CURVATURE PROPERTIES OF SOME CLASS OF WARPED PRODUCT MANIFOLDS

Dedicated to the memory of Professor Mileva Prvanović

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Some endomorphisms

Let (M, g) be a connected n-dimensional, $n \geq 3$, semi-Riemannian manifold of class C^∞ and ∇ its Levi-Civita connection.
We define on M the endomorphisms $X \wedge_A Y$, $\mathcal{R}(X,Y)$ and $\mathcal{C}(X,Y)$ by

$$
\begin{aligned}
(X \wedge_A Y)Z &= A(Y,Z)X - A(X,Z)Y, \\
\mathcal{R}(X,Y)Z &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \\
\mathcal{C}(X,Y) &= \mathcal{R}(X,Y) - \frac{1}{n-2}\left(X \wedge_g \mathcal{S}Y + \mathcal{S}X \wedge_g Y\right) \\
&\quad - \frac{\kappa}{(n-2)(n-1)} X \wedge_g Y,
\end{aligned}
$$

where $\Xi(M)$ is the Lie algebra of vector fields of M, $X, Y, Z \in \Xi(M)$, S - the Ricci tensor and $\mathcal{S}$ - the Ricci operator

$$
\begin{aligned}
S(X,Y) &= \mathrm{tr}\{Z \rightarrow \mathcal{R}(Z,X)Y\}, \\
g(\mathcal{S}X, Y) &= S(X,Y),
\end{aligned}
$$

$\kappa = \mathrm{tr}\,\mathcal{S}$ - the scalar curvature and A - a symmetric $(0,2)$-tensor.

Some $(0,4)$-tensors

The Riemann-Christoffel curvature tensor R,
the Weyl conformal curvature tensor C
and the $(0,4)$-tensor G of (M,g)
are defined by

$$\begin{aligned} R(X_1,X_2,X_3,X_4) &= g(\mathcal{R}(X_1,X_2)X_3,X_4), \\ C(X_1,X_2,X_3,X_4) &= g(\mathcal{C}(X_1,X_2)X_3,X_4), \\ G(X_1,X_2,X_3,X_4) &= g((X_1 \wedge_g X_2)X_3,X_4), \end{aligned}$$

respectively, where $X_1,\ldots,X_4 \in \Xi(M)$.

The Kulkarni-Nomizu product $E \wedge F$

For symmetric $(0,2)$-tensors E and F
we define their *Kulkarni-Nomizu product* $E \wedge F$ by

$$\begin{aligned}(E \wedge F)(X_1, X_2, X_3, X_4) &= E(X_1, X_4)F(X_2, X_3) + E(X_2, X_3)F(X_1, X_4)\\ &\quad - E(X_1, X_3)F(X_2, X_4) - E(X_2, X_4)F(X_1, X_3)\,,\end{aligned}$$

where $X_1, \ldots, X_4 \in \Xi(M)$.
Now the Weyl tensor C can be presented in the form

$$C = R - \frac{1}{n-2}\, g \wedge S + \frac{\kappa}{(n-2)(n-1)}\, G,$$

where

$$G = \frac{1}{2}\, g \wedge g\,.$$

The Kulkarni-Nomizu product $E \wedge T$

For symmetric $(0,2)$-tensor E and an $(0,k)$-tensor T, $k \geq 3$, we define their *Kulkarni-Nomizu product* $E \wedge T$ by (see, e.g., [DG])

$$
\begin{aligned}
& (E \wedge T)(X_1, X_2, X_3, X_4, Y_3, \ldots, Y_k) \\
= \; & E(X_1, X_4) T(X_2, X_3, Y_3, \ldots, Y_k) + E(X_2, X_3) T(X_1, X_4, Y_3, \ldots, Y_k) \\
& - E(X_1, X_3) T(X_2, X_4, Y_3, \ldots, Y_k) - E(X_2, X_4) T(X_1, X_3, Y_3, \ldots, Y_k),
\end{aligned}
$$

where $X_1, \ldots, X_4, Y_3, \ldots, Y_k \in \Xi(M)$.

[DG] R. Deszcz and M. Głogowska, On nonsemisymmetric Ricci-semisymmetric warped product hypersurfaces, Publ. Inst. Math. (Beograd) (N.S.) 72 (86) (2002), 81–93.

(1) Some $(0, k)$-tensors

For a symmetric $(0, 2)$-tensor A and a $(0, k)$-tensor T, $k \geq 1$, we define the $(0, k+2)$-tensors $R \cdot T$, $C \cdot T$ and $Q(A, T)$ by

$$
\begin{aligned}
& (R \cdot T)(X_1, \ldots, X_k; X, Y) \;=\; (\mathcal{R}(X, Y) \cdot T)(X_1, \ldots, X_k) \\
= \;& -T(\mathcal{R}(X, Y)X_1, X_2, \ldots, X_k) - \cdots - T(X_1, \ldots, X_{k-1}, \mathcal{R}(X, Y)X_k),
\end{aligned}
$$

$$
\begin{aligned}
& (C \cdot T)(X_1, \ldots, X_k; X, Y) \;=\; (\mathcal{C}(X, Y) \cdot T)(X_1, \ldots, X_k) \\
= \;& -T(\mathcal{C}(X, Y)X_1, X_2, \ldots, X_k) - \cdots - T(X_1, \ldots, X_{k-1}, \mathcal{C}(X, Y)X_k),
\end{aligned}
$$

$$
\begin{aligned}
& Q(A, T)(X_1, \ldots, X_k; X, Y) \;=\; ((X \wedge_A Y) \cdot T)(X_1, \ldots, X_k) \\
= \;& -T((X \wedge_A Y)X_1, X_2, \ldots, X_k) - \cdots - T(X_1, \ldots, X_{k-1}, (X \wedge_A Y)X_k),
\end{aligned}
$$

respectively. Setting in the above formulas $T = R$, $T = S$, $T = C$, $A = g$ or $A = S$ we obtain the tensors: $R \cdot R$, $R \cdot C$, $C \cdot R$, $C \cdot C$, $R \cdot S$ and $C \cdot S$, and $Q(g, R)$, $Q(g, C)$, $Q(S, R)$, $Q(S, C)$ and $Q(g, S)$.

(2) Some $(0, k)$-tensors - Tachibana tensors

Let A be a symmetric $(0, 2)$-tensor and T a $(0, k)$-tensor.
The tensor $Q(A, T)$ is called the *Tachibana tensor of A and T*, or the *Tachibana tensor* for short ([DGPSS]).

We like to point out that in some papers the tensor $Q(g, R)$ is called the *Tachibana tensor* (see, e.g., [HV], [JHSV], [JHP-TV]).

[DGPSS] R. Deszcz, M. Głogowska, M. Plaue, K. Sawicz, and M. Scherfner, On hypersurfaces in space forms satisfying particular curvature conditions of Tachibana type, Kragujevac J. Math. 35 (2011), 223-247.

[HV] S. Haesen and L. Verstraelen, Properties of a scalar curvature invariant depending on two planes, Manuscripta Math. 122 (2007), 59–72.

[JHSV] B. Jahanara, S. Haesen, Z. Sentürk and L. Verstraelen, On the parallel transport of the Ricci curvatures, J. Geom. Phys. 57 (2007), 1771–1777.

[JHP-TV] B. Jahanara, S. Haesen, M. Petrović-Torgasev and L. Verstraelen, On the Weyl curvature of Deszcz, Publ. Math. Debrecen 74 (2009), 417–431.

Some subsets of semi-Riemannian manifolds

Let (M, g), $n \geq 4$, be a semi-Riemannian manifold.
We define the following subset of M:

$$\mathcal{U}_R = \{x \in M \,|\, R \neq \tfrac{\kappa}{(n-1)n}\, G \text{ at } x\}, \quad G = \tfrac{1}{2}\, g \wedge g,$$

$$\mathcal{U}_S = \{x \in M \,|\, S \neq \tfrac{\kappa}{n}\, g \text{ at } x\},$$

$$\mathcal{U}_C = \{x \in M \,|\, C \neq 0 \text{ at } x\}.$$

We note that $\mathcal{U}_S \cup \mathcal{U}_C = \mathcal{U}_R$.

(1) Warped product manifolds

Let $(\overline{M}, \overline{g})$ and $(\widetilde{N}, \widetilde{g})$, $\dim \overline{M} = p$, $\dim N = n - p$, $1 \leq p < n$, be semi-Riemannian manifolds covered by systems of charts $\{U; x^a\}$ and $\{V; y^\alpha\}$, respectively. Let F be a positive smooth function on $\overline{M}$. The *warped product* $\overline{M} \times_F N$ of $(\overline{M}, \overline{g})$ and $(\widetilde{N}, \widetilde{g})$ is the product manifold $\overline{M} \times \widetilde{N}$ with the metric $g = \overline{g} \times_F \widetilde{g}$ defined by

$$\overline{g} \times_F \widetilde{g} \;\; = \;\; \pi_1^* \overline{g} + (F \circ \pi_1)\, \pi_2^* \widetilde{g},$$

where $\pi_1 : \overline{M} \times \widetilde{N} \longrightarrow \overline{M}$ and $\pi_2 : \overline{M} \times \widetilde{N} \longrightarrow \widetilde{N}$ are the natural projections on $\overline{M}$ and $\widetilde{N}$, respectively.
Let $\{U \times V; x^1, \ldots, x^p, x^{p+1} = y^1, \ldots, x^n = y^{n-p}\}$ be a product chart for $\overline{M} \times \widetilde{N}$. The local components g_{ij} of the metric $g = \overline{g} \times_F \widetilde{g}$ with respect to this chart are the following $g_{ij} = \overline{g}_{ab}$ if $i = a$ and $j = b$, $g_{ij} = F\, \widetilde{g}_{\alpha\beta}$ if $i = \alpha$ and $j = \beta$, and $g_{ij} = 0$ otherwise, where $a, b, c, d, f \in \{1, \ldots, p\}$, $\alpha, \beta, \gamma, \delta \in \{p+1, \ldots, n\}$ and $h, i, j, k, r, s \in \{1, 2, \ldots, n\}$.
We will denote by bars (resp., by tildes) tensors formed from $\overline{g}$ (resp., $\widetilde{g}$).

The local components

$$\Gamma^h_{ij} = \frac{1}{2} g^{hs}(\partial_i g_{js} + \partial_j g_{is} - \partial_s g_{ij}), \quad \partial_j = \frac{\partial}{\partial x^j},$$

of the Levi-Civita connection ∇ of $\overline{M} \times_F \widetilde{N}$ are the following (see, e.g., [K]):

$$\Gamma^a_{bc} = \overline{\Gamma}^a_{bc}, \quad \Gamma^\alpha_{\beta\gamma} = \widetilde{\Gamma}^\alpha_{\beta\gamma}, \quad \Gamma^a_{\alpha\beta} = -\frac{1}{2}\overline{g}^{ab} F_b \widetilde{g}_{\alpha\beta}, \quad \Gamma^\alpha_{a\beta} = \frac{1}{2F} F_a \delta^\alpha_\beta,$$

$$\Gamma^a_{\alpha b} = \Gamma^\alpha_{ab} = 0, \quad F_a = \partial_a F = \frac{\partial F}{\partial x^a}, \quad \partial_a = \frac{\partial}{\partial x^a}.$$

The local components

$$R_{hijk} = g_{hs} R^s_{ijk} = g_{hs}(\partial_k \Gamma^s_{ij} - \partial_j \Gamma^s_{ik} + \Gamma^r_{ij}\Gamma^s_{rk} - \Gamma^r_{ik}\Gamma^s_{rj}), \quad \partial_k = \frac{\partial}{\partial x^k},$$

of the Riemann-Christoffel curvature tensor R and the local components S_{ij} of the Ricci tensor S of the warped product $\overline{M} \times_F N$ which may not vanish identically are the following:

[K] G.I. Kruchkovich, On some class of Riemannian spaces (in Russian), Trudy sem. po vekt. i tenz. analizu, 11 (1961), 103–128.

(3) Warped product manifolds

$$
\begin{aligned}
R_{abcd} &= \overline{R}_{abcd}, \\
R_{\alpha ab\beta} &= -\frac{1}{2}\, T_{ab}\widetilde{g}_{\alpha\beta}, \\
R_{\alpha\beta\gamma\delta} &= F\widetilde{R}_{\alpha\beta\gamma\beta} - \frac{1}{4}\,\Delta_1 F\,\widetilde{G}_{\alpha\beta\gamma\delta}\,,
\end{aligned}
$$

$$
\begin{aligned}
S_{ab} &= \overline{S}_{ab} - \frac{n-p}{2}\,\frac{1}{F}\,T_{ab}, \\
S_{\alpha\beta} &= \tilde{S}_{\alpha\beta} - \frac{1}{2}\,(tr(T) + \frac{n-p-1}{2F}\Delta_1 F)\widetilde{g}_{\alpha\beta},
\end{aligned}
$$

where

$$
\begin{aligned}
T_{ab} &= \overline{\nabla}_b F_a - \frac{1}{2F} F_a F_b, \quad tr(T) = \overline{g}^{ab} T_{ab} = \Delta F - \frac{1}{2F}\,\Delta_1 F, \\
\Delta F &= \Delta_{\overline{g}} F = \overline{g}^{ab} \nabla_a F_b, \quad \Delta_1 F = \Delta_{1\overline{g}} F = \overline{g}^{ab} F_a F_b\,,
\end{aligned}
$$

and T is the $(0,2)$-tensor with the local components T_{ab}.

