CURVATURE PROPERTIES
OF SOME CLASS
OF WARPED PRODUCT MANIFOLDS

Dedicated to the memory of Professor Mileva Prvanović
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Some endomorphisms

Let \((M, g)\) be a connected \(n\)-dimensional, \(n \geq 3\), semi-Riemannian manifold of class \(C^\infty\) and \(\nabla\) its Levi-Civita connection. We define on \(M\) the endomorphisms \(X \wedge_A Y\), \(\mathcal{R}(X, Y)\) and \(\mathcal{C}(X, Y)\) by

\[
(X \wedge_A Y)Z = A(Y, Z)X - A(X, Z)Y,
\]
\[
\mathcal{R}(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z,
\]
\[
\mathcal{C}(X, Y) = \mathcal{R}(X, Y) - \frac{1}{n-2} (X \wedge g S Y + S X \wedge g Y)
\]
\[
- \frac{\kappa}{(n-2)(n-1)} X \wedge g Y,
\]
where \(\Xi(M)\) is the Lie algebra of vector fields of \(M\), \(X, Y, Z \in \Xi(M)\), \(S\) - the Ricci tensor and \(S\) - the Ricci operator

\[
S(X, Y) = \text{tr}\{Z \rightarrow \mathcal{R}(Z, X) Y\},
\]
\[
g(SX, Y) = S(X, Y),
\]
\[
\kappa = \text{tr} S\] - the scalar curvature and \(A\) - a symmetric \((0, 2)\)-tensor.
Some (0, 4)-tensors

The Riemann-Christoffel curvature tensor $R$, the Weyl conformal curvature tensor $C$ and the (0, 4)-tensor $G$ of $(M, g)$ are defined by

\[
R(X_1, X_2, X_3, X_4) = g(R(X_1, X_2)X_3, X_4),
\]

\[
C(X_1, X_2, X_3, X_4) = g(C(X_1, X_2)X_3, X_4),
\]

\[
G(X_1, X_2, X_3, X_4) = g((X_1 \wedge g X_2)X_3, X_4),
\]

respectively, where $X_1, \ldots, X_4 \in \Xi(M)$. 
The Kulkarni-Nomizu product $E \wedge F$

For symmetric $(0,2)$-tensors $E$ and $F$ we define their *Kulkarni-Nomizu product* $E \wedge F$ by

$$(E \wedge F)(X_1, X_2, X_3, X_4) = E(X_1, X_4)F(X_2, X_3) + E(X_2, X_3)F(X_1, X_4) - E(X_1, X_3)F(X_2, X_4) - E(X_2, X_4)F(X_1, X_3),$$

where $X_1, \ldots, X_4 \in \Xi(M)$.

Now the Weyl tensor $C$ can be presented in the form

$$C = R - \frac{1}{n-2} g \wedge S + \frac{\kappa}{(n-2)(n-1)} G,$$

where

$$G = \frac{1}{2} g \wedge g.$$
The Kulkarni-Nomizu product $E \wedge T$

For symmetric $(0, 2)$-tensor $E$ and an $(0, k)$-tensor $T$, $k \geq 3$, we define their *Kulkarni-Nomizu product* $E \wedge T$ by (see, e.g., [DG])

\[(E \wedge T)(X_1, X_2, X_3, X_4, Y_3, \ldots, Y_k) = E(X_1, X_4)T(X_2, X_3, Y_3, \ldots, Y_k) + E(X_2, X_3)T(X_1, X_4, Y_3, \ldots, Y_k) - E(X_1, X_3)T(X_2, X_4, Y_3, \ldots, Y_k) - E(X_2, X_4)T(X_1, X_3, Y_3, \ldots, Y_k),\]

where $X_1, \ldots, X_4, Y_3, \ldots, Y_k \in \Xi(M)$.

For a symmetric \((0, 2)\)-tensor \(A\) and a \((0, k)\)-tensor \(T\), \(k \geq 1\), we define the \((0, k + 2)\)-tensors \(R \cdot T\), \(C \cdot T\) and \(Q(A, T)\) by

\[
(R \cdot T)(X_1, \ldots, X_k; X, Y) = (\mathcal{R}(X, Y) \cdot T)(X_1, \ldots, X_k)
\]

\[
= -T(\mathcal{R}(X, Y)X_1, X_2, \ldots, X_k) - \cdots - T(X_1, \ldots, X_{k-1}, \mathcal{R}(X, Y)X_k),
\]

\[
(C \cdot T)(X_1, \ldots, X_k; X, Y) = (\mathcal{C}(X, Y) \cdot T)(X_1, \ldots, X_k)
\]

\[
= -T(\mathcal{C}(X, Y)X_1, X_2, \ldots, X_k) - \cdots - T(X_1, \ldots, X_{k-1}, \mathcal{C}(X, Y)X_k),
\]

\[
Q(A, T)(X_1, \ldots, X_k; X, Y) = ((X \wedge_A Y) \cdot T)(X_1, \ldots, X_k)
\]

\[
= -T((X \wedge_A Y)X_1, X_2, \ldots, X_k) - \cdots - T(X_1, \ldots, X_{k-1}, (X \wedge_A Y)X_k),
\]

respectively. Setting in the above formulas \(T = R\), \(T = S\), \(T = C\), \(A = g\) or \(A = S\) we obtain the tensors: \(R \cdot R\), \(R \cdot C\), \(C \cdot R\), \(C \cdot C\), \(R \cdot S\) and \(C \cdot S\), and \(Q(g, R)\), \(Q(g, C)\), \(Q(S, R)\), \(Q(S, C)\) and \(Q(g, S)\).
Some $(0, k)$-tensors - Tachibana tensors

Let $A$ be a symmetric $(0, 2)$-tensor and $T$ a $(0, k)$-tensor. The tensor $Q(A, T)$ is called the *Tachibana tensor of $A$ and $T$*, or the *Tachibana tensor* for short ([DGPSS]).

We like to point out that in some papers the tensor $Q(g, R)$ is called the *Tachibana tensor* (see, e.g., [HV], [JHSV], [JHP-TV]).


Some subsets of semi-Riemannian manifolds

Let \((M, g), \ n \geq 4\), be a semi-Riemannian manifold. We define the following subset of \(M\):

\[
\mathcal{U}_R = \{ x \in M | R \neq \frac{\kappa}{(n-1)n} G \text{ at } x \}, \quad G = \frac{1}{2} g \wedge g,
\]

\[
\mathcal{U}_S = \{ x \in M | S \neq \frac{\kappa}{n} g \text{ at } x \},
\]

\[
\mathcal{U}_C = \{ x \in M | C \neq 0 \text{ at } x \}.
\]

We note that \(\mathcal{U}_S \cup \mathcal{U}_C = \mathcal{U}_R\).
Warped product manifolds

Let $(\overline{M}, \overline{g})$ and $(\tilde{N}, \tilde{g})$, $\dim \overline{M} = p$, $\dim N = n - p$, $1 \leq p < n$, be semi-Riemannian manifolds covered by systems of charts $\{U; x^a\}$ and $\{V; y^\alpha\}$, respectively. Let $F$ be a positive smooth function on $\overline{M}$. The warped product $\overline{M} \times_F \tilde{N}$ of $(\overline{M}, \overline{g})$ and $(\tilde{N}, \tilde{g})$ is the product manifold $\overline{M} \times \tilde{N}$ with the metric $g = \overline{g} \times_F \tilde{g}$ defined by

$$\overline{g} \times_F \tilde{g} = \pi_1^* \overline{g} + (F \circ \pi_1) \pi_2^* \tilde{g},$$

where $\pi_1 : \overline{M} \times \tilde{N} \rightarrow \overline{M}$ and $\pi_2 : \overline{M} \times \tilde{N} \rightarrow \tilde{N}$ are the natural projections on $\overline{M}$ and $\tilde{N}$, respectively. Let $\{U \times V; x^1, \ldots, x^p, x^{p+1} = y^1, \ldots, x^n = y^{n-p}\}$ be a product chart for $\overline{M} \times \tilde{N}$. The local components $g_{ij}$ of the metric $g = \overline{g} \times_F \tilde{g}$ with respect to this chart are the following $g_{ij} = \overline{g}_{ab}$ if $i = a$ and $j = b$, $g_{ij} = F \tilde{g}_{\alpha\beta}$ if $i = \alpha$ and $j = \beta$, and $g_{ij} = 0$ otherwise, where $a, b, c, d, f \in \{1, \ldots, p\}$, $\alpha, \beta, \gamma, \delta \in \{p + 1, \ldots, n\}$ and $h, i, j, k, r, s \in \{1, 2, \ldots, n\}$. We will denote by bars (resp., by tildes) tensors formed from $\overline{g}$ (resp., $\tilde{g}$).
The local components
\[ \Gamma^h_{ij} = \frac{1}{2} g^{hs}(\partial_i g_{js} + \partial_j g_{is} - \partial_s g_{ij}), \quad \partial_j = \frac{\partial}{\partial x^j}, \]
of the Levi-Civita connection \( \nabla \) of \( \bar{M} \times_F \bar{N} \) are the following (see, e.g., [K]):

\[ \Gamma^a_{bc} = \tilde{\Gamma}^a_{bc}, \quad \Gamma^\alpha_{\beta\gamma} = \tilde{\Gamma}^\alpha_{\beta\gamma}, \quad \Gamma^a_{\alpha\beta} = -\frac{1}{2} \bar{g}^{ab} F_b \bar{g}_{\alpha\beta}, \quad \Gamma^\alpha_{a\beta} = \frac{1}{2F} F^\alpha_{\delta} \delta^\delta_{\beta}, \]

\[ \Gamma^\alpha_{\alpha\beta} = \Gamma^\alpha_{a\beta} = 0, \quad F_a = \partial_a F = \frac{\partial F}{\partial x^a}, \quad \partial_a = \frac{\partial}{\partial x^a}. \]

