G}$ are ST bases for the tensor R if and only if it satisfies

$$B - C + H - G \quad = \quad 0.$$

If these equivalent conditions are true, then, in any new basis $\mathcal{B}'$, the new components A', B', C', F', H', G' of the tensor R are the same as the original components A, B, C, F, H, G.

Proposition

Let $\mathcal{G} = H_1$ be the special representation of the group $\mathrm{SO}(2)$ described above and let R be any tensor from the family satisfying the corresponding homogeneous linear condition in a given ST basis $\mathcal{B}$. Let $p \in \mathcal{H}_5$ and $\mathcal{G}' = p^{-1}\mathcal{G}p$.

The matrix group $\mathcal{G}'$ is also one of the six special representations $G_1, \ldots, H_3$ of the group $\mathrm{SO}(2)$ and it transforms ST basis $\mathcal{B}p$ for the tensor R into new ST bases. In all these new bases $\mathcal{B}p\mathcal{G}'$, the tensor R satisfies the corresponding condition for the group $\mathcal{G}'$.

Example

Let $\mathcal{G} = H_1$ and let the tensor R in an ST basis $\mathcal{B}$ satisfies

$$B - C + H - G \quad = \quad 0.$$

For example, we use for p the transposition $p = (23)$.
The tensor R changes by the equations

$$\begin{gathered} A' = B, \quad B' = A, \quad C' = C, \\ F' = -H, \quad H' = -F, \quad G' = -G. \end{gathered}$$

We see that the tensor R in the new basis $\mathcal{B}p$ satisfies the equation

$$A' - C' - F' + G' \quad = \quad 0,$$

which corresponds to the group $\mathcal{G}' = G_3$. □

Theorem

Let R be an Einstein curvature tensor in an ST basis $\mathcal{B}$. The group $\mathrm{SO}(2)$ acts as a transformation group between ST bases if and only if the tensor R satisfies at least one of the following equations, each equation corresponds to a particular representation of the group $\mathrm{SO}(2)$.

$$\begin{aligned} H_1: &\quad B - C + H - G = 0, & \qquad G_1: &\quad B - C - H + G = 0, \\ H_2: &\quad A - B - H + F = 0, & \qquad G_2: &\quad A - B + H - F = 0, \\ H_3: &\quad A - C + F - G = 0, & \qquad G_3: &\quad A - C - F + G = 0. \end{aligned}$$

Classification

Let R be an Einstein curvature tensor in an ST basis $\mathcal{B}$.

- The group $\underline{\mathrm{SO}(2)}$
  R satisfies at least one of the equations.
- The torus group $\underline{T^2 = \mathrm{SO}(2) \times \mathrm{SO}(2)}$
  R satisfies at least one of the equations of the type $\overline{\mathrm{Sp}}(1)$
  and at least one of the equations of the type $\widetilde{\mathrm{Sp}}(1)$.
- The group $\underline{\mathrm{Sp}(1)}$
  R satisfies equations of the type $\overline{\mathrm{Sp}}(1)$
  or equations of the type $\widetilde{\mathrm{Sp}}(1)$. [2-stein]
- The group $\underline{\mathrm{U}(2) \simeq \mathrm{Sp}(1) \times \mathrm{SO}(2)}$
  R satisfies either equations of the type $\overline{\mathrm{Sp}}(1)$
  and at least one of the equations of the type $\widetilde{\mathrm{Sp}}(1)$
  or equations of the type $\widetilde{\mathrm{Sp}}(1)$
  and at least one of the equations of the type $\overline{\mathrm{Sp}}(1)$.
- The group $\underline{\mathrm{O}(4)}$.
  R satisfies $A = B = C$, $F = G = H = 0$. [constant curvature]

The set of all ST bases for a fixed tensor R

For each tensor R and a fixed ST basis $\mathcal{B}$, transformations of the type

$$p = gh, \qquad g \in \mathcal{G}, \quad h \in \mathcal{H}_5,$$

transform given ST basis $\mathcal{B}$ into a new ST basis.

These transformations can be composed:

- $p = gh$ $(g \in \mathcal{G}, h \in \mathcal{H}_5) \dots$ a transformation from an ST basis $\mathcal{B}$ into a new ST basis $\mathcal{B}'$ (written with respect to the basis $\mathcal{B}$)
- $p' = g'h'$ $(g' \in \mathcal{G}', h' \in \mathcal{H}_5) \dots$ a transformation from an ST basis $\mathcal{B}'$ into a new ST basis $\mathcal{B}''$ (written with respect to $\mathcal{B}'$)

Then $g' = h^{-1}\bar{g}h$ for some $\bar{g} \in \mathcal{G}$ and the transformation $p \circ p'$ from the ST basis $\mathcal{B}$ into the ST basis $\mathcal{B}''$ (with respect to $\mathcal{B}$) is

$$p \circ p' = (gh) \circ (g'h') = ghh^{-1}\bar{g}hh' = g\bar{g}hh'.$$

An open question

An open question remains whether the ST bases obtained this way from one given ST basis $\mathcal{B}$ are all possible ST bases for the given tensor R.

The discrete group $\mathcal{H}_5$ was found as transformations which can be applied to and ST basis of *arbitrary* tensor R and obtain a new ST basis.

It is not disproved, that for special tensors R there may exist more *discrete* transformations which can be taken for h above.

References

Dušek, Z and Kowalski, O.: Transformations between Singer Thorpe bases for 4-dimensional Einstein manifolds, Hokkaido Math. J., **44** (2015), 441–458.

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