(4) Warped product manifolds

The scalar curvature κ of $\overline{M} \times_F \widetilde{N}$ satisfies the following relation

$$\begin{aligned}\kappa &= \overline{\kappa} + \frac{1}{F}\widetilde{\kappa} - \frac{n-p}{F}\left(tr(T) + \frac{n-p-1}{4F}\Delta_1 F\right)\\ &= \overline{\kappa} + \frac{1}{F}\widetilde{\kappa} - \frac{n-p}{F}\left(\Delta F + \frac{n-p-3}{4F}\Delta_1 F\right),\end{aligned}$$

where

$$\begin{aligned}\Delta F &= \Delta_{\overline{g}} F = \overline{g}^{ab}\nabla_a F_b,\\ \Delta_1 F &= \Delta_{1\overline{g}} F = \overline{g}^{ab} F_a F_b\,.\end{aligned}$$

(1) $R = \frac{\phi}{2}\, S \wedge S + \mu\, g \wedge S + \frac{\eta}{2}\, g \wedge g$

Theorem (see, e.g., [DGJZ], [K1]).
Let (M, g), $n \geq 4$, be a semi-Riemannian manifold and let the following condition be satisfied on $\mathcal{U}_S \cap \mathcal{U}_C \subset M$

$$R \;=\; \frac{\phi}{2}\, S \wedge S + \mu\, g \wedge S + \frac{\eta}{2}\, g \wedge g \,, \tag{1}$$

where ϕ, μ and η are some functions on this set. Then on $\mathcal{U}_S \cap \mathcal{U}_C$ we have

$$S^2 \;=\; \alpha_1\, S + \alpha_2\, g, \quad \alpha_1 \;=\; \kappa + \frac{(n-2)\mu - 1}{\phi}, \quad \alpha_2 \;=\; \frac{\mu\kappa + (n-1)\eta}{\phi},$$

$$R \cdot R \;=\; L_R\, Q(g, R), \quad L_R \;=\; \frac{1}{\phi}\left((n-2)(\mu^2 - \phi\eta) - \mu\right),$$

$$R \cdot R \;=\; Q(S, R) + L\, Q(g, C), \quad L \;=\; L_R + \frac{\mu}{\phi} \;=\; \frac{n-2}{\phi}(\mu^2 - \phi\eta),$$

$$C \cdot R \;=\; L_C\, Q(g, R), \quad L_C \;=\; L_R + \frac{1}{n-2}\left(\frac{\kappa}{n-1} - \alpha_1\right),$$

$$C \cdot C \;=\; L_C\, Q(g, C).$$

$$R = \frac{\phi}{2}\, S \wedge S + \mu\, g \wedge S + \frac{\eta}{2}\, g \wedge g \qquad (2)$$

Moreover, we have on $\mathcal{U}_S \cap \mathcal{U}_C$ ([DGJZ], [K1])

$$\begin{aligned} R \cdot C &= L_R\, Q(g, C), \\ C \cdot R &= Q(S, C) + \left(L_R - \frac{\kappa}{n-1} \right) Q(g, C), \\ R \cdot C + C \cdot R &= Q(S, C) + \left(2L_R - \frac{\kappa}{n-1} \right) Q(g, C) \end{aligned}$$

and

$$C \cdot R - R \cdot C = Q(S, C) - \frac{\kappa}{n-1}\, Q(g, C).$$

[DGJZ] R. Deszcz, M. Głogowska, J. Jełowicki and G. Zafindratafa, Curvature properties of some class of warped product manifolds, Int. J. Geom. Meth. Modern Phys. 13 (2016), 1550135 (36 pages).

[K1] D. Kowalczyk, On the Reissner-Nordström-de Sitter type spacetimes, Tsukuba J. Math. 30 (2006), 363–381.

(3) $R = \frac{\phi}{2} S \wedge S + \mu\, g \wedge S + \frac{\eta}{2} g \wedge g$

Theorem ([DGHHY], Theorem 4.1)**.**
If (M, g), $n \geq 4$, is a semi-Riemannian manifold satisfying on the set $\mathcal{U}_S \cap \mathcal{U}_C \subset M$ the following conditions:

$$\begin{aligned} R \cdot R &= Q(S, R) + L\, Q(g, C), \\ C \cdot C &= L_C\, Q(g, C), \\ R \cdot S &= Q(g, D), \end{aligned}$$

where L and L_C are some functions on $\mathcal{U}_S \cap \mathcal{U}_C$ and D is a symmetric $(0, 2)$-tensor on this set, then the Roter type equation (1) holds on the set $\mathcal{U}$ of all points of $\mathcal{U}_S \cap \mathcal{U}_C$ at which $\operatorname{rank}(S - \tau\, g) > 1$ for any $\tau \in \mathbb{R}$.

[DGHHY] R. Deszcz, M. Głogowska, H. Hashiguchi, M. Hotloś and M. Yawata,
On semi-Riemannian manifolds satisfying some conformally invariant curvature condition,
Colloquium Math. 131 (2013), 149–170.

$$(4) \qquad R = \frac{\phi}{2}\, S \wedge S + \mu\, g \wedge S + \frac{\eta}{2}\, g \wedge g$$

Example ([DK], Example 4.1)**.** Let $S^p(\frac{1}{\sqrt{c_1}})$, be the p-dimensional, $p \geq 2$, standard sphere of radius $\frac{1}{\sqrt{c_1}}$, $c_1 = const. > 0$, with the standard metric $\overline{g}$. Let f be a non-constant function on $S^p(\frac{1}{\sqrt{c_1}})$ satisfying the following differential equation ([Obata])

$$\overline{\nabla}(df) + c_1\, f\, \overline{g} \;=\; 0\,.$$

We set $F = (f + c)^2$, where c is a non-zero constant such that $f + c$ is either positive or negative on $S^p(\frac{1}{\sqrt{c_1}})$.

Let $(\widetilde{N}, \widetilde{g})$, $n - p = \dim \widetilde{N} \geq 2$, be a semi-Riemannian space of constant curvature c_2. We consider the warped product $S^p(\frac{1}{\sqrt{c_1}}) \times_F \widetilde{N}$ of the manifolds $S^p(\frac{1}{\sqrt{c_1}})$ and $(\widetilde{N}, \widetilde{g})$ with the above defined warping function F.

[DK] R. Deszcz and D. Kowalczyk, On some class of pseudosymmetric warped products, Colloquium Math. 97 (2003), 7–22.

[Obata] M. Obata, Certain conditions for a Riemannian manifold to be isometric with a sphere, J. Math. Soc. Japan, 14 (1962), 333–340.

(5) $$R = \frac{\phi}{2}\, S \wedge S + \mu\, g \wedge S + \frac{\eta}{2}\, g \wedge g$$

We can check that the warped product

$$S^p\left(\frac{1}{\sqrt{c_1}}\right) \times_F \widetilde{N}$$

satisfies the Roter type equation (1). In particular, the warped product

$$S^p\left(\frac{1}{\sqrt{c_1}}\right) \times_F S^{n-p}\left(\frac{1}{\sqrt{c_2}}\right),$$

where $p \geq 2$, $n - p \geq 2$ and $c_1 > 0$, $c_2 > 0$, also satisfies (1).

Remark ([DK], Example 4.1). We also can prove that $S^p(\frac{1}{\sqrt{c_1}}) \times_F \widetilde{N}$ can be locally realized as a hypersurface in a semi-Riemannian space of constant curvature.

[DK] R. Deszcz and D. Kowalczyk, On some class of pseudosymmetric warped products, Colloquium Math. 97 (2003), 7–22.

(6) References; $R = \frac{\phi}{2} S \wedge S + \mu\, g \wedge S + \frac{\eta}{2} g \wedge g$

Warped products satisfying (1) were investigated among others in:

[D] R. Deszcz, On some Akivis-Goldberg type metrics, Publ. Inst. Math. (Beograd) (N.S.) 74 (88) (2003), 71–83.

[DGJZ] R. Deszcz, M. Głogowska, J. Jełowicki and G. Zafindratafa, Curvature properties of some class of warped product manifolds, Int. J. Geom. Meth. Modern Phys. 13 (2016), 1550135 (36 pages).

[DKow] R. Deszcz and D. Kowalczyk, On some class of pseudosymmetric warped products, Colloquium Math. 97 (2003), 7–22.

[DPSch] R. Deszcz, M. Plaue and M. Scherfner, On Roter type warped products with 1-dimensional fibres, J. Geom. Phys. 69 (2013), 1–11.

[DSch] R. Deszcz and M. Scherfner, On a particular class of warped products with fibres locally isometric to generalized Cartan hypersurfaces, Colloquium Math. 109 (2007), 13–29.

[K1] D. Kowalczyk, On the Reissner-Nordström-de Sitter type spacetimes, Tsukuba J. Math. 30 (2006), 363–381.

[K2] D. Kowalczyk, On some class of semisymmetric manifolds, Soochow J. Math. 27 (2001), 445–461.

(1) Some curvature identities

Let (M, g), $n \geq 4$, be a semi-Riemannian manifold.
We have on $\mathcal{U}_S \cap \mathcal{U}_C$ the following identity ([DGJZ]):

$$(C - R) \cdot (C - R) = \frac{1}{(n-2)^2} (g \wedge S - \frac{\kappa}{n-1} G) \cdot (g \wedge S - \frac{\kappa}{n-1} G).$$

This yields

$$\begin{aligned} & (n-2)^2 (C \cdot C - R \cdot C - C \cdot R + R \cdot R) \\ = \; & (g \wedge S) \cdot (g \wedge S) - \frac{\kappa}{n-1} G \cdot (g \wedge S). \end{aligned}$$

[DGJZ] R. Deszcz, M. Głogowska, J. Jełowicki and G. Zafindratafa, Curvature properties of some class of warped product manifolds, Int. J. Geom. Meth. Modern Phys. 13 (2016), 1550135 (36 pages).

(2) Some curvature identities

We also have on $\mathcal{U}_S \cap \mathcal{U}_C$ the following identities (see, e.g., [DGJZ], [K1]):

$$Q(S, g \wedge S) = -\frac{1}{2} Q(g, S \wedge S), \quad Q(g, g \wedge S) = -Q(S, G),$$

$$Q(S, R) = Q(S, C) - \frac{1}{n-2} Q(g, \frac{1}{2} S \wedge S) - \frac{\kappa}{(n-2)(n-1)} Q(S, G),$$

$$(g \wedge S) \cdot S = Q(g, S^2), \quad G \cdot S = Q(g, S), \quad S^2(X, Y) = S(\mathcal{S}X, Y),$$

$$(g \wedge S) \cdot (g \wedge S) = -Q(S^2, G), \; G \cdot (g \wedge S) = Q(g, g \wedge S) = -Q(S, G)$$

[DGJZ] R. Deszcz, M. Głogowska, J. Jełowicki and G. Zafindratafa, Curvature properties of some class of warped product manifolds, Int. J. Geom. Meth. Modern Phys. 13 (2016), 1550135 (36 pages).

[K1] D. Kowalczyk, On the Reissner-Nordström-de Sitter type spacetimes, Tsukuba J. Math. 30 (2006), 363–381.

(3) Some curvature identities

Theorem (cf. [DGJZ], Theorem 3.4)**.**
Let (M, g), $n \geq 4$, be a semi-Riemannian manifold.
(i) The following identity is satisfied on $\mathcal{U}_S \cap \mathcal{U}_C \subset M$

$$C \cdot R + R \cdot C \;=\; R \cdot R + C \cdot C - \frac{1}{(n-2)^2}\, Q(g, -\frac{\kappa}{n-1}\, g \wedge S + g \wedge S^2).$$

(ii) If the following curvature conditions

$$R \cdot R \;=\; Q(S, R) - L\, Q(g, C), \quad C \cdot C \;=\; L_C\, Q(g, C),$$

are satisfied on $\mathcal{U}_S \cap \mathcal{U}_C$ then

$$\begin{aligned} C \cdot R + R \cdot C \;=\; & Q(S, C) + (L + L_C)\, Q(g, C) \\ & - \frac{1}{(n-2)^2}\, Q(g, \frac{n-2}{2}\, S \wedge S - \kappa\, g \wedge S + g \wedge S^2), \quad (2) \end{aligned}$$

where L and L_C are some functions on $\mathcal{U}_S \cap \mathcal{U}_C$.

[DGJZ] R. Deszcz, M. Głogowska, J. Jełowicki and G. Zafindratafa, Curvature properties of some class of warped product manifolds, Int. J. Geom. Meth. Modern Phys. 13 (2016),

(4) Some curvature identities

([DGJZ]) Moreover, if $\operatorname{rank}(S - \alpha\, g) = 1$ on $\mathcal{U}_S \cap \mathcal{U}_C$, where α is some function on $\mathcal{U}_S \cap \mathcal{U}_C$, then on this set we have

$$\begin{aligned} \frac{1}{2}\, S \wedge S - \alpha\, g \wedge S + \alpha^2\, G &= 0, \\ S^2 + ((n-2)\alpha - \kappa)\, S + \alpha(\kappa - (n-1)\alpha)\, g &= 0, \end{aligned}$$

and now (2) reduces to

$$C \cdot R + R \cdot C = Q(S, C) + (L + L_C)\, Q(g, C). \tag{3}$$

In particular, if (M, g) is the Gödel spacetime then $\mathcal{U}_S \cap \mathcal{U}_C = M$ and (3) turns into

$$C \cdot R + R \cdot C = Q(S, C) + \frac{\kappa}{6}\, Q(g, C). \tag{4}$$

[DGJZ] R. Deszcz, M. Głogowska, J. Jełowicki and G. Zafindratafa, Curvature properties of some class of warped product manifolds, Int. J. Geom. Meth. Modern Phys. 13 (2016), 1550135 (36 pages).