The local components
\[ R_{hijk} = g_{hs} R^s_{ijk} = g_{hs}(\partial_k \Gamma^s_{ij} - \partial_j \Gamma^s_{ik} + \Gamma^r_{ij} \Gamma^s_{rk} - \Gamma^r_{ik} \Gamma^s_{rj}), \quad \partial_k = \frac{\partial}{\partial x^k}, \]
of the Riemann-Christoffel curvature tensor \( R \) and the local components \( S_{ij} \) of the Ricci tensor \( S \) of the warped product \( \bar{M} \times_F \bar{N} \) which may not vanish identically are the following:

Warped product manifolds

\[ R_{abcd} = \bar{R}_{abcd}, \]

\[ R_{\alpha ab\beta} = -\frac{1}{2} T_{ab} \bar{g}_{\alpha\beta}, \]

\[ R_{\alpha\beta\gamma\delta} = F \bar{R}_{\alpha\beta\gamma\beta} - \frac{1}{4} \Delta_1 F \bar{G}_{\alpha\beta\gamma\delta}, \]

\[ S_{ab} = \bar{S}_{ab} - \frac{n - p - 1}{2} \frac{1}{F} T_{ab}, \]

\[ S_{\alpha\beta} = \bar{S}_{\alpha\beta} - \frac{1}{2} \left( tr(T) + \frac{n - p - 1}{2F} \Delta_1 F \right) \bar{g}_{\alpha\beta}, \]

where

\[ T_{ab} = \bar{\nabla}_b F_a - \frac{1}{2F} F_a F_b, \quad tr(T) = \bar{g}^{ab} T_{ab} = \Delta F - \frac{1}{2F} \Delta_1 F, \]

\[ \Delta F = \Delta_{\bar{g}} F = \bar{g}^{ab} \nabla_a F_b, \quad \Delta_1 F = \Delta_{1\bar{g}} F = \bar{g}^{ab} F_a F_b, \]

and \( T \) is the \((0,2)\)-tensor with the local components \( T_{ab} \).
The scalar curvature $\kappa$ of $\overline{M} \times_F \tilde{N}$ satisfies the following relation

$$\kappa = \bar{\kappa} + \frac{1}{F} \tilde{\kappa} - \frac{n - p}{F} \left( \text{tr}(T) + \frac{n - p - 1}{4F} \Delta_1 F \right)$$

$$= \bar{\kappa} + \frac{1}{F} \tilde{\kappa} - \frac{n - p}{F} \left( \Delta F + \frac{n - p - 3}{4F} \Delta_1 F \right),$$

where

$$\Delta F = \Delta_{\bar{g}} F = \bar{g}^{ab} \nabla_a F_b,$$

$$\Delta_1 F = \Delta_{1\bar{g}} F = \bar{g}^{ab} F_a F_b.$$
Theorem (see, e.g., [DGJZ], [K1]).

Let \((M, g), n \geq 4\), be a semi-Riemannian manifold and let the following condition be satisfied on \(U_S \cap U_C \subset M\)

\[
R = \frac{\phi}{2} S \wedge S + \mu g \wedge S + \frac{\eta}{2} g \wedge g, \tag{1}
\]

where \(\phi, \mu\) and \(\eta\) are some functions on this set. Then on \(U_S \cap U_C\) we have

\[
S^2 = \alpha_1 S + \alpha_2 g, \quad \alpha_1 = \kappa + \frac{(n-2)\mu - 1}{\phi}, \quad \alpha_2 = \frac{\mu\kappa + (n-1)\eta}{\phi},
\]

\[
R \cdot R = L_R Q(g, R), \quad L_R = \frac{1}{\phi} ((n-2)(\mu^2 - \phi\eta) - \mu),
\]

\[
R \cdot R = Q(S, R) + L Q(g, C), \quad L = L_R + \frac{\mu}{\phi} = \frac{n-2}{\phi} (\mu^2 - \phi\eta),
\]

\[
C \cdot R = L_C Q(g, R), \quad L_C = L_R + \frac{1}{n-2} \left( \frac{\kappa}{n-1} - \alpha_1 \right),
\]

\[
C \cdot C = L_C Q(g, C).
\]
Moreover, we have on $\mathcal{U}_S \cap \mathcal{U}_C$ ([DGJZ], [K1])

\[
R \cdot C = L_R Q(g, C),
\]

\[
C \cdot R = Q(S, C) + \left(L_R - \frac{\kappa}{n-1}\right) Q(g, C),
\]

\[
R \cdot C + C \cdot R = Q(S, C) + \left(2L_R - \frac{\kappa}{n-1}\right) Q(g, C)
\]

and

\[
C \cdot R - R \cdot C = Q(S, C) - \frac{\kappa}{n-1} Q(g, C).
\]


Theorem ([DGHHY], Theorem 4.1).
If \((M, g), n \geq 4\), is a semi-Riemannian manifold satisfying on the set \(U_S \cap U_C \subset M\) the following conditions:

\[
R \cdot R = Q(S, R) + L Q(g, C), \\
C \cdot C = L_C Q(g, C), \\
R \cdot S = Q(g, D),
\]

where \(L\) and \(L_C\) are some functions on \(U_S \cap U_C\) and \(D\) is a symmetric \((0, 2)\)-tensor on this set, then the Roter type equation (1) holds on the set \(U\) of all points of \(U_S \cap U_C\) at which \(\text{rank} (S - \tau g) > 1\) for any \(\tau \in \mathbb{R}\).

[DGHHY] R. Deszcz, M. Głogowska, H. Hashiguchi, M. Hotloś and M. Yawata,
On semi-Riemannian manifolds satisfying some conformally invariant curvature condition,
\[ R = \frac{\phi}{2} S \wedge S + \mu g \wedge S + \frac{\eta}{2} g \wedge g \]

**Example** ([DK], Example 4.1). Let \( S^p\left(\frac{1}{\sqrt{c_1}}\right) \) be the \( p \)-dimensional, \( p \geq 2 \), standard sphere of radius \( \frac{1}{\sqrt{c_1}} \), \( c_1 = \text{const.} > 0 \), with the standard metric \( \bar{g} \). Let \( f \) be a non-constant function on \( S^p\left(\frac{1}{\sqrt{c_1}}\right) \) satisfying the following differential equation ([Obata])

\[
\nabla (df) + c_1 f \bar{g} = 0.
\]

We set \( F = (f + c)^2 \), where \( c \) is a non-zero constant such that \( f + c \) is either positive or negative on \( S^p\left(\frac{1}{\sqrt{c_1}}\right) \).

Let \( (\tilde{N}, \tilde{g}) \), \( n - p = \dim \tilde{N} \geq 2 \), be a semi-Riemannian space of constant curvature \( c_2 \). We consider the warped product \( S^p\left(\frac{1}{\sqrt{c_1}}\right) \times_F \tilde{N} \) of the manifolds \( S^p\left(\frac{1}{\sqrt{c_1}}\right) \) and \( (\tilde{N}, \tilde{g}) \) with the above defined warping function \( F \).


We can check that the warped product
\[
S^p \left( \frac{1}{\sqrt{c_1}} \right) \times_F \tilde{N}
\]
satisfies the Roter type equation (1). In particular, the warped product
\[
S^p \left( \frac{1}{\sqrt{c_1}} \right) \times_F S^{n-p} \left( \frac{1}{\sqrt{c_2}} \right),
\]
where \( p \geq 2, \ n - p \geq 2 \) and \( c_1 > 0, \ c_2 > 0 \), also satisfies (1).

**Remark** ([DK], Example 4.1). We also can prove that \( S^p \left( \frac{1}{\sqrt{c_1}} \right) \times_F \tilde{N} \) can be locally realized as a hypersurface in a semi-Riemannian space of constant curvature.

Warped products satisfying (1) were investigated among others in:


Some curvature conditions

(1) Some curvature identities

Let \((M, g)\), \(n \geq 4\), be a semi-Riemannian manifold. We have on \(\mathcal{U}_S \cap \mathcal{U}_C\) the following identity ([DGJZ]):

\[
(C - R) \cdot (C - R) = \frac{1}{(n-2)^2} (g \wedge S - \frac{\kappa}{n-1} G) \cdot (g \wedge S - \frac{\kappa}{n-1} G).
\]

This yields

\[
(n-2)^2 (C \cdot C - R \cdot C - C \cdot R + R \cdot R) = (g \wedge S) \cdot (g \wedge S) - \frac{\kappa}{n-1} G \cdot (g \wedge S).
\]

Some curvature identities

We also have on $\mathcal{U}_S \cap \mathcal{U}_C$ the following identities (see, e.g., [DGJZ], [K1]):

\[
Q(S, g \wedge S) = -\frac{1}{2} Q(g, S \wedge S), \quad Q(g, g \wedge S) = -Q(S, G),
\]

\[
Q(S, R) = Q(S, C) - \frac{1}{n-2} Q(g, \frac{1}{2} S \wedge S) - \frac{\kappa}{(n-2)(n-1)} Q(S, G),
\]

\[
(g \wedge S) \cdot S = Q(g, S^2), \quad G \cdot S = Q(g, S), \quad S^2(X, Y) = S(SX, Y),
\]

\[
(g \wedge S) \cdot (g \wedge S) = -Q(S^2, G), \quad G \cdot (g \wedge S) = Q(g, g \wedge S) = -Q(S, G)
\]


Some curvature conditions

**Theorem** (cf. [DGJZ], Theorem 3.4).
Let \((M, g), n \geq 4\), be a semi-Riemannian manifold.
(i) The following identity is satisfied on \(\mathcal{U}_S \cap \mathcal{U}_C \subset M\)
\[
C \cdot R + R \cdot C = R \cdot R + C \cdot C - \frac{1}{(n-2)^2} Q(g, -\frac{\kappa}{n-1} g \wedge S + g \wedge S^2).
\]
(ii) If the following curvature conditions
\[
R \cdot R = Q(S, R) - L Q(g, C), \quad C \cdot C = L_C Q(g, C),
\]
are satisfied on \(\mathcal{U}_S \cap \mathcal{U}_C\) then
\[
C \cdot R + R \cdot C = Q(S, C) + (L + L_C) Q(g, C)
- \frac{1}{(n-2)^2} Q(g, \frac{n-2}{2} S \wedge S - \kappa g \wedge S + g \wedge S^2),
\]
where \(L\) and \(L_C\) are some functions on \(\mathcal{U}_S \cap \mathcal{U}_C\).