(1) The Gödel spacetime

The *Gödel metric* (5) is given by ([G]):

$$\begin{aligned} ds^2 &= g_{ij} dx^i dx^j \\ &= a^2 \left(-(dx^1)^2 + \frac{1}{2} e^{2x^1} (dx^2)^2 - (dx^3)^2 + (dx^4)^2 + 2e^{x^1} dx^2 dx^4\right), \quad (5) \end{aligned}$$

where $x^i \in \mathbb{R}$, $i, j \in \{1, 2, 3, 4\}$, and a is a non-zero constant.
For the Gödel metric we have

$$(\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) + (\nabla_Z S)(X, Y) = 0,$$
$$S = \kappa \, \omega \otimes \omega, \quad \kappa = \frac{1}{a^2},$$

where ω is a 1-form and $\omega = (\omega_1, \omega_2, \omega_3, \omega_4) = (0, a \exp(x^1), 0, a)$.
We also note that

$$S^2 = \kappa \, S.$$

[G] K. Gödel, An example of a new type of cosmological solutions of Einstein's field equations of gravitation, Reviews Modern Physics, 21(3) (1949), 447–450.

(2) The Gödel spacetime

Moreover for the Gödel metric (5) we have ([DHJKS]):

$$\begin{aligned} R \cdot R &= Q(S, R), \\ R(\mathcal{S}X, Y, Z, W) &+ R(\mathcal{S}Z, Y, W, X) + R(\mathcal{S}W, Y, X, Z) \;=\; 0, \\ C \cdot C &= C \cdot conh(R) \;=\; \frac{\kappa}{6}\, Q(g, C), \\ conh(R) \cdot conh(R) &= conh(R) \cdot C \;=\; 0, \end{aligned}$$

where the tensor $conh(R)$ is defined by ([I])

$$conh(R) \;=\; R - \frac{1}{n-2}\, g \wedge S \;=\; C - \frac{\kappa}{(n-2)(n-1)}\, G.$$

[DHJKS] R. Deszcz, M. Hotlos, J. Jełowicki, H. Kundu and A.A. Shaikh, Curvature properties of Gödel metric, Int. J. Geom. Meth. Modern Phys. 11 (2014) 1450025 (20 pages).

[I] Y. Ishii, On conharmonic transformations, Tensor (N.S.) 7 (1957), 73–80.

(3) The Gödel spacetime

From $R \cdot R = Q(S, R)$ and $S^2 = \kappa S$ we obtain immediately

$$R \cdot R = \frac{1}{\kappa} Q(S^2, R) .$$

Thus the Gödel metric (5) satisfies a condition of the form

$$R \cdot R = L_2 Q(S^2, R) .$$

Conditions of the form $R \cdot R = L_p Q(S^p, R)$, $p = 1, 2, \dots,$ where L_p are some functions, were introduced and investigated in [P1] and [P2].
The tensors S^2, S^3, S^4, $\dots$, are defined by

$$S^2(X, Y) = S(\mathcal{S}X, Y),\ S^3(X, Y) = S^2(\mathcal{S}X, Y),\ S^4(X, Y) = S^3(\mathcal{S}X, Y), \dots$$

[P1] M. Prvanović, On SP-Sasakian manifold satisfying some curvature conditions, SUT Journal of Mathematics, 26 (1990), 201–206.

[P2] M. Prvanović, On a class of SP-Sasakian manifold, Note di Matematica, Lecce, 10 (1990), 325–334.

(4) The Gödel spacetime

Remark.

For any semi-Riemannian manifold (M, g), $n \geq 4$, we have (cf. [DGH])

$$\begin{aligned} conh(R) \cdot S &= C \cdot S - \frac{\kappa}{(n-2)(n-1)} Q(g, S), \\ R \cdot conh(R) &= R \cdot C, \\ conh(R) \cdot R &= C \cdot R - \frac{\kappa}{(n-2)(n-1)} Q(g, R), \\ conh(R) \cdot conh(R) &= C \cdot C - \frac{\kappa}{(n-2)(n-1)} Q(g, C). \end{aligned}$$

[DGH] R. Deszcz, M. Głogowska and M. Hotloś, Some identities on hypersurfaces in conformally flat spaces, in: Proceedings of the International Conference XVI Geometrical Seminar, Vrnjacka Banja, September, 20-25, 2010, Faculty of Science and Mathematics, University of Nis, Serbia, 2011, 34–39.

(1) Quasi-Einstein manifolds

We recall that the semi-Riemannian manifold (M, g), $n \geq 3$, is said to be a *quasi-Einstein manifold* if

$$\operatorname{rank}(S - \alpha\, g) \ = \ 1$$

on $\mathcal{U}_S \subset M$, where α is some function on this set (see, e.g., [DGHS]). Every warped product manifold $\overline{M} \times_F \widetilde{N}$ of an 1-dimensional $(\overline{M}, \overline{g})$ base manifold and a 2-dimensional manifold $(\widetilde{N}, \widetilde{g})$ or an $(n-1)$-dimensional Einstein manifold $(\widetilde{N}, \widetilde{g})$, $n \geq 4$, with a warping function F, is a quasi-Einstein manifold (see, e.g., [Ch-DDGP]).

[DGHS] R. Deszcz, M. Głogowska, M. Hotloś and Z. Sentürk, On certain quasi-Einstein semisymmetric hypersurfaces, Ann. Univ. Sci. Budapest. Eötvös Sect. Math. 41 (1998), 151–164.

[Ch-DDGP] J. Chojnacka-Dulas, R. Deszcz, M. Głogowska and M. Prvanović, On warped product manifolds satisfying some curvature conditions, J. Geom. Phys. 74 (2013), 328–341.

(2) Quasi-Einstein manifolds

Quasi-Einstein manifolds arose during the study of exact solutions of the Einstein field equations and the investigation on quasi-umbilical hypersurfaces of conformally flat spaces, see, e.g., [DGHSaw] and references therein. Quasi-Einstein hypersurfaces in semi-Riemannian spaces of constant curvature were studied among others in: [DGHS], [G] and [DHS].

[DGHSaw] R. Deszcz, M. Głogowska, M. Hotloś, and K. Sawicz, A Survey on Generalized Einstein Metric Conditions, Advances in Lorentzian Geometry: Proceedings of the Lorentzian Geometry Conference in Berlin, AMS/IP Studies in Advanced Mathematics 49, S.-T. Yau (series ed.), M. Plaue, A.D. Rendall and M. Scherfner (eds.), 2011, 27–46.

[DGHS] R. Deszcz, M. Głogowska, M. Hotloś and Z. Sentürk, On certain quasi-Einstein semisymmetric hypersurfaces, Ann. Univ. Sci. Budapest. Eötvös Sect. Math. 41 (1998), 151–164.

[G] M. Głogowska, On quasi-Einstein Cartan type hypersurfaces, J. Geom. Phys. 58 (2008), 599–614.

[DHS] R. Deszcz, M. Hotloś and Z. Sentürk, On curvature properties of certain quasi-Einstein hypersurfaces, Int. J. Math. 23 (2012), 1250073, 17 pp.

(1) Examples of 3-dimensional quasi-Einstein manifolds

Remark ([DGJZ]). (i) The Ricci tensor of the following 3-dimensional Riemannian manifolds $(\widetilde{N}, \widetilde{g})$: the Berger spheres, the Heisenberg group Nil_3, $\widetilde{PSL(2,\mathbb{R})}$ - the universal covering of the Lie group $PSL(2,\mathbb{R})$ and the Lie group Sol_3 ([LVW]), a Riemannian manifold isometric to an open part of the Cartan hypersurface ([DG]) and some three-spheres of Kaluza-Klein type ([CP]) have exactly two distinct eigenvalues.

[DGJZ] R. Deszcz, M. Głogowska, J. Jełowicki and G. Zafindratafa, Curvature properties of some class of warped product manifolds, Int. J. Geom. Meth. Modern Phys. 13 (2016), 1550135 (36 pages).

[LVW] H. Li, L. Vrancken, X. Wang, A new characterization of the Berger sphere in complex projective space, J. Geom. Phys. 92 (2015), 129–139.

[DG] R. Deszcz and M. Głogowska, Some nonsemisymmetric Ricci-semisymmetric warped product hypersurfaces, Publ. Inst. Math. (Beograd) (N.S.) 72 (86)) (2002), 81–93.

[CP] G. Calvaruso and D. Perrone, Geometry of Kaluza-Klein metrics on the 3-dimensional sphere, Annali di Mat. 192 (2013), 879–900.

(2) Examples of 3-dimensional quasi-Einstein manifolds

These manifolds are quasi-Einstein, and in a consequence, pseudosymmetric (see, e.g., [DVY]). For further examples of 3-dimensional quasi-Einstein manifolds we refer to [BDV] (Thurston geometries and warped product manifolds) and [K] (manifolds with constant Ricci principal curvatures).
(ii) We mention that recently pseudosymmetry type curvature conditions of four-dimensional Thurston geometries were investigated in [H].

[DVY] R. Deszcz, L. Verstraelen and S. Yaprak, Warped products realizing a certain condition of pseudosymmetry type imposed on the Weyl curvature tensor, Chinese J. Math. 22 (1994), 139–157.

[BDV] M. Belkhelfa, R. Deszcz and L. Verstraelen, Pseudosymmetry of 3-dimensional D'Atri space, Kyungpook Math. J. 46 (2006), 367–376.

[K] O. Kowalski, A classification of Riemannian 3-manifolds with constant principal Ricci curvatures .... , Nagoya Math. J. 132 (1993), 1–36.

[H] A. Hasni, Les géométries de Thurston et la pseudo-symétrie d'apreès R. Deszcz, Thèse de doctorat en mathématique, Université Abou Bakr Belkaid-Tlemcen, Faculté de Sciences, Département de Mathématiques, 2014.

(1) An example of a 5-dimensional quasi-Einstein manifold

Example.

(i) ([K1], [K2], [K3]) Let M be an open connected subset of $\mathbb{R}^5$ endowed with the metric g of the form

$$\begin{aligned} ds^2 &= g_{ij} dx^i dx^j \\ &= dx^2 + dy^2 + du^2 + dv^2 + \rho^2 (xdu - ydv + dz)^2 , \end{aligned}$$

where $\rho = \text{const.} \neq 0$.

[K1] O. Kowalski, Classifcation of generalized symmetric Riemannian spaces of dimension $n \geq 5$, Rozpr. Cesk. Akad. Ved, Rada Mat. Prir. Ved, 85(8) (1975), 1–61.

[K2] O. Kowalski, Generalized Symmetric Spaces, Lecture Notes in Mathematics, Springer Verlag, Berlin Heidelberg New York, 1980.

[K3] O. Kowalski, Generalized Symmetric Spaces, MIR, Moscow, 1984. (in Russian)

(2) An example of a 5-dimensional quasi-Einstein manifold

(ii) ([SDHJK])

The manifold (M, g) is a non-conformally flat manifold with cyclic parallel Ricci tensor, i.e. $\nabla_X S(Y,Z) + \nabla_Y S(Z,X) + \nabla_Z S(X,Y) = 0$, satisfying:

$$S \;=\; \frac{\kappa}{2}\, g - \frac{3\kappa}{2}\, \eta \otimes \eta\, , \quad \eta \;=\; (0, 0, -\rho, -x\rho, y\rho)\, , \quad \kappa \;=\; \rho^2\, .$$

$$\begin{aligned} C \cdot S &= 0\, , \\ R \cdot R &= -\frac{\kappa}{4}\, Q(g, R)\, , \\ C \cdot C &= C \cdot R\, , \\ C \cdot R &= -\frac{1}{3}\, Q(S, C) - \frac{\kappa}{3}\, Q(g, C)\, , \\ R \cdot C - C \cdot R &= \frac{1}{3}\, Q(S, C) + \frac{\kappa}{12}\, Q(g, C)\, . \end{aligned}$$

[SDHJK] A.A. Shaikh, R. Deszcz, M. Hotloś, J. Jełowicki, and H. Kundu, On pseudosymmetric manifolds, Publ. Math. Debrecen 86 (2015), 433–456.

(3) An example of a 5-dimensional quasi-Einstein manifold

(iii) We also have

$$
\begin{aligned}
R \cdot C + C \cdot R &= -\frac{1}{3}\, Q(S, C) - \frac{7\kappa}{12}\, Q(g, C)\,, \\
S^2 &= -\frac{\kappa}{2}\, S + \frac{\kappa^2}{2}\, g\,, \\
R \cdot R &= -\frac{1}{2\kappa}\, Q(S^2, R) - \frac{1}{4}\, Q(S, R)\,, \\
S \cdot R &= 2\kappa\, R - \frac{\kappa}{2} g \wedge S + \frac{\kappa^2}{4}\, g \wedge g\,.
\end{aligned}
$$

The $(0, 4)$-tensor $S \cdot R$ is defined by

$$
\begin{aligned}
(S \cdot R)(X, Y, W, Z) &= R(\mathcal{S}X, Y, W, Z) + R(X, \mathcal{S}Y, W, Z) \\
&\quad + R(X, Y, \mathcal{S}W, Z) + R(X, Y, W, \mathcal{S}Z)\,.
\end{aligned}
$$

(1) Warped product manifolds with 1-dimensional base manifold and the conformally flat quasi-Einstein fiber

Theorem ([DGJZ], Theorem 4.3)**.**
Let $\overline{M} \times_F \widetilde{N}$ be the warped product manifold with an 1-dimensional manifold $(\overline{M}, \overline{g})$, $\overline{g}_{11} = \pm 1$, and an $(n-1)$-dimensional quasi-Einstein semi-Riemannian manifold $(\widetilde{N}, \widetilde{g})$, $n \geq 4$, and a warping function F, and let $(\widetilde{N}, \widetilde{g})$ be a conformally flat manifold, when $n \geq 5$. Then

$$\begin{aligned} C \cdot C &= L_C\, Q(g, C), \\ R \cdot R - Q(S, R) &= L\, Q(g, C), \\ C \cdot R + R \cdot C &= Q(S, C) + (L_C + L)\, Q(g, C) \\ &\quad - \frac{1}{(n-2)^2}\, Q(g, \frac{n-2}{2}\, S \wedge S - \kappa\, g \wedge S + g \wedge S^2) \end{aligned}$$

on $\mathcal{U}_S \cap \mathcal{U}_C \subset \overline{M} \times_F \widetilde{N}$.

[DGJZ] R. Deszcz, M. Głogowska, J. Jełowicki and G. Zafindratafa, Curvature properties of some class of warped product manifolds, Int. J. Geom. Meth. Modern Phys. 13 (2016), 1550135 (36 pages).

(2) Warped product manifolds with 1-dimensional base manifold and the conformally flat quasi-Einstein fiber

Theorem ([DGJZ], Theorem 4.4)**.**
Let $\overline{M} \times \widetilde{N}$ be the product manifold with an 1-dimensional manifold $(\overline{M}, \overline{g})$, $\overline{g}_{11} = \pm 1$, and an $(n-1)$-dimensional quasi-Einstein semi-Riemannian manifold $(\widetilde{N}, \widetilde{g})$, $n \geq 4$, satisfying $\mathrm{rank}\,(\widetilde{S} - \rho\,\widetilde{g}) = 1$ on $\mathcal{U}_{\widetilde{S}} \subset \widetilde{M}$, where ρ is some function on $\mathcal{U}_{\widetilde{S}}$, and let $(\widetilde{N}, \widetilde{g})$ be a conformally flat manifold, when $n \geq 5$. Then on $\mathcal{U}_S \cap \mathcal{U}_C \subset \overline{M} \times \widetilde{N}$ we have

$$
\begin{aligned}
(n-3)(n-2)\rho\, C &= \frac{n-2}{2}\, S \wedge S - \kappa\, g \wedge S \\
&\quad + (n-2)\rho \left(\frac{2\kappa}{n-1} - \rho \right) G + g \wedge S^2 , \\
C \cdot R + R \cdot C &= Q(S, C) + \left(\frac{\kappa}{(n-2)(n-1)} - \rho \right) Q(g, C).
\end{aligned}
$$

[DGJZ] R. Deszcz, M. Głogowska, J. Jełowicki and G. Zafindratafa, Curvature properties of some class of warped product manifolds, Int. J. Geom. Meth. Modern Phys. 13 (2016),

(1) Warped products with 2-dimensional base

Let $\overline{M} \times_F \widetilde{N}$ be the warped product manifold with a 2-dimensional semi-Riemannian manifold $(\overline{M}, \overline{g})$ and an $(n-2)$-dimensional semi-Riemannian manifold $(\widetilde{N}, \widetilde{g})$, $n \geq 4$, and a warping function F, and let $(\widetilde{N}, \widetilde{g})$ be a space of constant curvature, when $n \geq 5$.
Let S_{hk} and C_{hijk} be the local components of the Ricci tensor S and the tensor Weyl conformal curvature tensor C of $\overline{M} \times_F \widetilde{N}$, respectively. We have

$$S_{ad} = \frac{\overline{\kappa}}{2} g_{ab} - \frac{n-2}{2F} T_{ab}, \quad S_{\alpha\beta} = \tau_1 g_{\alpha\beta}, \quad S_{a\alpha} = 0, \tag{6}$$

$$\tau_1 = \frac{\widetilde{\kappa}}{(n-2)F} - \frac{\mathrm{tr}(T)}{2F} - (n-3)\frac{\Delta_1 F}{4F^2},$$

$$\Delta_1 F = \Delta_{1\overline{g}} F = \overline{g}^{ab} F_a F_b,$$

$$T_{ab} = \overline{\nabla}_a F_b - \frac{1}{2F} F_a F_b, \quad \mathrm{tr}(T) = \overline{g}^{ab} T_{ab},$$

where T is the $(0,2)$-tensor with the local components T_{ab}.

(2) Warped products with 2-dimensional base

$$
\begin{aligned}
C_{abcd} &= \frac{n-3}{n-1}\,\rho_1\,G_{abcd} = \frac{n-3}{n-1}\,\rho_1\,(g_{ad}g_{bc} - g_{ac}g_{bd}),\\
C_{\alpha bc\beta} &= -\frac{n-3}{(n-2)(n-1)}\,\rho_1\,G_{\alpha bc\beta} = -\frac{n-3}{(n-2)(n-1)}\,\rho_1\,g_{bc}g_{\alpha\beta},\\
C_{\alpha\beta\gamma\delta} &= \frac{2}{(n-2)(n-1)}\,\rho_1\,G_{\alpha\beta\gamma\delta} = \frac{2}{(n-2)(n-1)}\,\rho_1(g_{\alpha\delta}g_{\beta\gamma} - g_{\alpha\gamma}g_{\beta\delta})\\
C_{abc\delta} &= C_{ab\alpha\beta} = C_{a\alpha\beta\gamma} = 0, \qquad (7
\end{aligned}
$$

where

$$
\begin{aligned}
G_{hijk} &= g_{hk}g_{ij} - g_{hj}g_{ik},\\
\Delta_1 F &= \Delta_{1\,\overline{g}} F = \overline{g}^{ab} F_a F_b, \quad \Delta F = \overline{g}^{ab}\overline{\nabla}_a F_b,
\end{aligned}
$$

$$
\rho_1 = \frac{\overline{\kappa}}{2} + \frac{\widetilde{\kappa}}{(n-3)(n-2)F} + \frac{1}{2F}\left(\Delta F - \frac{\Delta_1 F}{F}\right).
$$

(3) Warped products with 2-dimensional base

If we set

$$\rho = \frac{2(n-3)}{n-1}\,\rho_1 \tag{8}$$

then (7) turns into ([DGJZ])

$$\begin{aligned} C_{abcd} &= \frac{\rho}{2}\, G_{abcd}, \\ C_{\alpha bc\beta} &= -\frac{\rho}{2(n-2)}\, G_{\alpha bc\beta}, \\ C_{\alpha\beta\gamma\delta} &= \frac{\rho}{(n-3)(n-2)}\, G_{\alpha\beta\gamma\delta}, \\ C_{abc\delta} &= C_{ab\alpha\beta} = C_{a\alpha\beta\gamma} = 0. \end{aligned} \tag{9}$$

[DGJZ] R. Deszcz, M. Głogowska, J. Jełowicki and G. Zafindratafa, Curvature properties of some class of warped product manifolds, Int. J. Geom. Meth. Modern Phys. 13 (2016), 1550135 (36 pages).

(4) Warped products with 2-dimensional base

Further, by making use of the formulas for the local components $(C \cdot C)_{hijklm}$ and $Q(g, C)_{hijklm}$ of the tensors $C \cdot C$ and $Q(g, C)$, i.e.

$$\begin{aligned}(C \cdot C)_{hijklm} &= g^{rs}(C_{rijk}C_{shlm} + C_{hrjk}C_{silm} + C_{hirk}C_{sjlm} + C_{hijr}C_{sklm}),\\ Q(g, C)_{hijklm} &= g_{hl}C_{mijk} + g_{il}C_{hmjk} + g_{jl}C_{himk} + g_{kl}C_{hijm}\\ &\quad - g_{hm}C_{lijk} - g_{im}C_{hljk} - g_{jm}C_{hilk} - g_{km}C_{hijl},\end{aligned}$$

we obtain

$$\begin{aligned}(C \cdot C)_{\alpha abcd\beta} &= -\frac{(n-1)\rho^2}{4(n-2)^2} g_{\alpha\beta} G_{dabc},\\ (C \cdot C)_{a\alpha\beta\gamma d\delta} &= \frac{(n-1)\rho^2}{4(n-2)^2(n-3)} g_{ad} G_{\delta\alpha\beta\gamma},\\ Q(g, C)_{\alpha abcd\beta} &= \frac{(n-1)\rho}{2(n-2)} g_{\alpha\beta} G_{dabc},\\ Q(g, C)_{a\alpha\beta\gamma d\delta} &= -\frac{(n-1)\rho}{2(n-2)(n-3)} g_{ad} G_{\delta\alpha\beta\gamma}.\end{aligned}$$

(5) Warped products with 2-dimensional base

Theorem ([DGJZ], Theorem 7.1).
Let $\overline{M} \times_F \widetilde{N}$ be the warped product manifold with a 2-dimensional semi-Riemannian manifold $(\overline{M}, \overline{g})$ and an $(n-2)$-dimensional semi-Riemannian manifold $(\widetilde{N}, \widetilde{g})$, $n \geq 4$, and a warping function F, and let $(\widetilde{N}, \widetilde{g})$ be a space of constant curvature, when $n \geq 5$.
1. The following equation is satisfied on the set $\mathcal{U}_C \subset \overline{M} \times_F \widetilde{N}$

$$C \cdot C \;=\; L_C \, Q(g, C)\,, \quad L_C \;=\; -\frac{\rho}{2(n-2)}, \tag{10}$$

$$\rho \;=\; \frac{2(n-3)}{n-1}\left(\frac{\overline{\kappa}}{2} + \frac{\widetilde{\kappa}}{(n-3)(n-2)F} + \frac{1}{2F}\left(\Delta F - \frac{\Delta_1 F}{F}\right)\right).$$

Remark. The above result, for $n = 4$, was proved in [D] (Theorem 2).

[D] R. Deszcz, On four-dimensional warped product manifolds satisfying certain pseudosymmetry curvature conditions, Colloquium Math. 62 (1991), 103–120.

[DGJZ] R. Deszcz, M. Głogowska, J. Jełowicki and G. Zafindratafa, Curvature properties of some class of warped product manifolds, Int. J. Geom. Meth. Modern Phys. 13 (2016),

(6) Warped products with 2-dimensional base

2. The following equation is satisfied on the set $\mathcal{U}_C \subset \overline{M} \times_F \widetilde{N}$

$$R \cdot R \;=\; Q(S,R) + L\, Q(g,C),$$

where L is some function on $\mathcal{U}_C$. Precisely,

$$L \;=\; -\frac{n-2}{(n-1)\rho}\left(\overline{\kappa}\left(\tau_1 + \frac{\mathrm{tr}(T)}{2F}\right) + \frac{n-3}{4F^2}(\mathrm{tr}(T^2) - (\mathrm{tr}(T))^2)\right), \quad (11)$$

$$\begin{aligned} \tau_1 &= \frac{\widetilde{\kappa}}{(n-2)F} - \frac{\mathrm{tr}(T)}{2F} - (n-3)\,\frac{\Delta_1 F}{4F^2}, \\ \Delta_1 F &= \Delta_{1\,\overline{g}} F \;=\; \overline{g}^{ab} F_a F_b, \\ T_{ab} &= \overline{\nabla}_a F_b - \frac{1}{2F} F_a F_b, \quad \mathrm{tr}(T) \;=\; \overline{g}^{ab} T_{ab}, \end{aligned}$$

where T is the $(0,2)$-tensor with the local components T_{ab}.
The tensor T^2 is defined by $T^2_{ad} = T_{ac}\overline{g}^{cd} T_{db}$ and $\mathrm{tr}(T^2) = \overline{g}^{ab} T^2_{ab}$.

(7) Warped products with 2-dimensional base

3. The following equation is satisfied on the set $\mathcal{U}_C \subset \overline{M} \times_F \widetilde{N}$

$$\begin{aligned} C \cdot R + R \cdot C &= Q(S, C) + (L_C + L)\, Q(g, C) \\ &\quad - \frac{1}{(n-2)^2}\, Q(g, \frac{n-2}{2}\, S \wedge S - \kappa\, g \wedge S + g \wedge S^2). \end{aligned}$$

where L_C and L are functions defined by (10) and (11), respectively, i.e.

$$L_C = -\frac{n-3}{(n-2)(n-1)} \left(\frac{\overline{\kappa}}{2} + \frac{\widetilde{\kappa}}{(n-3)(n-2)F} + \frac{1}{2F} \left(\Delta F - \frac{\Delta_1 F}{F} \right) \right),$$

$$L = -\frac{n-2}{(n-1)\rho} \left(\overline{\kappa} \left(\tau_1 + \frac{\mathrm{tr}(T)}{2F} \right) + \frac{n-3}{4F^2} (\mathrm{tr}(T^2) - (\mathrm{tr}(T))^2) \right).$$

(8) Warped products with 2-dimensional base

We have (see, eq. (6))

$$S_{ad} \;=\; \frac{\overline{\kappa}}{2}\, g_{ab} - \frac{n-2}{2F}\, T_{ab}, \quad S_{\alpha\beta} \;=\; \tau_1\, g_{\alpha\beta}, \quad S_{a\alpha} \;=\; 0,$$

$$\tau_1 \;=\; \frac{\widetilde{\kappa}}{(n-2)F} - \frac{\mathrm{tr}(T)}{2F} - (n-3)\,\frac{\Delta_1 F}{4F^2}.$$

We define on $\mathcal{U}_S \subset \overline{M} \times_F \widetilde{N}$ the $(0,2)$-tensor A by

$$A \;=\; S - \tau_1\, g\,.$$

We can check that $\mathrm{rank}(A) = 2$ at a point of $\mathcal{U}_S$ if and only if $\mathrm{tr}(A^2) - (\mathrm{tr}(A))^2 \neq 0$ at this point ([DGJZ], Section 6). Now, at all points of $\mathcal{U}_S$, at which $\mathrm{rank}(A) = 2$, we can define the function τ_2 by

$$\tau_2 \;=\; (\mathrm{tr}(A^2) - (\mathrm{tr}(A))^2)^{-1}\,.$$

Further, let V be the set of all points of $\mathcal{U}_S \cap \mathcal{U}_C$ at which: $\mathrm{rank}(A) = 2$ and S_{ad} is not proportional to g_{ad}.