Some curvature conditions

([DGJZ]) Moreover, if \( \text{rank} (S - \alpha g) = 1 \) on \( \mathcal{U}_S \cap \mathcal{U}_C \), where \( \alpha \) is some function on \( \mathcal{U}_S \cap \mathcal{U}_C \), then on this set we have

\[
\frac{1}{2} S \wedge S - \alpha g \wedge S + \alpha^2 G = 0,
\]
\[
S^2 + ((n - 2)\alpha - \kappa) S + \alpha(\kappa - (n - 1)\alpha) g = 0,
\]

and now (2) reduces to

\[
C \cdot R + R \cdot C = Q(S, C) + (L + L_C) Q(g, C).
\]

(3)

In particular, if \((M, g)\) is the Gödel spacetime then \( \mathcal{U}_S \cap \mathcal{U}_C = M \) and (3) turns into

\[
C \cdot R + R \cdot C = Q(S, C) + \frac{\kappa}{6} Q(g, C).
\]

(4)

The Gödel spacetime

The Gödel metric (5) is given by ([G]):

\[ ds^2 = g_{ij}dx^i dx^j \]

\[ = a^2 \left( -(dx^1)^2 + \frac{1}{2}e^{2x^1}(dx^2)^2 - (dx^3)^2 + (dx^4)^2 + 2e^{x^1}dx^2 dx^4 \right), \]

where \( x^i \in \mathbb{R}, i,j \in \{1,2,3,4\} \), and \( a \) is a non-zero constant.

For the Gödel metric we have

\[ (\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) = 0, \]

\[ S = \kappa \omega \otimes \omega, \quad \kappa = \frac{1}{a^2}, \]

where \( \omega \) is a 1-form and \( \omega = (\omega_1, \omega_2, \omega_3, \omega_4) = (0, a \exp(x^1), 0, a) \).

We also note that

\[ S^2 = \kappa S. \]

Moreover for the G"odel metric (5) we have ([DHJKS]):

\[ R \cdot R = Q(S, R), \]
\[ R(SX, Y, Z, W) + R(SZ, Y, W, X) + R(SW, Y, X, Z) = 0, \]
\[ C \cdot C = C \cdot conh(R) = \frac{\kappa}{6} Q(g, C), \]
\[ conh(R) \cdot conh(R) = conh(R) \cdot C = 0, \]

where the tensor\ conh(R)\ is defined by ([I])

\[ conh(R) = R - \frac{1}{n-2} g \wedge S = C - \frac{\kappa}{(n-2)(n-1)} G. \]


Some curvature conditions

(3) The Gödel spacetime

From $R \cdot R = Q(S, R)$ and $S^2 = \kappa S$ we obtain immediately

$$R \cdot R = \frac{1}{\kappa} Q(S^2, R).$$

Thus the Gödel metric (5) satisfies a condition of the form

$$R \cdot R = L_2 Q(S^2, R).$$

Conditions of the form $R \cdot R = L_p Q(S^p, R)$, $p = 1, 2, \ldots$, where $L_p$ are some functions, were introduced and investigated in [P1] and [P2]. The tensors $S^2, S^3, S^4, \ldots$, are defined by

$$S^2(X, Y) = S(SX, Y), \quad S^3(X, Y) = S^2(SX, Y), \quad S^4(X, Y) = S^3(SX, Y), \ldots$$


**Remark.**

For any semi-Riemannian manifold \((M, g), n \geq 4\), we have (cf. [DGH])

\[
\begin{align*}
conh(R) \cdot S &= C \cdot S - \frac{\kappa}{(n-2)(n-1)} Q(g, S), \\
R \cdot conh(R) &= R \cdot C, \\
conh(R) \cdot R &= C \cdot R - \frac{\kappa}{(n-2)(n-1)} Q(g, R), \\
conh(R) \cdot conh(R) &= C \cdot C - \frac{\kappa}{(n-2)(n-1)} Q(g, C).
\end{align*}
\]

Some curvature conditions

Quasi-Einstein manifolds

We recall that the semi-Riemannian manifold \((M, g)\), \(n \geq 3\), is said to be a **quasi-Einstein manifold** if

\[
\text{rank } (S - \alpha g) = 1
\]
on \(\mathcal{U}_S \subset M\), where \(\alpha\) is some function on this set (see, e.g., [DGHS]).

Every warped product manifold \(\bar{M} \times_F \tilde{N}\) of an 1-dimensional \((\bar{M}, \bar{g})\) base manifold and a 2-dimensional manifold \((\tilde{N}, \tilde{g})\) or an \((n - 1)\)-dimensional Einstein manifold \((\tilde{N}, \tilde{g})\), \(n \geq 4\), with a warping function \(F\), is a quasi-Einstein manifold (see, e.g., [Ch-DDGP]).


(2) **Quasi-Einstein manifolds**

Quasi-Einstein manifolds arose during the study of exact solutions of the Einstein field equations and the investigation on quasi-umbilical hypersurfaces of conformally flat spaces, see, e.g., [DGHSaw] and references therein. Quasi-Einstein hypersurfaces in semi-Riemannian spaces of constant curvature were studied among others in: [DGHS], [G] and [DHS].


Examples of 3-dimensional quasi-Einstein manifolds

Remark ([DGJZ]). (i) The Ricci tensor of the following 3-dimensional Riemannian manifolds \( (\tilde{N}, \tilde{g}) \): the Berger spheres, the Heisenberg group \( \text{Nil}_3 \), \( \text{PSL}(2, \mathbb{R}) \) - the universal covering of the Lie group \( \text{PSL}(2, \mathbb{R}) \) and the Lie group \( \text{Sol}_3 \) ([LVW]), a Riemannian manifold isometric to an open part of the Cartan hypersurface ([DG]) and some three-spheres of Kaluza-Klein type ([CP]) have exactly two distinct eigenvalues.

References:


(2) **Examples of 3-dimensional quasi-Einstein manifolds**

These manifolds are quasi-Einstein, and in a consequence, pseudosymmetric (see, e.g., [DVY]). For further examples of 3-dimensional quasi-Einstein manifolds we refer to [BDV] (Thurston geometries and warped product manifolds) and [K] (manifolds with constant Ricci principal curvatures).

(ii) We mention that recently pseudosymmetry type curvature conditions of four-dimensional Thurston geometries were investigated in [H].


An example of a 5-dimensional quasi-Einstein manifold

Example.
(i) ([K1], [K2], [K3]) Let \( M \) be an open connected subset of \( \mathbb{R}^5 \) endowed with the metric \( g \) of the form

\[
ds^2 = g_{ij} \, dx^i \, dx^j = dx^2 + dy^2 + du^2 + dv^2 + \rho^2 \left(xdu - ydv + dz\right)^2,
\]

where \( \rho = \text{const.} \neq 0. \)

(2) An example of a 5-dimensional quasi-Einstein manifold

(ii) ([SDHJK])

The manifold \((M, g)\) is a non-conformally flat manifold with cyclic parallel Ricci tensor, i.e. \(\nabla_X S(Y, Z) + \nabla_Y S(Z, X) + \nabla_Z S(X, Y) = 0\), satisfying:

\[
S = \frac{\kappa}{2} g - \frac{3\kappa}{2} \eta \otimes \eta, \quad \eta = (0, 0, -\rho, -x\rho, y\rho), \quad \kappa = \rho^2.
\]

\[
C \cdot S = 0,
\]

\[
R \cdot R = -\frac{\kappa}{4} Q(g, R),
\]

\[
C \cdot C = C \cdot R,
\]

\[
C \cdot R = -\frac{1}{3} Q(S, C) - \frac{\kappa}{3} Q(g, C),
\]

\[
R \cdot C - C \cdot R = \frac{1}{3} Q(S, C) + \frac{\kappa}{12} Q(g, C).
\]

(3) An example of a 5-dimensional quasi-Einstein manifold

(iii) We also have

\[
R \cdot C + C \cdot R = -\frac{1}{3} Q(S, C) - \frac{7\kappa}{12} Q(g, C),
\]

\[
S^2 = -\frac{\kappa}{2} S + \frac{\kappa^2}{2} g,
\]

\[
R \cdot R = -\frac{1}{2\kappa} Q(S^2, R) - \frac{1}{4} Q(S, R),
\]

\[
S \cdot R = 2\kappa R - \frac{\kappa}{2} g \wedge S + \frac{\kappa^2}{4} g \wedge g.
\]

The (0, 4)-tensor \( S \cdot R \) is defined by

\[
\]
Warped product manifolds with 1-dimensional base manifold and the conformally flat quasi-Einstein fiber

Theorem ([DGJZ], Theorem 4.3). Let $\overline{M} \times_F \tilde{N}$ be the warped product manifold with an 1-dimensional manifold $(\overline{M}, \overline{g})$, $\overline{g}_{11} = \pm 1$, and an $(n - 1)$-dimensional quasi-Einstein semi-Riemannian manifold $(\tilde{N}, \tilde{g})$, $n \geq 4$, and a warping function $F$, and let $(\tilde{N}, \tilde{g})$ be a conformally flat manifold, when $n \geq 5$. Then

$$C \cdot C = L_C Q(g, C),$$
$$R \cdot R - Q(S, R) = L Q(g, C),$$
$$C \cdot R + R \cdot C = Q(S, C) + (L_C + L) Q(g, C)$$
$$- \frac{1}{(n-2)^2} Q(g, \frac{n-2}{2} S \wedge S - \kappa g \wedge S + g \wedge S^2)$$

on $U_S \cap U_C \subset \overline{M} \times_F \tilde{N}$.

Warped product manifolds with 1-dimensional base manifold and the conformally flat quasi-Einstein fiber

**Theorem** ([DGJZ], Theorem 4.4).

Let $\overline{M} \times \tilde{N}$ be the product manifold with an 1-dimensional manifold $(\overline{M}, \bar{g}), \bar{g}_{11} = \pm 1$, and an $(n-1)$-dimensional quasi-Einstein semi-Riemannian manifold $(\tilde{N}, \tilde{g}), n \geq 4$, satisfying $\text{rank} (\tilde{S} - \rho \tilde{g}) = 1$ on $U_{\tilde{S}} \subset \tilde{M}$, where $\rho$ is some function on $U_{\tilde{S}}$, and let $(\tilde{N}, \tilde{g})$ be a conformally flat manifold, when $n \geq 5$. Then on $U_{S} \cap U_{C} \subset \overline{M} \times \tilde{N}$ we have

\[
(n - 3)(n - 2)\rho \ C = \frac{n - 2}{2} \ S \wedge S - \kappa \ g \wedge S \\
+ (n - 2)\rho \left( \frac{2\kappa}{n - 1} - \rho \right) \ G + g \wedge S^2,
\]

\[
C \cdot R + R \cdot C = Q(S, C) + \left( \frac{\kappa}{(n - 2)(n - 1) - \rho} \right) Q(g, C).
\]

Warped products with 2-dimensional base

Let $\overline{M} \times_F \tilde{N}$ be the warped product manifold with a 2-dimensional semi-Riemannian manifold $(\overline{M}, \overline{g})$ and an $(n-2)$-dimensional semi-Riemannian manifold $(\tilde{N}, \tilde{g})$, $n \geq 4$, and a warping function $F$, and let $(\tilde{N}, \tilde{g})$ be a space of constant curvature, when $n \geq 5$.

Let $S_{hk}$ and $C_{hijk}$ be the local components of the Ricci tensor $S$ and the tensor Weyl conformal curvature tensor $C$ of $\overline{M} \times_F \tilde{N}$, respectively. We have

$$S_{ad} = \frac{\overline{\kappa}}{2} g_{ab} - \frac{n-2}{2F} T_{ab}, \quad S_{\alpha\beta} = \tau_1 g_{\alpha\beta}, \quad S_{a\alpha} = 0,$$

where $T_{ab}$ is the $(0,2)$-tensor with the local components $T_{ab}$.

$$\tau_1 = \frac{\tilde{\kappa}}{(n-2)F} - \frac{\text{tr}(T)}{2F} - (n-3) \frac{\Delta_1 F}{4F^2},$$

$$\Delta_1 F = \Delta_1 \tilde{g} F = \overline{g}^{ab} F_a F_b,$$

$$T_{ab} = \nabla_a F_b - \frac{1}{2F} F_a F_b, \quad \text{tr}(T) = \overline{g}^{ab} T_{ab},$$

where $T$ is the $(0,2)$-tensor with the local components $T_{ab}$. 
Warped products with 2-dimensional base manifold

\( C_{abcd} = \frac{n - 3}{n - 1} \rho_1 \ G_{abcd} = \frac{n - 3}{n - 1} \rho_1 \ (g_{ad}g_{bc} - g_{ac}g_{bd}), \)

\( C_{\alpha bc\beta} = -\frac{n - 3}{(n - 2)(n - 1)} \rho_1 \ G_{\alpha bc\beta} = -\frac{n - 3}{(n - 2)(n - 1)} \rho_1 \ g_{bc}g_{\alpha\beta}, \)

\( C_{\alpha\beta\gamma\delta} = \frac{2}{(n - 2)(n - 1)} \rho_1 \ G_{\alpha\beta\gamma\delta} = \frac{2}{(n - 2)(n - 1)} \rho_1 \ (g_{\alpha\delta}g_{\beta\gamma} - g_{\alpha\gamma}g_{\beta\delta}) \)

\( C_{abc\delta} = C_{ab\alpha\beta} = C_{a\alpha\beta\gamma} = 0, \)

where

\( G_{hijk} = g_{hk}g_{ij} - g_{hj}g_{ik}, \)

\( \Delta_1 F = \Delta_1 \bar{g}F = \bar{g}^{ab}F_aF_b, \quad \Delta F = \bar{g}^{ab}\nabla_aF_b, \)

\( \rho_1 = \frac{\kappa}{2} + \frac{\tilde{\kappa}}{(n - 3)(n - 2)}F + \frac{1}{2F} \left( \Delta F - \frac{\Delta_1 F}{F} \right). \)
If we set

$$\rho = \frac{2(n-3)}{n-1} \rho_1$$

(8)

then (7) turns into ([DGJZ])

$$C_{abcd} = \frac{\rho}{2} G_{abcd},$$

$$C_{\alpha bc\beta} = -\frac{\rho}{2(n-2)} G_{\alpha bc\beta},$$

$$C_{\alpha \beta \gamma \delta} = \frac{\rho}{(n-3)(n-2)} G_{\alpha \beta \gamma \delta},$$

$$C_{abc\delta} = C_{ab\alpha\beta} = C_{a\alpha\beta\gamma} = 0.$$  

(9)

Further, by making use of the formulas for the local components \((C \cdot C)_{hijklm}\) and \(Q(g, C)_{hijklm}\) of the tensors \(C \cdot C\) and \(Q(g, C)\), i.e.

\[
(C \cdot C)_{hijklm} = g^{rs}(C_{rijk}C_{shlm} + C_{hrjk}C_{silm} + C_{hirk}C_{sjlm} + C_{hijr}C_{sklm}),
\]

\[
Q(g, C)_{hijklm} = g_{hl}C_{mijk} + g_{il}C_{hmjk} + g_{jl}C_{himk} + g_{kl}C_{hijm} - g_{hm}C_{lijk} - g_{im}C_{hljk} - g_{jm}C_{hilk} - g_{km}C_{hijl},
\]

we obtain

\[
(C \cdot C)_{\alpha abcd\beta} = -\frac{(n-1)\rho^2}{4(n-2)^2} g_{\alpha \beta} G_{dabc},
\]

\[
(C \cdot C)_{a\alpha \beta \gamma \delta} = \frac{(n-1)\rho^2}{4(n-2)^2(n-3)} g_{ad} G_{\delta \alpha \beta \gamma},
\]

\[
Q(g, C)_{\alpha abcd\beta} = \frac{(n-1)\rho}{2(n-2)} g_{\alpha \beta} G_{dabc},
\]

\[
Q(g, C)_{a\alpha \beta \gamma \delta} = -\frac{(n-1)\rho}{2(n-2)(n-3)} g_{ad} G_{\delta \alpha \beta \gamma}.
\]
Warped products with 2-dimensional base

**Theorem** ([DGJZ], Theorem 7.1).

Let $\bar{M} \times_F \tilde{N}$ be the warped product manifold with a 2-dimensional semi-Riemannian manifold $(\bar{M}, \bar{g})$ and an $(n-2)$-dimensional semi-Riemannian manifold $(\tilde{N}, \tilde{g})$, $n \geq 4$, and a warping function $F$, and let $(\tilde{N}, \tilde{g})$ be a space of constant curvature, when $n \geq 5$.

1. The following equation is satisfied on the set $\mathcal{U}_C \subset \bar{M} \times_F \tilde{N}$

   $$C \cdot C = L_C \, Q(g, C), \quad L_C = -\frac{\rho}{2(n-2)},$$

   (10)

   $$\rho = \frac{2(n-3)}{n-1} \left( \frac{\tilde{\kappa}}{2} + \frac{\tilde{\kappa}}{(n-3)(n-2)F} + \frac{1}{2F} \left( \Delta F - \frac{\Delta_1 F}{F} \right) \right).$$

**Remark.** The above result, for $n = 4$, was proved in [D] (Theorem 2).


Warped products with 2-dimensional base

2. The following equation is satisfied on the set $\mathcal{U}_C \subset \overline{M} \times F \tilde{N}$

$$R \cdot R = Q(S, R) + L Q(g, C),$$

where $L$ is some function on $\mathcal{U}_C$. Precisely,

$$L = - \frac{n-2}{(n-1)\rho} \left( \tilde{\kappa} \left( \tau_1 + \frac{\text{tr}(T)}{2F} \right) + \frac{n-3}{4F^2} (\text{tr}(T^2) - (\text{tr}(T))^2) \right), \quad (11)$$

$$\tau_1 = \frac{\tilde{\kappa}}{(n-2)F} - \frac{\text{tr}(T)}{2F} - (n-3) \frac{\Delta_1 F}{4F^2},$$

$$\Delta_1 F = \Delta_1 g F = \bar{g}^{ab} F_a F_b,$$

$$T_{ab} = \nabla_a F_b - \frac{1}{2F} F_a F_b, \quad \text{tr}(T) = \bar{g}^{ab} T_{ab},$$

where $T$ is the $(0,2)$-tensor with the local components $T_{ab}$. The tensor $T^2$ is defined by $T^2_{ad} = T_{ac} \bar{g}^{cd} T_{db}$ and $\text{tr}(T^2) = \bar{g}^{ab} T^2_{ab}$. 
(7) Warped products with 2-dimensional base

3. The following equation is satisfied on the set $\mathcal{U}_C \subset \overline{M} \times F \tilde{N}$

$$C \cdot R + R \cdot C = Q(S, C) + (L_C + L) Q(g, C)$$

$$- \frac{1}{(n-2)^2} Q(g, \frac{n-2}{2} S \wedge S - \kappa g \wedge S + g \wedge S^2).$$

where $L_C$ and $L$ are functions defined by (10) and (11), respectively, i.e.