(9) Warped products with 2-dimensional base

4. On the set $V \subset \mathcal{U}_S \cap \mathcal{U}_C$ we have:

$$C = -\frac{(n-1)\rho\tau_2}{(n-3)(n-2)} \left(\frac{n-2}{2} S \wedge S - \kappa g \wedge S + g \wedge S^2 - \frac{\mathrm{tr}(S^2) - \kappa^2}{n-1} G \right)$$

and

$$R \cdot C + C \cdot R = Q(S, C) + \left(L - \frac{\rho}{2(n-2)} + \frac{n-3}{(n-2)(n-1)\rho\tau_2} \right) Q(g, C).$$

Remark. At all points of the set $\mathcal{U}_S \cap \mathcal{U}_C$, at which S_{ad} is proportional to g_{ad} and $\mathrm{rank}(A) = 2$, the Weyl tensor C is a linear combination of the Kulkarni-Nomizu products $S \wedge S$, $g \wedge S$ and $g \wedge g$.

(10) Warped products with 2-dimensional base

Further, on V we also have

$$\begin{aligned} R \cdot C &= Q(S, C) + \left(L + \frac{n-3}{(n-2)(n-1)\rho\tau_2} \right) Q(g, C) \\ &+ \frac{(n-1)\,\rho\tau_2}{(n-2)^2}\, g \wedge Q(S, S^2) \\ &+ \frac{1}{(n-2)^2}\, Q((\frac{\rho}{2} + (n-1)\,\rho\tau_1^2\tau_2)\,S - (n-1)\,\rho\tau_1\tau_2\,S^2, G), \end{aligned}$$

and

$$\begin{aligned} C \cdot R &= -\frac{1}{(n-2)^2}\, Q((\frac{\rho}{2} + (n-1)\,\rho\tau_1^2\tau_2)\,S - (n-1)\,\rho\tau_1\tau_2\,S^2, G) \\ &- \frac{(n-1)\,\rho\tau_2}{(n-2)^2}\, g \wedge Q(S, S^2) \\ &- \frac{\rho}{2(n-2)}\, Q(g, C). \end{aligned}$$

(11) Warped products with 2-dimensional base

Theorem ([DGJZ], Theorem 6.2)**.** Let $\overline{M} \times_F \widetilde{N}$ be the warped product manifold with a 2-dimensional semi-Riemannian manifold $(\overline{M}, \overline{g})$ and an $(n-2)$-dimensional semi-Riemannian manifold $(\widetilde{N}, \widetilde{g})$, $n \geq 4$, and a warping function F, and let $(\widetilde{N}, \widetilde{g})$ be an Einstein, when $n \geq 5$. On the set $V \subset \mathcal{U}_S \cap \mathcal{U}_C$ we have:

$$\begin{aligned} R \cdot S &= (\phi_1 - 2\tau_1\phi_2 + \tau_1^2\phi_3)\, Q(g, S) \\ &\quad + (\phi_2 - \tau_1\phi_3)\, Q(g, S^2) + \phi_3\, Q(S, S^2), \end{aligned}$$

$$\phi_1 = \frac{2\tau_1 - \overline{\kappa}}{2(n-2)}, \quad \phi_2 = \frac{1}{n-2}, \quad \phi_3 = \frac{\tau_2(2\kappa - \overline{\kappa} - 2(n-1)\tau_1)}{n-2}.$$

Remark. At all points of the set $\mathcal{U}_S \cap \mathcal{U}_C$, at which S_{ad} is proportional to g_{ad} and $\mathrm{rank}(A) = 2$, we have $R \cdot S = L_S\, Q(g, S)$, for some function L_S.

(1) Some 4-dimensional warped products metrics

We define on $\overline{M} = \{(t, r) \in \mathbb{R}^2 \,|\, r > 0\}$ the metric tensor $\overline{g}$ by

$$\overline{g}_{11} = -h, \;\; \overline{g}_{12} = \overline{g}_{21} = 0, \;\; \overline{g}_{22} = h^{-1}, \;\; h = h(t, r),$$

where h is a smooth positive (or negative) function on $\overline{M}$.

Let $F = F(t, r) = f^2(t, r)$ be a positive smooth function on $\overline{M}$.

Let $\overline{M} \times_F \widetilde{N}$ be the warped product of $(\overline{M}, \overline{g})$ and the 2-dimensional unit standard sphere $(\widetilde{N}, \widetilde{g})$, with the warping function F.

The warped product metric $\overline{g} \times_F \widetilde{g}$ of $\overline{M} \times_F \widetilde{N}$ is the following

$$ds^2 = -h(t, r)\, dt^2 + \frac{1}{h(t, r)}\, dr^2 + f^2(t, r)\,(d\theta^2 + \sin^2\theta\, d\phi^2). \quad (12)$$

(2) Some 4-dimensional warped products metrics

The metric (12), i.e. the metric

$$ds^2 = -h(t,r)\,dt^2 + \frac{1}{h(t,r)}\,dr^2 + f^2(t,r)\,(d\theta^2 + \sin^2\theta\,d\phi^2),$$

satisfies on the set $\mathcal{U}_S \cap \mathcal{U}_C \subset \overline{M} \times_F \widetilde{N}$ the following conditions

$$\begin{aligned}
\mathbf{R}\cdot\mathbf{R} - \mathbf{Q}(\mathbf{S},\mathbf{R}) &= \phi_1\,\mathbf{Q}(\mathbf{g},\mathbf{C}), \quad \mathbf{C}\cdot\mathbf{C} = \phi_2\,\mathbf{Q}(\mathbf{g},\mathbf{C}),\\
\mathbf{R} &= \phi_3\,\mathbf{g}\wedge\mathbf{g} + \phi_4\,\mathbf{g}\wedge\mathbf{S} + \phi_5\,\mathbf{S}\wedge\mathbf{S} + \phi_6\,\mathbf{g}\wedge\mathbf{S}^2,\\
\mathbf{S}\cdot\mathbf{R} &= \phi_7\,\mathbf{g}\wedge\mathbf{g} + \phi_8\,\mathbf{g}\wedge\mathbf{S} + \phi_9\,\mathbf{S}\wedge\mathbf{S} + \phi_{10}\,\mathbf{R},\\
\mathbf{R}\cdot\mathbf{C} + \mathbf{C}\cdot\mathbf{R} &= \mathbf{Q}(\mathbf{S},\mathbf{C}) + \phi\,\mathbf{Q}(\mathbf{g},\mathbf{C}),\\
\mathbf{C}\cdot\mathbf{S} &= \phi_{11}\,\mathbf{Q}(\mathbf{g},\mathbf{S}) + \phi_{12}\,\mathbf{Q}(\mathbf{g},\mathbf{S}^2) + \phi_{13}\,\mathbf{Q}(\mathbf{S},\mathbf{S}^2),
\end{aligned}$$

where ϕ, $\phi_1,\dots,\phi_{13}$ are some functions.

(3) Some 4-dimensional warped products metrics

Special cases of the metric (12), i.e. of the metric

$$ds^2 \;=\; -h(t,r)\,dt^2 + \frac{1}{h(t,r)}\,dr^2 + f^2(t,r)\,(d\theta^2 + \sin^2\theta\,d\phi^2)\,.$$

We assume that $f(t,r) = r > 0$.
If $h(t,r) = 1 - \frac{2\,m(t)}{r}$, $m = m(t) > 0$, then (12) reduces to the *Vaidya metric*.
If $h(t,r) = 1 - \frac{2\,m}{r} + \frac{e^2}{r^2} - \frac{\Lambda}{3}\,r^2$, $m = const. > 0$, $e = const.$, $\Lambda = const.$, then (12) reduces to the *Reissner-Nordström-de Sitter metric*.
If $h(t,r) = 1 - \frac{2\,m}{r} + \frac{e^2}{r^2}$, $m = const. > 0$, $e = const. \neq 0$, then (12) reduces to the *Reissner-Nordström metric*.
If $h(t,r) = 1 - \frac{2\,m}{r} - \frac{\Lambda}{3}\,r^2$, $m = const. > 0$, $\Lambda = const. \neq 0$, then (12) reduces to the *Kottler metric*.
If $h(t,r) = 1 - \frac{2\,m}{r}$ then (12) reduces to the *Schwarzschild metric*.

(4) The Schwarzschild and the Kottler spacetimes

- $\overline{M} \times_F \widetilde{N}$ is the Schwarzschild spacetime, if

$$h(r) = 1 - \frac{2m}{r}, \quad m = \text{const.} > 0.$$

We have: $S = 0$, $R \cdot R = L_R\, Q(g, R)$, for some function L_R, and

$$R \cdot C = C \cdot R.$$

- $\overline{M} \times_F \widetilde{N}$ is the Kottler spacetime, if

$$h(r) = 1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}, \quad m = \text{const.} > 0, \quad \Lambda = \text{const.} \neq 0;$$

We have: $S = \frac{\kappa}{4}\, g$, $R \cdot R = L_R\, Q(g, R)$, for some function L_R, and

$$R \cdot C - C \cdot R = \frac{\kappa}{12}\, Q(g, R).$$

(1) Curvature properties of some metric ([Hall], eq. (21))

We consider the metric ([Hall], eq. (21))

$$\begin{aligned} ds^2 &= dt^2 + R^2(t)\,(dr^2 + f^2(r)\,(d\theta^2 + \sin^2\theta\, d\phi^2)) \\ &= (dt^2 + R^2(t)\,dr^2) + (f(r)\,R(t))^2\,(d\theta^2 + \sin^2\theta\, d\phi^2)\,. \end{aligned} \quad (13)$$

The metric (13) satisfies the following curvature conditions

$$\begin{aligned} \mathbf{R}\cdot\mathbf{R} - \mathbf{Q}(\mathbf{S},\mathbf{R}) &= (2\,R^{''}/R)\,\mathbf{Q}(\mathbf{g},\mathbf{C})\,, \\ \mathbf{C}\cdot\mathbf{C} &= ((f^{'\,2} - f\,f^{''} - 1)/(6\,(f\,R)^2))\,\mathbf{Q}(\mathbf{g},\mathbf{C})\,, \end{aligned}$$

where $f^{'} = \frac{df}{dr}$, $f^{''} = \frac{df^{'}}{dr}$, $R^{'} = \frac{dR}{dt}$, $R^{''} = \frac{dR^{'}}{dt}$.
We also have

$$\kappa = (6\,f^2\,R\,R^{''} + 6\,(f\,R^{'})^2 + 4\,f\,f^{''} + 2\,f^{'\,2} - 2)\,(f\,R)^{-2}\,.$$

[Hall] G.S. Hall, Projective Structure in Differential Geometry, in: Proceedings of the International Conference XVI Geometrical Seminar, September, 20-25, Vrnjacka Banja, September, 20-25, 2010, Faculty of Science and Mathematics, University of Nis, Serbia, 2011, 40–45.

(2) Curvature properties of some metric ([Hall], eq. (21))

We have

$$\mathbf{R} \;=\; \frac{\phi_1}{2}\,\mathbf{g}\wedge\mathbf{g} + \phi_2\,\mathbf{g}\wedge\mathbf{S} + \phi_3\,\mathbf{S}\wedge\mathbf{S} + \phi_4\,\mathbf{g}\wedge\mathbf{S}^2\,,$$

with

$$\begin{aligned}
\phi_1 \;=\;& (-((7\,f^2\,f'' + 3\,f\,f'^{\,2} - 3\,f)\,R + 10\,R\,f^3\,R'^{\,2})\,R'' - (f'^{\,2} - 1)\,f'' \\
& -(3\,f^2\,f'' + f\,f'^{\,2} - f)\,R'^{\,2} - 6\,f^3\,R^2\,R''^{\,2} - 2\,f^3\,R'^{\,4} - f\,f''^{\,2}) \\
& /((-f^2\,f'' - f\,f'^{\,2} + f)\,R^2 + 2\,f^3\,R^3\,R'' - 2\,f^3\,R^2\,R'^{\,2})\,,\\
\phi_2 \;=\;& (8\,f^2\,R\,R'' + 4\,f^2\,R'^{\,2} + 3\,f\,f'' + f'^{\,2} - 1) \\
& /(4\,f^2\,R\,R'' - 4\,f^2\,R'^{\,2} - 2\,f\,f'' - 2\,f'^{\,2} + 2)\,,\\
\phi_3 \;=\;& \phi_4 \;=\; -(f\,R)^2/(4\,f^2\,R\,R'' - 4\,f^2\,R'^{\,2} - 2\,f\,f'' - 2\,f'^{\,2} + 2)\,,
\end{aligned}$$

where $f' = \frac{df}{dr}$, $f'' = \frac{df'}{dr}$, $R' = \frac{dR}{dt}$, $R'' = \frac{dR'}{dt}$.