$$L_C = - \frac{n-3}{(n-2)(n-1)} \left( \frac{\bar{\kappa}}{2} + \frac{\tilde{\kappa}}{(n-3)(n-2)F} + \frac{1}{2F} \left( \Delta F - \Delta_1 F \right) \right),$$

$$L = - \frac{n-2}{(n-1)\rho} \left( \frac{\bar{\kappa}}{2} \left( \tau_1 + \frac{\text{tr}(T)}{2F} \right) + \frac{n-3}{4F^2} (\text{tr}(T^2) - (\text{tr}(T))^2) \right).$$
(8) Warped products with 2-dimensional base manifold

We have (see, eq. (6))

\[ S_{ad} = \frac{\kappa}{2} g_{ab} - \frac{n-2}{2F} T_{ab}, \quad S_{\alpha\beta} = \tau_1 g_{\alpha\beta}, \quad S_{a\alpha} = 0, \]

\[ \tau_1 = \frac{\kappa}{(n-2)F} - \frac{\text{tr}(T)}{2F} - (n-3) \frac{\Delta_1 F}{4F^2}. \]

We define on \( \mathcal{U}_{S} \subset \bar{M} \times F \bar{N} \) the \((0,2)\)-tensor \( A \) by

\[ A = S - \tau_1 g. \]

We can check that \( \text{rank}(A) = 2 \) at a point of \( \mathcal{U}_{S} \) if and only if \( \text{tr}(A^2) - (\text{tr}(A))^2 \neq 0 \) at this point ([DGJZ], Section 6). Now, at all points of \( \mathcal{U}_{S} \), at which \( \text{rank}(A) = 2 \), we can define the function \( \tau_2 \) by

\[ \tau_2 = \left( \text{tr}(A^2) - (\text{tr}(A))^2 \right)^{-1}. \]

Further, let \( V \) be the set of all points of \( \mathcal{U}_{S} \cap \mathcal{U}_{C} \) at which: \( \text{rank}(A) = 2 \) and \( S_{ad} \) is not proportional to \( g_{ad} \).
Warped products with 2-dimensional base

4. On the set $V \subset \mathcal{U}_S \cap \mathcal{U}_C$ we have:

$$C = -\frac{(n-1)\rho \tau_2}{(n-3)(n-2)} \left( \frac{n-2}{2} S \wedge S - \kappa g \wedge S + g \wedge S^2 - \frac{\text{tr}(S^2) - \kappa^2}{n-1} G \right)$$

and

$$R \cdot C + C \cdot R = Q(S, C) + \left( L - \frac{\rho}{2(n-2)} + \frac{n-3}{(n-2)(n-1)\rho \tau_2} \right) Q(g, C).$$

Remark. At all points of the set $\mathcal{U}_S \cap \mathcal{U}_C$, at which $S_{ad}$ is proportional to $g_{ad}$ and $\text{rank}(A) = 2$, the Weyl tensor $C$ is a linear combination of the Kulkarni-Nomizu products $S \wedge S$, $g \wedge S$ and $g \wedge g$. 

Further, on $V$ we also have

$$R \cdot C = Q(S, C) + \left( L + \frac{n-3}{(n-2)(n-1)\rho^2} \right) Q(g, C)$$

$$+ \frac{(n-1)\rho^2}{(n-2)^2} g \wedge Q(S, S^2)$$

$$+ \frac{1}{(n-2)^2} Q((\frac{\rho}{2} + (n-1)\rho\tau^2) S - (n-1)\rho\tau S^2, G),$$

and

$$C \cdot R = -\frac{1}{(n-2)^2} Q((\frac{\rho}{2} + (n-1)\rho\tau^2) S - (n-1)\rho\tau S^2, G)$$

$$- \frac{(n-1)\rho^2}{(n-2)^2} g \wedge Q(S, S^2)$$

$$- \frac{\rho}{2(n-2)} Q(g, C).$$
Warped products with 2-dimensional base

**Theorem** ([DGJZ], Theorem 6.2). Let $\bar{M} \times_F \tilde{N}$ be the warped product manifold with a 2-dimensional semi-Riemannian manifold $(\bar{M}, \bar{g})$ and an $(n-2)$-dimensional semi-Riemannian manifold $(\tilde{N}, \tilde{g})$, $n \geq 4$, and a warping function $F$, and let $(\tilde{N}, \tilde{g})$ be an Einstein, when $n \geq 5$. On the set $V \subset U_S \cap U_C$ we have:

$$R \cdot S = (\phi_1 - 2\tau_1\phi_2 + \tau_1^2\phi_3) Q(g, S) + (\phi_2 - \tau_1\phi_3) Q(g, S^2) + \phi_3 Q(S, S^2),$$

where

$$\phi_1 = \frac{2\tau_1 - \kappa}{2(n - 2)}, \quad \phi_2 = \frac{1}{n - 2}, \quad \phi_3 = \frac{\tau_2(2\kappa - \kappa - 2(n - 1)\tau_1)}{n - 2}.$$

**Remark.** At all points of the set $U_S \cap U_C$, at which $S_{ad}$ is proportional to $g_{ad}$ and $\text{rank}(A) = 2$, we have $R \cdot S = L_S Q(g, S)$, for some function $L_S$. 
Some 4-dimensional warped products metrics

We define on \( \overline{M} = \{(t, r) \in \mathbb{R}^2 \mid r > 0\} \) the metric tensor \( \overline{g} \) by

\[
\overline{g}_{11} = -h, \quad \overline{g}_{12} = \overline{g}_{21} = 0, \quad \overline{g}_{22} = h^{-1}, \quad h = h(t, r),
\]

where \( h \) is a smooth positive (or negative) function on \( \overline{M} \).

Let \( F = F(t, r) = f^2(t, r) \) be a positive smooth function on \( \overline{M} \).

Let \( \overline{M} \times_F \tilde{N} \) be the warped product of \( (\overline{M}, \overline{g}) \) and the 2-dimensional unit standard sphere \( (\tilde{N}, \tilde{g}) \), with the warping function \( F \).

The warped product metric \( \overline{g} \times_F \tilde{g} \) of \( \overline{M} \times_F \tilde{N} \) is the following

\[
ds^2 = -h(t, r) \, dt^2 + \frac{1}{h(t, r)} \, dr^2 + f^2(t, r) \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right). \quad (12)\]
Some 4-dimensional warped products metrics

The metric (12), i.e. the metric

\[ ds^2 = -h(t, r) \, dt^2 + \frac{1}{h(t, r)} \, dr^2 + f^2(t, r) \left( d\theta^2 + \sin^2 \theta \, d\varphi^2 \right), \]

satisfies on the set \( \mathcal{U}_S \cap \mathcal{U}_C \subset \overline{M} \times F \tilde{N} \) the following conditions

\[
\begin{align*}
R \cdot R - Q(S, R) &= \phi_1 Q(g, C), & C \cdot C &= \phi_2 Q(g, C), \\
R &= \phi_3 g \wedge g + \phi_4 g \wedge S + \phi_5 S \wedge S + \phi_6 g \wedge S^2, \\
S \cdot R &= \phi_7 g \wedge g + \phi_8 g \wedge S + \phi_9 S \wedge S + \phi_{10} R, \\
R \cdot C + C \cdot R &= Q(S, C) + \phi Q(g, C), \\
C \cdot S &= \phi_{11} Q(g, S) + \phi_{12} Q(g, S^2) + \phi_{13} Q(S, S^2),
\end{align*}
\]

where \( \phi, \phi_1, \ldots, \phi_{13} \) are some functions.
(3) **Some 4-dimensional warped products metrics**

Special cases of the metric (12), i.e. of the metric

\[ ds^2 = -h(t, r) \, dt^2 + \frac{1}{h(t, r)} \, dr^2 + f^2(t, r) \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right) . \]

We assume that \( f(t, r) = r > 0 \).

If \( h(t, r) = 1 - \frac{2m(t)}{r} \), \( m = m(t) > 0 \), then (12) reduces to the **Vaidya metric**.

If \( h(t, r) = 1 - \frac{2m}{r} + \frac{e^2}{r^2} - \frac{\Lambda}{3} \, r^2 \), \( m = \text{const.} > 0 \), \( e = \text{const.} \), \( \Lambda = \text{const.} \), then (12) reduces to the **Reissner-Nordström-de Sitter metric**.

If \( h(t, r) = 1 - \frac{2m}{r} + \frac{e^2}{r^2} \), \( m = \text{const.} > 0 \), \( e = \text{const.} \neq 0 \), then (12) reduces to the **Reissner-Nordström metric**.

If \( h(t, r) = 1 - \frac{2m}{r} - \frac{\Lambda}{3} \, r^2 \), \( m = \text{const.} > 0 \), \( \Lambda = \text{const.} \neq 0 \), then (12) reduces to the **Kottler metric**.

If \( h(t, r) = 1 - \frac{2m}{r} \) then (12) reduces to the **Schwarzschild metric**.
The Schwarzschild and the Kottler spacetimes

- \( \overline{M} \times_F \tilde{N} \) is the Schwarzschild spacetime, if
  \[
  h(r) = 1 - \frac{2m}{r}, \quad m = \text{const.} > 0.
  \]
  We have: \( S = 0, \ R \cdot R = L_R Q(g, R) \), for some function \( L_R \), and
  \[
  R \cdot C = C \cdot R.
  \]

- \( \overline{M} \times_F \tilde{N} \) is the Kottler spacetime, if
  \[
  h(r) = 1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}, \quad m = \text{const.} > 0, \ \Lambda = \text{const.} \neq 0;
  \]
  We have: \( S = \frac{\kappa}{4} g, \ R \cdot R = L_R Q(g, R) \), for some function \( L_R \), and
  \[
  R \cdot C - C \cdot R = \frac{\kappa}{12} Q(g, R).
  \]
Curvature properties of some metric ([Hall], eq. (21))

We consider the metric ([Hall], eq. (21))

\[ ds^2 = dt^2 + R^2(t) (dr^2 + f^2(r) (d\theta^2 + \sin^2 \theta \, d\phi^2)) \]
\[ = (dt^2 + R^2(t) \, dr^2) + (f(r) \, R(t))^2 (d\theta^2 + \sin^2 \theta \, d\phi^2). \quad (13) \]

The metric (13) satisfies the following curvature conditions

\[ R \cdot R - Q(S, R) = (2 R'' / R) \, Q(g, C), \]
\[ C \cdot C = ((f' - f \, f'') - 1)/(6 (f \, R)^2)) \, Q(g, C), \]

where \( f' = df/dr, f'' = df'/dr, R' = dR/dt, R'' = dR'/dt \).

We also have

\[ \kappa = (6 f^2 \, R \, R'' + 6 (f \, R')^2 + 4 f \, f'' + 2 f' - 2) (f \, R)^{-2}. \]

Some 4-dimensional warped product metrics

(2) Curvature properties of some metric ([Hall], eq. (21))

We have

\[ R = \frac{\phi_1}{2} \mathbf{g} \wedge \mathbf{g} + \phi_2 \mathbf{g} \wedge \mathbf{S} + \phi_3 \mathbf{S} \wedge \mathbf{S} + \phi_4 \mathbf{g} \wedge \mathbf{S}^2, \]

with

\[ \phi_1 = \frac{-((-7f^2f'' + 3ff'r' - 3f)R + 10Rf^3R'^2)R'' - (f'^2 - 1)f''}{((-7f^2f'' + ff' - f)R^2 - 6f^3R^2R''^2 - 2f^3R^4 - ff''^2)} \]

\[ \phi_2 = \frac{(8f^2RR'' + 4f^2R'^2 + 3ff'' + f'^2 - 1)}{(4f^2RR'' - 4f^2R'^2 - 2ff'' - 2f'^2 + 2)}, \]

\[ \phi_3 = \phi_4 = \frac{-(fR)^2}{(4f^2RR'' - 4f^2R'^2 - 2ff'' - 2f'^2 + 2)}, \]

where \( f' = \frac{df}{dr}, f'' = \frac{df'}{dr}, R' = \frac{dR}{dt}, R'' = \frac{dR'}{dt}. \)

We have

\[ S \cdot R = \frac{\phi_1}{2} g \wedge g + \phi_2 g \wedge S + \phi_3 S \wedge S + \phi_4 R, \]

\[ \phi_1 = (((12 f^3 f'' - 6 f^2 f'f' + 6 f^2) R + 6 f^4 R R') R'') + ((-12 f f'f' + 12 f) f'' + (-24 f^3 f'' - 24 f^2 f'f' + 24 f^2 R'') - 24 f^4 R^4 - 6 f^2 f''f' - 6 f^4 + 12 f'f' - 6) R'' + 18 f^4 R^2 R''^3) / ((-f^3 f'' - f^2 f'f' + f^2) R^3 + 2 f^4 R^4 R'' - 2 f^4 R^3 R'') , \]

\[ \phi_2 = ((-10 f^2 R'f' - 2 f f'' - 8 f'f' + 8) R'' - 10 f^2 R R'') / ((-f f'' - f^2 + 1) R + 2 f^2 R^2 R'' - 2 f^2 R R'') , \]

\[ \phi_3 = (f^2 RR''^2 - f^2 R^2 - f'f' + 1) / (2 R f^2 R'' - 2 f^2 R^2 - f f'' - f'f' + 1) \]

\[ \phi_4 = ((-12 f^2 R'f' - 6 f f'' - 6 f'f' + 6) / ((f R)^2) . \]
Curvature properties of some metric ([Hall], eq. (21))

We have

\[ R \cdot C + C \cdot R = Q(S, C) + \phi Q(g, C), \]

with

\[ \phi = \frac{3 f^2 R R'' + 3 (f R')^2 + f f'' - 2 f'^2 - 2}{(3 f R^2)}. \]

Some 4-dimensional warped product metrics

Curvature properties of some metric ([Hall], eq. (21))

We have

\[ C \cdot S = \phi_1 Q(g, S) + \phi_2 Q(g, S^2) + \phi_3 Q(S, S^2), \]

\[ \phi_1 = \frac{-((4 f^3 f'' + 8 f^2 f'^2 - 8 f^2) R + 12 f^4 R R''^2) R'' - (6 f f'^2 - 6 f) f'' - (14 f^3 f'' + 10 f^2 f'^2 - 10 f^2) R'^2 - 3 f^4 R^2 R''^2 - 12 f^4 R'^4 - 4 f^2 f''^2 - 2 f f'^4 + 4 f'^2 - 2)}{((-6 f^3 f'' - 6 f^2 f'^2 + 6 f^2) R^2 + 12 f^4 R^3 R'' - 12 f^4 R^2 R'^2)} \]

\[ \phi_2 = \frac{(R f^2 R'' + 2 f^2 R'^2 + f f'' + f'^2 - 1)}{(4 f^2 R R'' - 4 f^2 R'^2 - 2 f f'' - 2 f'^2 + 2)} \]

\[ \phi_3 = \frac{-(f R)^2}{(4 f^2 R R'' - 4 f^2 R'^2 - 2 f f'' - 2 f'^2 + 2)} \]

Curvature properties of some metric ([Hall], eq. (22))

We consider the metric ([Hall], eq. (22))

\[ ds^2 = (1 + e R^2(t))^{-2} dt^2 + (1 + e R^2(t))^{-1} R^2(t) (dr^2 + f^2(r) (d\theta^2 + \sin^2 \theta d\phi^2)) \], (14)

where \( e = \text{const.} \), and its extension

\[ ds^2 = P(t) dt^2 + S(t) (dr^2 + f^2(r) (d\theta^2 + \sin^2 \theta d\phi^2)) \]

\[ = (P(t) dt^2 + S(t) dr^2) + S(t) f^2(r) (d\theta^2 + \sin^2 \theta d\phi^2)) \]. (15)

Curvature properties of some metric ([Hall], eq. (22))

The metric (15), i.e. the metric

\[
ds^2 = P(t) \, dt^2 + S(t) \, (dr^2 + f^2(r) \, (d\theta^2 + \sin^2 \theta \, d\phi^2))
\]

satisfies the following curvature conditions

\[
R \cdot R - Q(S, R) = \phi_1 Q(g, C),
\]
\[
C \cdot C = \phi_2 Q(g, C),
\]
\[
R = \phi_3 g \wedge g + \phi_4 g \wedge S + \phi_5 S \wedge S + \phi_6 g \wedge S^2,
\]
\[
S \cdot R = \phi_7 g \wedge g + \phi_8 g \wedge S + \phi_9 S \wedge S + \phi_{10} R,
\]
\[
R \cdot C + C \cdot R = Q(S, C) + \phi Q(g, C),
\]
\[
C \cdot S = \phi_{11} Q(g, S) + \phi_{12} Q(g, S^2) + \phi_{13} Q(S, S^2),
\]

where \(\phi, \phi_1, \ldots, \phi_{13}\) are some functions.
Some 4-dimensional warped product metrics

The condition: \((*)\) \(R \cdot R - Q(S, R) = L \cdot Q(g, C)\)

**Theorem ([DDP]).**
Let \((\tilde{N}, \tilde{g})\) be a semi-Riemannian manifold,
\[\overline{M} = (a; b) \subset \mathbb{R}, \; a < b, \; \overline{g}_{11} = \varepsilon = \pm 1,\]
\(F : (a; b) \rightarrow \mathbb{R}_+\) a smooth function,
\[F'' = \frac{dF'}{dt}, \quad F' = \frac{dF}{dt}, \quad t \in (a; b).\]

(i) Then the warped product \(\overline{M} \times_F \tilde{N}\), \(\dim \tilde{N} = 3\), satisfies \((*)\) with
\[L = \frac{\varepsilon}{F} \left( F'' - \frac{(F')^2}{2F} \right).\]

(2) The condition: 
\( R \cdot R - Q(S, R) = L Q(g, C) \)

(ii) (DDP) Let \((\tilde{N}, \tilde{g})\), \(\dim \tilde{N} = n - 1 \geq 4\), be a manifold satisfying
\[
\tilde{R} \cdot \tilde{R} - Q(\tilde{S}, \tilde{R}) = -(n - 3)k Q(\tilde{g}, \tilde{C}) , \quad k = \text{const} .
\]
Then the manifold \(M \times F \tilde{N}\) satisfies \((\ast)\) with \(L = \frac{(n-2)\varepsilon}{2F} \left( F'' - \frac{(F')^2}{2F} \right)\)
and \(F\) satisfying
\[
F F'' - (F')^2 + 2\varepsilon k F = 0 .
\]

Remark. (i) (DV) On every hypersurface \(\tilde{N}\) immersed isometrically in a semi-Riemannian space of constant curvature \(N_{S}^n(c)\), \(n - 1 \geq 4\),
the condition (16) is satisfied with \(k = c = \frac{\tau}{(n-1)n}\),
where \(\tau\) is the scalar curvature of the ambient space.


(3) The condition: (*) \( R \cdot R - Q(S, R) = L Q(g, C) \)

(ii) ([DSch]) The following functions

\[
F(t) = \varepsilon k \left( t + \frac{\varepsilon l}{k} \right)^2, \quad \varepsilon k > 0,
\]

\[
F(t) = \frac{l}{2} \left( \exp \left( \pm \frac{m}{2} t \right) - \frac{2 \varepsilon k}{l m^2} \exp \left( \mp \frac{m}{2} t \right) \right)^2, \quad l > 0, \quad m \neq 0,
\]

\[
F(t) = \frac{2 \varepsilon k}{l^2} (1 + \sin(l t + m)), \quad \varepsilon k > 0, \quad l \neq 0,
\]

where \( k, l, m \) are constants and \( t \in (a; b) \), are solutions of (17), i.e. of the equation

\[
F F'' - (F')^2 + 2\varepsilon k F = 0.
\]

Some 4-dimensional warped product metrics

The condition: $Q(g, R)$

Theorem.
Let $(\tilde{N}, \tilde{g})$ be a semi-Riemannian manifold, $\overline{M} = (a; b)$, $a < b$,

$\tilde{g}_{11} = \varepsilon = \pm 1$, $F : (a; b) \rightarrow \mathbb{R}_+$ a smooth function, $F'' = \frac{dF'}{dt}$,

$F' = \frac{dF}{dt}$, $t \in (a; b)$.