[Hall] G.S. Hall, Projective Structure in Differential Geometry, in: Proceedings of the International Conference XVI Geometrical Seminar, September, 20-25, Vrnjacka Banja, September, 20-25, 2010, Faculty of Science and Mathematics, University of Nis, Serbia,

(3) Curvature properties of some metric ([Hall], eq. (21))

We have

$$\mathbf{S}\cdot\mathbf{R} = \frac{\phi_1}{2}\,\mathbf{g}\wedge\mathbf{g} + \phi_2\,\mathbf{g}\wedge\mathbf{S} + \phi_3\,\mathbf{S}\wedge\mathbf{S} + \phi_4\,\mathbf{R}\,,$$

$$\begin{aligned}
\phi_1 &= (((12\,f^3\,f'' - 6\,f^2\,f'^{\,2} + 6\,f^2)\,R + 6\,f^4\,R\,R'^{\,2})\,R''^{\,2} \\
&\quad +((-12\,f\,f'^{\,2} + 12\,f)\,f'' + (-24\,f^3\,f'' - 24\,f^2\,f'^{\,2} + 24\,f^2\,R'^{\,2} \\
&\quad -24\,f^4\,R'^{\,4} - 6\,f^2\,f''^{\,2} - 6\,f'^{\,4} + 12\,f'^{\,2} - 6)\,R'' + 18\,f^4\,R^2\,R''^{\,3}) \\
&\quad /((-f^3\,f'' - f^2\,f'^{\,2} + f^2)\,R^3 + 2\,f^4\,R^4\,R'' - 2\,f^4\,R^3\,R'^{\,2})\,, \\
\phi_2 &= (-(-10\,f^2\,R'^{\,2} - 2\,f\,f'' - 8\,f'^{\,2} + 8)\,R'' - 10\,f^2\,R\,R''^{\,2}) \\
&\quad /((-f\,f'' - f^{\,2} + 1)\,R + 2\,f^2\,R^2\,R''^2 - 2\,f^2\,R\,R'^{\,2})\,, \\
\phi_3 &= (f^2\,R\,R''^2 - f^2\,R'^{\,2} - f'^{\,2} + 1) \\
&\quad /(2\,R\,f^2\,R'' - 2\,f^2\,R'^{\,2} - f\,f'' - f'^{\,2} + 1) \\
\phi_4 &= (-12\,f^2\,R'^{\,2} - 6\,f\,f'' - 6\,f'^{\,2} + 6)/((f\,R)^2)\,.
\end{aligned}$$

(4) Curvature properties of some metric ([Hall], eq. (21))

We have

$$\mathbf{R} \cdot \mathbf{C} + \mathbf{C} \cdot \mathbf{R} \; = \; \mathbf{Q}(\mathbf{S}, \mathbf{C}) + \phi \, \mathbf{Q}(\mathbf{g}, \mathbf{C}) \, ,$$

with

$$\phi \; = \; (3 \, f^2 \, R \, R^{''} + 3 \, (f \, R^{'})^2 + f \, f^{''} - 2 \, f^{' \, 2} - 2)/(3 \, (f \, R)^2) \, .$$

[Hall] G.S. Hall, Projective Structure in Differential Geometry, in: Proceedings of the International Conference XVI Geometrical Seminar, September, 20-25, Vrnjacka Banja, September, 20-25, 2010, Faculty of Science and Mathematics, University of Nis, Serbia, 2011, 40–45.

(5) Curvature properties of some metric ([Hall], eq. (21))

We have

$$\mathbf{C} \cdot \mathbf{S} \;=\; \phi_1\, \mathbf{Q}(\mathbf{g}, \mathbf{S}) + \phi_2\, \mathbf{Q}(\mathbf{g}, \mathbf{S^2}) + \phi_3\, \mathbf{Q}(\mathbf{S}, \mathbf{S^2})\,,$$

$$\begin{aligned}
\phi_1 &= (-((4\, f^3\, f'' + 8\, f^2\, f'^{\,2} - 8\, f^2)\, R + 12\, f^4\, R\, R'^{\,2})\, R'' \\
&\quad -(6\, f\, f'^{\,2} - 6\, f)\, f'' - (14\, f^3\, f'' + 10\, f^2\, f'^{\,2} - 10\, f^2)\, R'^{\,2} \\
&\quad -3\, f^4\, R^2\, R''^{\,2} - 12\, f^4\, R'^{\,4} - 4\, f^2\, f''^{\,2} - 2\, f'^{\,4} + 4\, f'^{\,2} - 2) \\
&\quad /((-6\, f^3\, f'' - 6\, f^2\, f'^{\,2} + 6\, f^2)\, R^2 + 12\, f^4\, R^3\, R'' - 12\, f^4\, R^2\, R'^{\,2}) \\
\phi_2 &= (R\, f^2\, R'' + 2\, f^2\, R'^{\,2} + f\, f'' + f'^{\,2} - 1) \\
&\quad /(4\, f^2\, R\, R'' - 4\, f^2\, R'^{\,2} - 2\, f\, f'' - 2\, f'^{\,2} + 2) \\
\phi_3 &= -(f\, R)^2/(4\, f^2\, R\, R'' - 4\, f^2\, R'^{\,2} - 2\, f\, f'' - 2\, f'^{\,2} + 2)
\end{aligned}$$

[Hall] G.S. Hall, Projective Structure in Differential Geometry, in: Proceedings of the International Conference XVI Geometrical Seminar, September, 20-25, Vrnjacka Banja, September, 20-25, 2010, Faculty of Science and Mathematics, University of Nis, Serbia, 2011, 40–45.

(1) Curvature properties of some metric ([Hall], eq. (22))

We consider the metric ([Hall], eq. (22))

$$
\begin{aligned}
ds^2 &= (1 + e\,R^2(t))^{-2}\,dt^2 \\
&\quad +(1 + e\,R^2(t))^{-1}\,R^2(t)\,(dr^2 + f^2(r)\,(d\theta^2 + \sin^2\theta\,d\phi^2))\,, \quad (14)
\end{aligned}
$$

where $e = const.$, and its extension

$$
\begin{aligned}
ds^2 &= P(t)\,dt^2 + S(t)\,(dr^2 + f^2(r)\,(d\theta^2 + \sin^2\theta\,d\phi^2)) \\
&= (P(t)\,dt^2 + S(t)\,dr^2) + S(t)\,f^2(r)\,(d\theta^2 + \sin^2\theta\,d\phi^2))\,. \quad (15)
\end{aligned}
$$

[Hall] G.S. Hall, Projective Structure in Differential Geometry, in: Proceedings of the International Conference XVI Geometrical Seminar, September, 20-25, Vrnjacka Banja, September, 20-25, 2010, Faculty of Science and Mathematics, University of Nis, Serbia, 2011, 40–45.

(2) Curvature properties of some metric ([Hall], eq. (22))

The metric (15), i.e. the metric

$$ds^2 = P(t)\, dt^2 + S(t)\, (dr^2 + f^2(r)\, (d\theta^2 + \sin^2\theta\, d\phi^2)) .$$

satisfies the following curvature conditions

$$\begin{aligned}
\mathbf{R}\cdot\mathbf{R} - \mathbf{Q}(\mathbf{S},\mathbf{R}) &= \phi_1\, \mathbf{Q}(\mathbf{g},\mathbf{C}) ,\\
\mathbf{C}\cdot\mathbf{C} &= \phi_2\, \mathbf{Q}(\mathbf{g},\mathbf{C}) ,\\
\mathbf{R} &= \phi_3\, \mathbf{g}\wedge\mathbf{g} + \phi_4\, \mathbf{g}\wedge\mathbf{S} + \phi_5\, \mathbf{S}\wedge\mathbf{S} + \phi_6\, \mathbf{g}\wedge\mathbf{S}^2 ,\\
\mathbf{S}\cdot\mathbf{R} &= \phi_7\, \mathbf{g}\wedge\mathbf{g} + \phi_8\, \mathbf{g}\wedge\mathbf{S} + \phi_9\, \mathbf{S}\wedge\mathbf{S} + \phi_{10}\, \mathbf{R} ,\\
\mathbf{R}\cdot\mathbf{C} + \mathbf{C}\cdot\mathbf{R} &= \mathbf{Q}(\mathbf{S},\mathbf{C}) + \phi\, \mathbf{Q}(\mathbf{g},\mathbf{C}) ,\\
\mathbf{C}\cdot\mathbf{S} &= \phi_{11}\, \mathbf{Q}(\mathbf{g},\mathbf{S}) + \phi_{12}\, \mathbf{Q}(\mathbf{g},\mathbf{S}^2) + \phi_{13}\, \mathbf{Q}(\mathbf{S},\mathbf{S}^2) ,
\end{aligned}$$

where ϕ, $\phi_1, \ldots, \phi_{13}$ are some functions.

(1) The condition: $(*)$ $R \cdot R - Q(S,R) = L\, Q(g,C)$

Theorem ([DDP]).
Let $(\widetilde{N}, \widetilde{g})$ be a semi-Riemannian manifold,
$\overline{M} = (a;b) \subset \mathbb{R}$, $a < b$, $\overline{g}_{11} = \varepsilon = \pm 1$,
$F : (a;b) \to \mathbb{R}_+$ a smooth function,
$F'' = \frac{dF'}{dt}$, $F' = \frac{dF}{dt}$, $t \in (a;b)$.

(i) Then the warped product $\overline{M} \times_F \widetilde{N}$, $\dim \widetilde{N} = 3$, satisfies $(*)$ with

$$L = \frac{\varepsilon}{F} \left(F'' - \frac{(F')^2}{2\,F} \right) .$$

[DDP] F. Defever, R. Deszcz and M. Prvanović, On warped product manifolds satisfying some curvature condition of pseudosymmetry type, Bull. Greek Math. Soc., 36 (1994), 43–67.

(2) The condition: $(*)$ $R \cdot R - Q(S, R) = L\, Q(g, C)$

(ii) ([DDP]) Let $(\widetilde{N}, \widetilde{g})$, $\dim \widetilde{N} = n - 1 \geq 4$, be a manifold satisfying

$$\widetilde{R} \cdot \widetilde{R} - Q(\widetilde{S}, \widetilde{R}) = -(n-3)k\, Q(\widetilde{g}, \widetilde{C}), \quad k = const. \tag{16}$$

Then the manifold $\overline{M} \times_F \widetilde{N}$ satisfies $(*)$ with $L = \frac{(n-2)\varepsilon}{2F}\left(F'' - \frac{(F')^2}{2F}\right)$ and F satisfying

$$F F'' - (F')^2 + 2\varepsilon k F = 0. \tag{17}$$

Remark. (i) ([DV]) On every hypersurface $\widetilde{N}$ immersed isometrically in a semi-Riemannian space of constant curvature $N_s^n(c)$, $n - 1 \geq 4$, the condition (16) is satisfied with $k = c = \frac{\tau}{(n-1)n}$, where τ is the scalar curvature of the ambient space.

[DDP] F. Defever, R. Deszcz and M. Prvanović, On warped product manifolds satisfying some curvature condition of pseudosymmetry type, Bull. Greek Math. Soc., 36(1994),43–67.

[DV] R. Deszcz and L. Verstraelen, Hypersurfaces of semi-Riemannian conformally flat manifolds, in: Geometry and Topology of Submanifolds, III, World Sci., River Edge, NJ, 1991, 131–147.

(3) The condition: $(*)\ \ R \cdot R - Q(S,R) \ = \ L\, Q(g,C)$

(ii) ([DSch]) The following functions

$$F(t) \ = \ \varepsilon k \left(t + \frac{\varepsilon l}{k} \right)^2 , \qquad \varepsilon k > 0 \, ,$$

$$F(t) \ = \ \frac{l}{2} \left(\exp\left(\pm \frac{m}{2} t \right) - \frac{2 \varepsilon k}{l\, m^2} \exp\left(\mp \frac{m}{2} t \right) \right)^2 , \qquad l > 0 \, , \quad m \neq 0 \, ,$$

$$F(t) \ = \ \frac{2 \varepsilon k}{l^2} \left(1 + \sin(l\, t + m) \right) , \qquad \varepsilon k > 0 \, , \quad l \neq 0 \, ,$$

where k, l and m are constants and $t \in (a;b)$,
are solutions of (17), i.e. of the equation

$$F\, F^{''} - (F^{'})^2 + 2 \varepsilon k\, F \ = \ 0 \, .$$

[DSch] R. Deszcz and M. Scherfner, On a particular class of warped products with fibres locally isometric to generalized Cartan hypersurfaces, Colloquium Math. 109 (2007), 13–29.

(1) The condition: $(**)$ $R \cdot R = L_R \, Q(g, R)$

Theorem.
Let $(\widetilde{N}, \widetilde{g})$ be a semi-Riemannian manifold, $\overline{M} = (a; b)$, $a < b$, $\overline{g}_{11} = \varepsilon = \pm 1$, $F : (a; b) \longrightarrow \mathbb{R}_+$ a smooth function, $F^{''} = \frac{dF^{'}}{dt}$, $F^{'} = \frac{dF}{dt}$, $t \in (a; b)$.

(i) ([DDV]) If $(\widetilde{N}, \widetilde{g})$, $\dim \widetilde{N} = n - 1 \geq 3$, is a semi-Riemannian space of constant curvature then the warped product $\overline{M} \times_F \widetilde{N}$, is a conformally flat manifold satisfying $(**)$ with $L_R = -\varepsilon \left(\frac{F^{''}}{2F} - \frac{(F^{'})^2}{4\, F^2} \right)$. Moreover,

$$\operatorname{rank} \left(S - \left(\frac{\kappa}{n-1} - L_R \right) g \right) = 1 \, .$$

[DDV] J. Deprez, R. Deszcz and L. Verstraelen, Examples of pseudosymmetric conformally flat warped products, Chinese J. Math., 17 (1989), 51–65.