(i) ([DDV]) If $(\tilde{N}, \tilde{g})$, dim $\tilde{N} = n - 1 \geq 3$, is a semi-Riemannian space of constant curvature then the warped product $\overline{M} \times_F \tilde{N}$, is a conformally flat manifold satisfying $(\ast\ast)$ with $L_R = -\varepsilon \left( \frac{F''}{2F} - \frac{(F')^2}{4F^2} \right)$. Moreover,

$$\text{rank} \left( S - \left( \frac{\kappa}{n - 1} - L_R \right) g \right) = 1.$$ 

(2) The condition: 
\[ R \cdot R = L_R Q(g, R) \]

(ii) ([DSch]) Let \((\tilde{N}, \tilde{g})\), \(\dim \tilde{N} = n - 1 \geq 3\), be a manifold satisfying
\[ \tilde{R} \cdot \tilde{R} = k Q(\tilde{g}, \tilde{R}), \quad k = \text{const}. \] 

The warped product \(M \times_F \tilde{N}\) satisfies (**) with \(L_R = \varepsilon \left( \frac{(F')^2}{4F^2} - \frac{F''}{2F} \right)\) and the function \(F\) satisfying
\[ FF'' - (F')^2 + 2\varepsilon k F = 0. \]

Remark. ([DVY]) On 3-dimensional Cartan hypersurface the condition (18), with \(k = \frac{\tilde{\kappa}}{12}\), where \(\tilde{\kappa}\) is the scalar curvature of the ambient space.


The condition: \((***)\) \(R \cdot S = L_S Q(g, S)\)

**Theorem.** Let \((\widetilde{N}, \tilde{g})\) a semi-Riemannian manifold, \(\overline{M} = (a; b), a < b, \overline{g}_{11} = \varepsilon = \pm 1, F : (a; b) \rightarrow \mathbb{R}_+\) a smooth function, \(F'' = \frac{dF'}{dt}, F' = \frac{dF}{dt}, t \in (a; b).\)

(i) ([DH]) If \((\widetilde{N}, \tilde{g})\), \(\dim \widetilde{N} = n - 1 \geq 3\), is a semi-Riemannian Einsteinian manifold then the warped product \(\overline{M} \times_F \widetilde{N}\), is a manifold satisfying \((***)\) with \(L_S = \varepsilon \left( \frac{(F')^2}{4F^2} - \frac{F''}{2F} \right)\). Moreover, we have ([Ch-DDGP])

\[
\text{rank} \left( S - \left( \frac{\kappa}{n-1} - L_S \right) g \right) = 1,
\]

\[
(n - 2) (R \cdot C - C \cdot R) = Q(S - L_S g, R).
\]


Remark 1 ([SDHJK]). The Schwarzschild spacetime, the Kottler spacetime, the Reissner-Nordström spacetime, as well as some Friedmann-Lemaître-Robertson-Walker spacetimes (FLRW spacetimes) are the ”oldest” examples of non-semisymmetric pseudosymmetric warped product manifolds (cf. [DHV], [HV]). The Schwarzschild spacetime was discovered in 1916 by Schwarzschild and independently by Droste during their study on solutions of Einstein’s equations (see, e.g., [P]).


Remark 2 ([SDHJK]). Pseudosymmetric manifolds

We note that [DG] is the first paper, in which manifolds satisfying

$$ R \cdot R = L_R Q(g, R) $$

were called pseudosymmetric manifolds. We also mention that in [WG] it was proved that fibers of semisymmetric warped products are pseudosymmetric (cf. [HV], Section 7).


**Remark ([Saw]).** According to [H] and [Saw] the generalized curvature tensor $B$ on $M$ satisfies the *Ricci-type equation* if on $M$ we have

$$R \cdot B = B \cdot B,$$

or

$$C \cdot B = B \cdot B.$$ 

If either $B = C$ or $B = R - C$ or $B = R$ or $B = C - R$ satisfies the Ricci-type equation then ([Saw])

$$R \cdot C = C \cdot C,$$

$$C \cdot R = C \cdot C,$$

$$R \cdot C = R \cdot R,$$

$$C \cdot R = R \cdot R,$$

(19)

respectively.

Hypersurfaces in a semi-Riemannian space of constant curvature $N^s_{n+1}(c)$, $n \geq 4$, satisfying Ricci-type equations (19) were investigated in [Saw].


Some generalizations of the Roter type equation

Example.
(i) We define the metric $g$ on $M = \{(t, r, \phi, z) : t > 0, r > 0\} \subset \mathbb{R}^4$ by (cf. [RT], Section 1)

$$ds^2 = (dt + H(r) d\phi)^2 - D^2(r) d\phi^2 - dr^2 - dz^2,$$

where $H$ and $D$ are certain functions on $M$. If

$$H(r) = \frac{2\sqrt{2}}{m} \sinh^2\left(\frac{mr}{2}\right)$$

and

$$D(r) = \frac{2}{m} \sinh\left(\frac{mr}{2}\right) \cosh\left(\frac{mr}{2}\right)$$

then $g$ is the Gödel metric (e.g. see [RT], eq. (1.6)).

Some generalizations of the Roter type equation

(ii) ([DHJKS]) The metric $g$ defined by (20) is the product metric of a 3-dimensional metric and a 1-dimensional metric. Thus $R \cdot R = Q(S, R)$ on $M$. The Riemann-Christoffel curvature tensor $R$ of $(M, g)$ is expressed by a linear combination of the Kulkarni-Nomizu products formed by $S$ and $S^2$, i.e. by the tensors $S \wedge S$, $S \wedge S^2$ and $S^2 \wedge S^2$,

$$R = \phi_1 S \wedge S + \phi_2 S \wedge S^2 + \phi_3 S^2 \wedge S^2,$$

$$\phi_1 = \frac{D^2}{\tau} (2D^2 H''^2 - 4DD' H' H'' - 3H'4 + 2H'^2(4DD'' + D'2) - 8D^2 D''2),$$

$$\phi_2 = \frac{2D^4}{\tau} (H'^2 - 4DD''), \quad \phi_3 = -\frac{4D^6}{\tau}, \quad H' = \frac{dH}{dr}, \quad H'' = \frac{dH'}{dr},$$

$$\tau = (H'^2 - 2DD'')(D^2 H''^2 - 2DD' H' H'' - H'4 + 2DD'' H'2 + D'2 H'2),$$

provided that the function $\tau$ is non-zero at every point of $M$. 

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(iii) If $H(r) = ar^2$, $a = \text{const.} \neq 0$ and $D(r) = r$
then (20) turns into ([RT], eq. (3.20))

$$ds^2 = (dt + ar^2 d\phi)^2 - r^2 d\phi^2 - dr^2 - dz^2. \tag{21}$$

The spacetime $(M, g)$ with the metric $g$ defined by (21) is called
the Som-Raychaudhuri solution of the Einstein field equations [SR].
For the metric (21) the function $\tau$ is non-zero at every point of $M$.

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Some generalizations of the Roter type equation

Example.
We define on $M = \{(x, y, z, t) : x > 0, y > 0, z > 0, t > 0\} \subset \mathbb{R}^4$ the metric $g$ by ([DK])

$$ds^2 = \exp(y) \, dx^2 + (x \, z)^2 \, dy^2 + dz^2 - dt^2.$$  \hspace{1cm} (22)

We have on $M$ ([DGJP-TZ]):

$$\text{rank} (S) = \ldots = \text{rank} (S^4) = 3, \quad \kappa = 1/(2 \, x^2 \, z^2),$$
$$\omega(X) \, R(Y, Z) + \omega(Y) \, R(Z, X) + \omega(Z) \, R(X, Y) = 0,$$
$$R \cdot R = Q(S, R),$$

where the 1-form $\omega$ is defined by $\omega(\partial_x) = \omega(\partial_y) = 1, \omega(\partial_z) = \omega(\partial_t) = 0.$


Some generalizations of the Roter type equation

Moreover, for the metric (22) we have on $M$ ([DGJP-TZ]):

$$R = \phi_1 S \wedge S + \phi_2 S \wedge S^2 + \phi_3 S^2 \wedge S^2,$$

$$\phi_1 = \frac{(16 x^2 z^4 + z^2 (4 x^2 + 1) \exp(y))}{(8 z^2 + 2 \exp(y))},$$

$$\phi_2 = -\frac{4 x^2 z^4 \exp(y)}{(4 z^2 + \exp(y))},$$

$$\phi_3 = \frac{8 x^4 z^6 \exp(y)}{(4 z^2 + \exp(y))},$$

$$Q(S, S^2 \wedge S^2) = Q(S^3 - \exp(y)/(2 xz^2) S^2, S \wedge S),$$

and

Pseudosymmetry

Let \((M, g), \ n \geq 3,\) be a Riemannian manifold. We assume that the set \(\mathcal{U}_R \subset M\) is non-empty and let \(p \in \mathcal{U}_R\). Let \(\pi = u \wedge v\) and \(\overline{\pi} = x \wedge y\) be planes of \(T_pM\), where \(u, v \in T_pM\) form an orthonormal basis of \(\pi\) and \(x, y \in T_pM\) form an orthonormal basis of \(\overline{\pi}\). The plane \(\pi\) is said to be curvature-dependent with respect to the plane \(\overline{\pi}\) ([HV], Definition 2) if \(Q(g, R)(u, v, v, u; x, y) \neq 0\). According to [HV](Definition 3), we define at \(p\) the sectional curvature of Deszcz \(L_R(p, \pi, \overline{\pi})\) of the plane \(\pi\) with respect to the plane \(\overline{\pi}\) by

\[
L_R(p, \pi, \overline{\pi}) = \frac{(R \cdot R)(u, v, v, u; x, y)}{Q(g, R)(u, v, v, u; x, y)}.
\]

In [HV](Theorem 3) it was proved that a Riemannian manifold \((M, g), \ n \geq 3,\) is pseudosymmetric if and only if all the double sectional curvatures \(L_R(p, \pi, \overline{\pi})\) are the same at every point \(p \in \mathcal{U}_R \subset M\), i.e. for all curvature-dependent planes \(\pi\) and \(\overline{\pi}\) at \(p\), \(L_R(p, \pi, \overline{\pi}) = L_R(p)\) for some function \(L_R\) on \(\mathcal{U}_R\).
Ricci-pseudosymmetry

Let \((M, g), n \geq 3\), be a Riemannian manifold. We assume that the set \(\mathcal{U}_S \subset M\) is non-empty and let \(p \in \mathcal{U}_S\). A direction \(d\), spanned by a vector \(v \in T_p M\), is said to be curvature dependent on a plane \(\pi = x \wedge y \subset T_p M\) if \(Q(g, S)(v, v; x, y) \neq 0\), where \(x, y \in T_p M\) form an orthonormal basis of \(\pi\). According to [JHSV] (Definition 6), we define at \(p\) the Ricci curvature of Deszcz \(L_S(p, d, \pi)\) of the curvature-dependent direction \(d\) and the plane \(\pi\) by

\[
L_S(p, d, \pi) = \frac{(R \cdot S)(v, v; x, y)}{Q(g, S)(v, v; x, y)}.
\]

In [JHSV] (Theorem 10) it was stated that a Riemannian manifold \((M, g), n \geq 3\), is Ricci-pseudosymmetric if and only if all the Ricci curvatures of Deszcz are the same at every point \(p \in \mathcal{U}_S \subset M\), i.e. for all curvature-dependent directions \(d\) with respect to planes \(\pi\) we have \(L_S(p, d, \pi) = L_S(p)\) for some function \(L_S\) on \(\mathcal{U}_S\).
References; pseudosymmetry, Ricci-pseudosymmetry, Weyl-pseudosymmetry


[SDHJK] A.A. Shaikh, R. Deszcz, M. Hotloś, J. Jełowicki, and H. Kundu, On pseudosym
Pseudosymmetric manifolds

A semi-Riemannian manifold $(M, g)$, $n \geq 3$, is said to be pseudosymmetric if at every point of $M$ the tensors $R \cdot R$ and $Q(g, R)$ are linearly dependent.

The manifold $(M, g)$ is pseudosymmetric if and only if

$$R \cdot R = L_R Q(g, R)$$

holds on $\mathcal{U}_R$, where $L_R$ is some function on this set.

Every semisymmetric manifold ($R \cdot R = 0$) is pseudosymmetric. The converse statement is not true.
Pseudosymmetric manifolds of constant type

According to [BKV], a pseudosymmetric manifold \((M, g), \ n \geq 3, \ (R \cdot R = L_R \ Q(g, R))\) is said to be pseudosymmetric space of constant type if the function \(L_R\) is constant on \(U_R \subset M\).

**Theorem** (cf. [D]). Every type number two hypersurface \(M\) isometrically immersed in a semi-Riemannian space of constant curvature \(N_{2n+1}(c), \ n \geq 3\), is a pseudosymmetric space of constant type. Precisely,

\[
R \cdot R = \frac{\tilde{\kappa}}{n(n + 1)} \ Q(g, R),
\]

holds on \(U_R \subset M\), where \(\tilde{\kappa}\) is the scalar curvature of the ambient space.


A semi-Riemannian manifold $(M, g)$, $n \geq 3$, is said to be \textit{Ricci-pseudosymmetric} if at every point of $M$ the tensors $R \cdot S$ and $Q(g, S)$ are linearly dependent.

The manifold $(M, g)$ is Ricci-pseudosymmetric if and only if

$$R \cdot S = L_S Q(g, S)$$

holds on $\mathcal{U}_S$, where $L_S$ is some function on this set.

Every \textit{Ricci-semisymmetric} manifold ($R \cdot S = 0$) is Ricci-pseudosymmetric. The converse statement is not true.
According to [G], a Ricci-pseudosymmetric manifold $(M, g), \ n \geq 3, \ (R \cdot S = L_S Q(g, S))$ is said to be a **Ricci-pseudosymmetric manifold of constant type** if the function $L_S$ is constant on $\mathcal{U}_S \subset M$.

Theorem (cf. [DY]). If $M$ is a hypersurface in a Riemannian space of constant curvature $N^{n+1}(c), n \geq 3$, such that at every point of $M$ there are principal curvatures $0, \ldots, 0, \lambda, \ldots, \lambda, -\lambda, \ldots, -\lambda$, with the same multiplicity of $\lambda$ and $-\lambda$, and $\lambda$ is a positive function on $M$, then $M$ is a Ricci-pseudosymmetric manifold of constant type. Precisely,

$$R \cdot S = \frac{\kappa}{n(n+1)} Q(g, S)$$

holds on $M$. In particular, every Cartan hypersurface is a Ricci-pseudosymmetric manifold of constant type.

A semi-Riemannian manifold $(M, g)$, $n \geq 4$, is said to be \textit{Weyl-pseudosymmetric} if at every point of $M$ the tensors $R \cdot C$ and $Q(g, C)$ are linearly dependent.

The manifold $(M, g)$ is Weyl-pseudosymmetric if and only if

$$R \cdot C = L_C Q(g, C)$$

holds on $\mathcal{U}_C$, where $L_C$ is some function on this set.
Every pseudosymmetric manifold \( (R \cdot R = L_R Q(g, R)) \)
is Weyl-pseudosymmetric \( (R \cdot C = L_R Q(g, C)) \).
In particular, every semisymmetric manifold \( (R \cdot R = 0) \)
is Weyl-semisymmetric \( (R \cdot C = 0) \).

If \( \dim M \geq 5 \) the converse statement are true. Precisely, if
\[
R \cdot C = L_C Q(g, C), \quad \text{resp.} \quad R \cdot C = 0,
\]
is satisfied on \( \mathcal{U}_C \subset M \), then
\[
R \cdot R = L_C Q(g, R), \quad \text{resp.} \quad R \cdot R = 0,
\]
holds on \( \mathcal{U}_C \).
Further results on curvature conditions of pseudosymmetry type

(3) **Weyl-pseudosymmetric manifolds**

An example of a 4-dimensional Riemannian manifold satisfying $R \cdot C = 0$ with non-zero tensor $R \cdot R$ was found by A. Derdziński ([D]).

An example of a 4-dimensional submanifold in a 6-dimensional Euclidean space $\mathbb{E}^6$ satisfying $R \cdot C = 0$ with non-zero tensor $R \cdot R$ was found by G. Zafindratafa ([Z]).


Further results on curvature conditions of pseudosymmetry type

Weyl-pseudosymmetric manifolds

For further results on 4-dimensional semi-Riemannian manifolds satisfying $R \cdot C = 0$ or $R \cdot C = L Q(g, C)$ we refer to the following papers:


Further results on curvature conditions of pseudosymmetry type

(1) Relations between some classes of manifolds

Inclusions between mentioned classes of manifolds can be presented in the following diagram ([DGHS]).
We mention that all inclusions are strict, provided that $n \geq 4$.

Further results on curvature conditions of pseudosymmetry type

(2) Relations between some classes of manifolds, $n \geq 4$

$R \cdot S = L_S Q(g, S) \supset R \cdot R = L_R Q(g, R) \subset R \cdot C = L_C Q(g, C)$

$R \cdot S = 0 \supset R \cdot R = 0 \subset R \cdot C = 0$

$\nabla S = 0 \supset \nabla R = 0 \subset \nabla C = 0$

$S = \frac{\kappa}{n} g \supset R = \frac{\kappa}{(n-1)n} G \subset C = 0$
Relations between some classes of manifolds; References


Further results on curvature conditions of pseudosymmetry type

(4) Relations between some classes of manifolds, $n \geq 4$

We also have

\[
C \cdot S = L_S Q(g, S) \supset C \cdot R = L_R Q(g, R) \subset C \cdot C = L_C Q(g, C)
\]

\[
C \cdot S = 0 \supset C \cdot R = 0 \subset C \cdot C = 0
\]

\[
S = \frac{\kappa}{n} g \supset R = \frac{\kappa}{(n-1)n} G \subset C = 0
\]
Further results on curvature conditions of pseudosymmetry type

(5) Relations between some classes of manifolds, \( n \geq 4 \)

Remark ([MADEO]). Let \((M, g), n \geq 4\), be a semi-Riemannian manifold satisfying \( C \cdot R = L Q(g, R) \) on \( \mathcal{U}_C \subset M \). From this we get on \( \mathcal{U}_C \)

\[ C \cdot S = L Q(g, S) \]

Further, we have

\[
C \cdot C = C \cdot \left( R - \frac{1}{n-2} g \wedge S + \frac{\kappa}{(n-2)(n-1)} G \right)
\]

\[
= C \cdot R - \frac{1}{n-2} g \wedge (C \cdot S) + \frac{\kappa}{(n-2)(n-1)} C \cdot G
\]

\[
= L Q(g, R) - \frac{L}{n-2} g \wedge Q(g, S)
\]

\[
= L Q(g, R) - \frac{L}{n-2} Q(g, g \wedge S) = L Q(g, R - \frac{1}{n-2} g \wedge S)
\]

\[
= L Q(g, R - \frac{1}{n-2} g \wedge S + \frac{\kappa}{(n-2)(n-1)} G) = L Q(g, C).
\]