(2) The condition: $(**)\ \ R \cdot R \ = \ L_R\, Q(g, R)$

(ii) ([DSch]) Let $(\widetilde{N}, \widetilde{g})$, $\dim \widetilde{N} = n - 1 \geq 3$, be a manifold satisfying

$$\widetilde{R} \cdot \widetilde{R} \ = \ k\, Q(\widetilde{g}, \widetilde{R})\,, \quad k \ = \ const\,. \tag{18}$$

The warped product $\overline{M} \times_F \widetilde{N}$ satisfies $(**)$ with $L_R \ = \varepsilon \left(\frac{(F^{'})^2}{4\,F^2} - \frac{F^{''}}{2F} \right)$ and the function F satisfying

$$F\, F^{''} - (F^{'})^2 + 2\varepsilon k\, F \ = \ 0\,.$$

Remark. ([DVY]) On 3-dimensional Cartan hypersurface the condition (18), with $k \ = \ \frac{\widetilde{\kappa}}{12}$, where $\widetilde{\kappa}$ is the scalar curvature of the ambient space.

[DSch] R. Deszcz and M. Scherfner, On a particular class of warped products with fibres locally isometric to generalized Cartan hypersurfaces, Colloquium Math. 109 (2007), 13–29.

[DVY] R. Deszcz, L. Verstraelen and S. Yaprak, Pseudosymmetric hypersurfaces in 4-dimensional spaces of constant curvature, Bull. Inst. Math. Acad. Sinica 22 (1994), 167–179.

(1) The condition: $(***)$ $R \cdot S = L_S\, Q(g, S)$

Theorem. Let $(\widetilde{N}, \widetilde{g})$ a semi-Riemannian manifold, $\overline{M} = (a; b)$, $a < b$, $\overline{g}_{11} = \varepsilon = \pm 1$, $F : (a; b) \to \mathbb{R}_+$ a smooth function, $F'' = \frac{dF'}{dt}$, $F' = \frac{dF}{dt}$, $t \in (a; b)$.

(i) ([DH]) If $(\widetilde{N}, \widetilde{g})$, $\dim \widetilde{N} = n - 1 \geq 3$, is a semi-Riemannian Einsteinian manifold then the warped product $\overline{M} \times_F \widetilde{N}$, is a manifold satisfying $(***)$ with $L_S = \varepsilon \left(\frac{(F')^2}{4\,F^2} - \frac{F''}{2F} \right)$. Moreover, we have ([Ch-DDGP])

$$\begin{aligned} \operatorname{rank} \left(S - \left(\frac{\kappa}{n-1} - L_S \right) g \right) &= 1 \,, \\ (n-2)\,(R \cdot C - C \cdot R) &= Q(S - L_S\, g, R) \,. \end{aligned}$$

[DH] R. Deszcz and M. Hotloś, Remarks on Riemannian manifolds satisfying certain curvature condition imposed on the Ricci tensor, Pr. Nauk. Pol. Szczec., 11 (1988), 23–34.

[Ch-DDGP] J. Chojnacka-Dulas, R. Deszcz, M. Głogowska and M. Prvanović, On warped products manifolds satisfying some curvature conditions, J. Geom. Physics, 74 (2013), 328–341.

(1) Remark 1 ([SDHJK]). The Schwarzschild spacetime

It seems that the Schwarzschild spacetime, the Kottler spacetime, the Reissner-Nordström spacetime, as well as some Friedmann-Lemaître -Robertson-Walker spacetimes (FLRW spacetimes) are the "oldest" examples of non-semisymmetric pseudosymmetric warped product manifolds (cf. [DHV], [HV]). The Schwarzschild spacetime was discovered in 1916 by Schwarzschild and independently by Droste during their study on solutions of Einstein's equations (see, e.g., [P]).

[DHV] R. Deszcz, S. Haesen and L. Verstraelen, On natural symmetries, in: Topics in Differential Geometry, Eds. A. Mihai, I. Mihai and R. Miron, Editura Academiei Române, 2008.

[HV] S. Haesen and L. Verstraelen, Natural intrinsic geometrical symmetries, Symmetry, Integrability Geom. Methods Appl. 5 (2009), 086, 14 pp.

[P] V. Perlick, Gravitational Lensing from a Spacetime Perspective, Living Rev. Relativity 7 (2004), 9. doi: 10.12942/lrr-2004-9. http://www.livingreviews.org/lrr-2004-9.

[SDHJK] A.A. Shaikh, R. Deszcz, M. Hotloś, J. Jełowicki, and H. Kundu, On pseudosymmetric manifolds, Publ. Math. Debrecen 86 (2015), 433–456.

(2) Remark 2 ([SDHJK]). Pseudosymmetric manifolds

We note that [DG] is the first paper, in which manifolds satisfying

$$R \cdot R \;=\; L_R\, Q(g, R)$$

were called pseudosymmetric manifolds.
We also mention that in [WG] it was proved that fibers of semisymmetric warped products are pseudosymmetric (cf. [HV], Section 7).

[DG] R. Deszcz and W. Grycak, On some class of warped product manifolds, Bull. Inst. Math. Acad. Sinica 15 (1987), 311–322.

[WG] W. Grycak, On semi-decomposable 2-recurrent Riemannian spaces, Sci. Papers Inst. Math. Wrocław Techn. Univ. 16 (1976), 15–25.

[HV] S. Haesen and L. Verstraelen, Natural intrinsic geometrical symmetries, Symmetry, Integrability Geom. Methods Appl. 5 (2009), 086, 14 pp.

Remark ([Saw]). According to [H] and [Saw] the generalized curvature tensor B on M satisfies the *Ricci-type equation* if on M we have

$$R \cdot B \;=\; B \cdot B, \quad \text{or} \quad C \cdot B \;=\; B \cdot B.$$

If either $B = C$ or $B = R - C$ or $B = R$ or $B = C - R$ satisfies the Ricci-type equation then ([Saw])

$$\begin{aligned} R \cdot C &= C \cdot C, \\ C \cdot R &= C \cdot C, \\ R \cdot C &= R \cdot R, \\ C \cdot R &= R \cdot R, \end{aligned} \tag{19}$$

respectively.
Hypersurfaces in a semi-Riemannian space of constant curvature $N_s^{n+1}(c)$, $n \geq 4$, satisfying Ricci-type equations (19) were investigated in [Saw].

[H] B.M. Haddow, Characterization of Rieman tensors using Ricci-type equations, J. Math. Phys. 35 (1994), 3587–3593.

[Saw] K. Sawicz, Hypersurfaces in spaces of constant curvature satisfying some Ricci-type equations, Colloquium Math. 101 (2004), 183–201.

(1) Some extension of the Gödel metric

Example.

(i) We define the metric g on $M = \{(t, r, \phi, z) : t > 0, r > 0\} \subset \mathbb{R}^4$ by (cf. [RT], Section 1)

$$ds^2 = (dt + H(r)\, d\phi)^2 - D^2(r)\, d\phi^2 - dr^2 - dz^2, \qquad (20)$$

where H and D are certain functions on M. If

$$H(r) = \frac{2\sqrt{2}}{m} sinh^2(\frac{mr}{2})$$

and

$$D(r) = \frac{2}{m} sinh(\frac{mr}{2})\, cosh(\frac{mr}{2})$$

then g is the Gödel metric (e.g. see [RT], eq. (1.6)).

[RT] M.J. Rebouças and J. Tiomno, Homogeneity of Riemannian space-times of Gödel type, Phys. Rev. D, 28 (1983), 1251–1264.

(2) Some extension of the Gödel metric

(ii) ([DHJKS]) The metric g defined by (20) is the product metric of a 3-dimensional metric and a 1-dimensional metric. Thus $R \cdot R = Q(S, R)$ on M. The Riemann-Christoffel curvature tensor R of (M, g) is expressed by a linear combination of the Kulkarni-Nomizu products formed by S and S^2, i.e. by the tensors $S \wedge S$, $S \wedge S^2$ and $S^2 \wedge S^2$,

$$\begin{aligned} R &= \phi_1\, S \wedge S + \phi_2\, S \wedge S^2 + \phi_3\, S^2 \wedge S^2, \\ \phi_1 &= \frac{D^2}{\tau}(2D^2 H''^2 - 4DD'H'H'' - 3H'^4 + 2H'^2(4DD'' + D'^2) - 8D^2 D''^2), \\ \phi_2 &= \frac{2D^4}{\tau}(H'^2 - 4DD''), \quad \phi_3 = -\frac{4D^6}{\tau}, \quad H' = \frac{dH}{dr}, \quad H'' = \frac{dH'}{dr}, \\ \tau &= (H'^2 - 2DD'')(D^2 H''^2 - 2DD'H'H'' - H'^4 + 2DD''H'^2 + D'^2 H'^2), \end{aligned}$$

provided that the function τ is non-zero at every point of M.

(3) Some extension of the Gödel metric

(iii) If $H(r) = ar^2$, $a = const. \neq 0$ and $D(r) = r$ then (20) turns into ([RT], eq. (3.20))

$$ds^2 = (dt + ar^2\, d\phi)^2 - r^2\, d\phi^2 - dr^2 - dz^2. \tag{21}$$

The spacetime (M, g) with the metric g defined by (21) is called the Som-Raychaudhuri solution of the Einstein field equations [SR]. For the metric (21) the function τ is non-zero at every point of M.

[SR] M.M. Som and A.K. Raychaudhuri, Cylindrically symmetric charged dust distributions in rigid rotation in General Relativity, Proc. R. Soc. London A, 304, 1476, 81 (1968), 81–86.

(1) Some extension of the Roter type equation

Example.

We define on $M = \{(x, y, z, t) \,:\, x > 0, y > 0, z > 0, t > 0\} \subset \mathbb{R}^4$ the metric g by ([DK])

$$ds^2 \;=\; \exp(y)\, dx^2 + (x\, z)^2\, dy^2 + dz^2 - dt^2. \qquad (22)$$

We have on M ([DGJP-TZ]):

$$\mathrm{rank}\,(S) = \ldots = \mathrm{rank}\,(S^4) = 3, \quad \kappa = 1/(2\, x^2\, z^2),$$

$$\omega(X)\,\mathcal{R}(Y, Z) + \omega(Y)\,\mathcal{R}(Z, X) + \omega(Z)\,\mathcal{R}(X, Y) \;=\; 0,$$

$$R \cdot R \;=\; Q(S, R),$$

where the 1-form ω is defined by $\omega(\partial_x) = \omega(\partial_y) = 1$, $\omega(\partial_z) = \omega(\partial_t) = 0$.

[DK] P. Debnath and A. Konar, On super quasi-Einstein manifold, Publ. Inst. Math. (Beograd) (N.S.) 89(103) (2011), 95–104.

[DGJP-TZ] R. Deszcz, M. Głogowska, J. Jełowicki, M. Petrović-Torgasev, and G. Zafindratafa, On curvature and Weyl compatible tensors, Publ. Inst. Math. (Beograd) (N.S.) 94(108) (2013), 111–124.

(2) Some extension of the Roter type equation

Moreover, for the metric (22) we have on M ([DGJP-TZ]):

$$\begin{aligned}
R &= \phi_1\, S \wedge S + \phi_2\, S \wedge S^2 + \phi_3\, S^2 \wedge S^2,\\
\phi_1 &= (16\, x^2\, z^4 + z^2\, (4\, x^2 + 1)\, \exp(y))/(8\, z^2 + 2\, \exp(y)),\\
\phi_2 &= -4\, x^2\, z^4\, \exp(y)/(4\, z^2 + \exp(y)),\\
\phi_3 &= 8\, x^4\, z^6\, \exp(y)/(4\, z^2 + \exp(y)),\\
Q(S, S^2 \wedge S^2) &= Q(S^3 - \exp(y)/(2\, xz^2)\, S^2, S \wedge S),
\end{aligned}$$

and

$$R(\mathcal{S}^p X, Y, Z, W) + R(\mathcal{S}^p Z, Y, W, X) + R(\mathcal{S}^p W, Y, X, Z) = 0, \;\; p \geq 1.$$

[DGJP-TZ] R. Deszcz, M. Głogowska, J. Jełowicki, M. Petrović-Torgasev, and G. Zafindratafa, On curvature and Weyl compatible tensors, Publ. Inst. Math. (Beograd) (N.S.) 94(108) (2013), 111–124.

Pseudosymmetry

Let (M, g), $n \geq 3$, be a Riemannian manifold.
We assume that the set $\mathcal{U}_R \subset M$ is non-empty and let $p \in \mathcal{U}_R$.
Let $\pi = u \wedge v$ and $\overline{\pi} = x \wedge y$ be planes of T_pM, where $u, v \in T_pM$ form an orthonormal basis of π and $x, y \in T_pM$ form an orthonormal basis of $\overline{\pi}$. The plane π is said to be curvature-dependent with respect to the plane $\overline{\pi}$ ([HV], Definition 2) if $Q(g, R)(u, v, v, u; x, y) \neq 0$. According to [HV](Definition 3), we define at p the *sectional curvature of Deszcz* $L_R(p, \pi, \overline{\pi})$ of the plane π with respect to the plane $\overline{\pi}$ by

$$L_R(p, \pi, \overline{\pi}) \quad = \quad \frac{(R \cdot R)(u, v, v, u; x, y)}{Q(g, R)(u, v, v, u; x, y)} .$$

In [HV](Theorem 3) it was proved that a Riemannian manifold (M, g), $n \geq 3$, is pseudosymmetric if and only if all the double sectional curvatures $L_R(p, \pi, \overline{\pi})$ are the same at every point $p \in \mathcal{U}_R \subset M$, i.e. for all curvature-dependent planes π and $\overline{\pi}$ at p, $L_R(p, \pi, \overline{\pi}) = L_R(p)$ for some function L_R on $\mathcal{U}_R$.

Ricci-pseudosymmetry

Let (M, g), $n \geq 3$, be a Riemannian manifold.
We assume that the set $\mathcal{U}_S \subset M$ is non-empty and let $p \in \mathcal{U}_S$. A direction d, spanned by a vector $v \in T_pM$, is said to be curvature dependent on a plane $\overline{\pi} = x \wedge y \subset T_pM$ if $Q(g, S)(v, v; x, y) \neq 0$, where $x, y \in T_pM$ form an orthonormal basis of $\overline{\pi}$. According to [JHSV] (Definition 6), we define at p the *Ricci curvature of Deszcz* $L_S(p, d, \overline{\pi})$ of the curvature-dependent direction d and the plane $\overline{\pi}$ by

$$L_S(p, d, \overline{\pi}) = \frac{(R \cdot S)(v, v; x, y)}{Q(g, S)(v, v; x, y)} .$$

In [JHSV](Theorem 10) it was stated that a Riemannian manifold (M, g), $n \geq 3$, is Ricci-pseudosymmetric if and only if all the Ricci curvatures of Deszcz are the same at every point $p \in \mathcal{U}_S \subset M$, i.e. for all curvature-dependent directions d with respect to planes $\overline{\pi}$ we have $L_S(p, d, \overline{\pi}) = L_S(p)$ for some function L_S on $\mathcal{U}_S$.

References; pseudosymmetry, Ricci-pseudosymmetry, Weyl-pseudosymmetry

[DGHS] R. Deszcz, M. Głogowska, M. Hotloś, and K. Sawicz, A Survey on Generalized Einstein Metric Conditions, Advances in Lorentzian Geometry: Proceedings of the Lorentzian Geometry Conference in Berlin, AMS/IP Studies in Advanced Mathematics 49, S.-T. Yau (series ed.), M. Plaue, A.D. Rendall and M. Scherfner (eds.), 2011, 27–46.

[HV1] S. Haesen and L. Verstraelen, Properties of a scalar curvature invariant depending on two planes, Manuscripta Math. 122 (2007), 59–72.

[HV2] S. Haesen and L. Verstraelen, Natural intrinsic geometrical symmetries, Symmetry, Integrability Geom. Methods Appl. 5 (2009), 086, 14 pp.

[JHP-TV] B. Jahanara, S. Haesen, M. Petrović-Torgasev and L. Verstraelen, On the Weyl curvature of Deszcz, Publ. Math. Debrecen 74 (2009), 417–431.

[JHSV] B. Jahanara, S. Haesen, Z. Sentürk and L. Verstraelen, On the parallel transport of the Ricci curvatures, J. Geom. Physics 57 (2007), 1771–1777.

[DHV] R. Deszcz, S. Haesen and L. Verstraelen, On natural symmetries, in: Topics in Differential Geometry, Eds. A. Mihai, I. Mihai and R. Miron, Ed. Academiei Române, 2008.

[SDHJK] A.A. Shaikh, R. Deszcz, M. Hotloś, J. Jełowicki, and H. Kundu, On pseudosym-

Pseudosymmetric manifolds

A semi-Riemannian manifold (M, g), $n \geq 3$, is said to be *pseudosymmetric* if at every point of M the tensors $R \cdot R$ and $Q(g, R)$ are linearly dependent.

The manifold (M, g) is pseudosymmetric if and only if

$$R \cdot R \;=\; L_R\, Q(g, R) \tag{23}$$

holds on $\mathcal{U}_R$, where L_R is some function on this set.

Every *semisymmetric* manifold ($R \cdot R = 0$) is pseudosymmetric.
The converse statement is not true.

Pseudosymmetric manifolds of constant type

According to [BKV], a pseudosymmetric manifold (M, g), $n \geq 3$, $(R \cdot R = L_R\, Q(g, R))$ is said to be *pseudosymmetric space of constant type* if the function L_R is constant on $\mathcal{U}_R \subset M$.

Theorem (cf. [D]). Every type number two hypersurface M isometrically immersed in a semi-Riemannian space of constant curvature $N_s^{n+1}(c)$, $n \geq 3$, is a pseudosymmetric space of constant type. Precisely,

$$R \cdot R \;=\; \frac{\widetilde{\kappa}}{n(n+1)}\, Q(g, R),$$

holds on $\mathcal{U}_R \subset M$, where $\widetilde{\kappa}$ is the scalar curvature of the ambient space.

[BKV] E. Boeckx, O. Kowalski, L. Vanhecke, Riemannian manifolds of Conullity Two, World Sci., Singapore.

[D] F. Defever, R. Deszcz, P. Dhooghe, L. Verstraelen and S. Yaprak, On Ricci-pseudo -symmetric hypersurfaces in spaces of constant curvature, Results in Math. 27 (1995), 227–236.

Ricci-pseudosymmetric manifolds

A semi-Riemannian manifold (M, g), $n \geq 3$, is said to be *Ricci-pseudosymmetric* if at every point of M the tensors $R \cdot S$ and $Q(g, S)$ are linearly dependent.

The manifold (M, g) is Ricci-pseudosymmetric if and only if

$$R \cdot S = L_S \, Q(g, S) \qquad (24)$$

holds on $\mathcal{U}_S$, where L_S is some function on this set.

Every *Ricci-semisymmetric* manifold $(R \cdot S = 0)$ is Ricci-pseudosymmetric. The converse statement is not true.

(1) Ricci-pseudosymmetric manifolds of constant type

According to [G], a Ricci-pseudosymmetric manifold (M, g), $n \geq 3$, $(R \cdot S = L_S \, Q(g, S))$ is said to be *Ricci-pseudosymmetric manifold of constant type* if the function L_S is constant on $\mathcal{U}_S \subset M$.

[G] M. Głogowska, Curvature conditions on hypersurfaces with two distinct principal curvatures, in: Banach Center Publ. 69, Inst. Math. Polish Acad. Sci., 2005, 133–143.

(2) Ricci-pseudosymmetric manifolds of constant type

Theorem (cf. [DY]). If M is a hypersurface in a Riemannian space of constant curvature $N^{n+1}(c)$, $n \geq 3$, such that at every point of M there are principal curvatures $0, \ldots, 0, \lambda, \ldots, \lambda, -\lambda, \ldots, -\lambda$, with the same multiplicity of λ and $-\lambda$, and λ is a positive function on M, then M is a Ricci-pseudosymmetric manifold of constant type. Precisely,

$$R \cdot S \;=\; \frac{\widetilde{\kappa}}{n(n+1)}\, Q(g, S)$$

holds on M. In particular, every Cartan hypersurface is a Ricci-pseudosymmetric manifold of constant type.

[DY] R. Deszcz and S. Yaprak, Curvature properties of Cartan hypersurfaces, Colloquium Math. 67 (1994), 91–98.

(1) Weyl-pseudosymmetric manifolds

A semi-Riemannian manifold (M, g), $n \geq 4$, is said to be *Weyl-pseudosymmetric* if at every point of M the tensors $R \cdot C$ and $Q(g, C)$ are linearly dependent.

The manifold (M, g) is Weyl-pseudosymmetric if and only if

$$R \cdot C \;=\; L_C \, Q(g, C)$$

holds on $\mathcal{U}_C$, where L_C is some function on this set.

(2) Weyl-pseudosymmetric manifolds

Every pseudosymmetric manifold ($R \cdot R = L_R\, Q(g, R)$)
is Weyl-pseudosymmetric ($R \cdot C = L_R\, Q(g, C)$).
In particular, every semisymmetric manifold ($R \cdot R = 0$)
is Weyl-semisymmetric ($R \cdot C = 0$).

If $\dim M \geq 5$ the converse statement are true. Precisely, if
$R \cdot C = L_C\, Q(g, C)$, resp. $R \cdot C = 0$, is satisfied on $\mathcal{U}_C \subset M$,
then
$R \cdot R = L_C\, Q(g, R)$, resp. $R \cdot R = 0$, holds on $\mathcal{U}_C$.

(3) Weyl-pseudosymmetric manifolds

An example of a 4-dimensional Riemannian manifold satisfying $R \cdot C = 0$ with non-zero tensor $R \cdot R$ was found by A. Derdziński ([D]).

An example of a 4-dimensional submanifold in a 6-dimensional Euclidean space $\mathbb{E}^6$ satisfying $R \cdot C = 0$ with non-zero tensor $R \cdot R$ was found by G. Zafindratafa ([Z]).

[D] A. Derdziński, Examples de métriques de Kaehler et d'Einstein autoduales sur le plan complexe, in: Géométrie riemannianne en dimension 4 (Seminaire Arthur Besse 1978/79), Cedic/Fernand Nathan, Paris 1981, 334–346.

[Z] G. Zafindratafa, Sous-variétés soumises à des conditions de courbure, Thèse principale de Doctorat Légal en Sciences, Faculteit Wetenschappen, Katholieke Universiteit Leuven, Belgium, 1991.

[G] W. Grycak, Riemannian manifolds with a symmetry condition imposed on the 2-nd derivative of the conformal curvature tensor, Tensor (N.S.) 46 (1987), 287–290.

(4) Weyl-pseudosymmetric manifolds

For further results on 4-dimensional semi-Riemannian manifolds satisfying $R \cdot C = 0$ or $R \cdot C = L\, Q(g, C)$ we refer to the following papers:

[DG] R. Deszcz and W. Grycak, On manifolds satisfying some curvature conditions, Colloquium Math. 57 (1989), 89–92.

[D1] R. Deszcz, Examples of four-dimensional Riemannian manifolds satisfying some pseudosymmetry curvature conditions, in: Geometry and Topology of Submanifolds, II, World Sci., Teaneck, NJ, 1990, 134–143.

[D2] R. Deszcz, On four-dimensional warped product manifolds satisfying certain pseudosymmetry curvature conditions, Colloquium Math. 62 (1991), 103–120.

[DY] R. Deszcz and S. Yaprak, Curvature properties of certain pseudosymmetric manifolds, Publ. Math. Debrecen 45 (1994), 333–345.

[DH] R. Deszcz and M. Hotloś, On a certain extension of class of semisymmetric manifolds, Publ. Inst. Math. (Beograd) (N.S.) 63 (77) (1998), 115–130.

(1) Relations between some classes of manifolds

Inclusions between mentioned classes of manifolds can be presented in the following diagram ([DGHS]).
We mention that all inclusions are strict, provided that $n \geq 4$.

[DGHS] R. Deszcz, M. Głogowska, M. Hotloś, and K. Sawicz, A Survey on Generalized Einstein Metric Conditions, Advances in Lorentzian Geometry: Proceedings of the Lorentzian Geometry Conference in Berlin, AMS/IP Studies in Advanced Mathematics 49, S.-T. Yau (series ed.), M. Plaue, A.D. Rendall and M. Scherfner (eds.), 2011, 27–46.

(2) Relations between some classes of manifolds, $n \geq 4$

$R \cdot S = L_S Q(g, S)$	$\supset$	$R \cdot R = L_R Q(g, R)$	$\subset$	$R \cdot C = L_C Q(g, C)$
$\cup$		$\cup$		$\cup$
$R \cdot S = 0$	$\supset$	$R \cdot R = 0$	$\subset$	$R \cdot C = 0$
$\cup$		$\cup$		$\cup$
$\nabla S = 0$	$\supset$	$\nabla R = 0$	$\subset$	$\nabla C = 0$
$\cup$		$\cup$		$\cup$
$S = \frac{\kappa}{n} g$	$\supset$	$R = \frac{\kappa}{(n-1)n} G$	$\subset$	$C = 0$

(3) Relations between some classes of manifolds; References

[D] R. Deszcz, On pseudosymmetric spaces, Bull. Soc. Math. Belg. 44 (1992), Ser. A, Fasc. 1, 1–34.

[BDGHKV] M. Belkhelfa, R. Deszcz, M. Głogowska, M. Hotloś, D. Kowalczyk, and L. Verstraelen, On some type of curvature conditions, in: Banach Center Publ. 57, Inst. Math. Polish Acad. Sci., 2002, 179–194.

[DGHV] R. Deszcz, M. Głogowska, M. Hotloś, and K. Sawicz, A Survey on Generalized Einstein Metric Conditions, in: Advances in Lorentzian Geometry: Proceedings of the Lorentzian Geometry Conference in Berlin, AMS/IP Studies in Advanced Mathematics 49, S.-T. Yau (series ed.), M. Plaue, A.D. Rendall and M. Scherfner (eds.), 2011, 27–46.

[SDHJK] A.A. Shaikh, R. Deszcz, M. Hotloś, J. Jełowicki, and H. Kundu, On pseudosymmetric manifolds, Publ. Math. Debrecen 86 (2015), 433–456.

(4) Relations between some classes of manifolds, $n \geq 4$

We also have

$C \cdot S = L_S Q(g, S)$	$\supset$	$C \cdot R = L_R Q(g, R)$	$\subset$	$C \cdot C = L_C Q(g, C)$
$\cup$		$\cup$		$\cup$
$C \cdot S = 0$	$\supset$	$C \cdot R = 0$	$\subset$	$C \cdot C = 0$
$\cup$		$\cup$		$\cup$
$S = \frac{\kappa}{n} g$	$\supset$	$R = \frac{\kappa}{(n-1)n} G$	$\subset$	$C = 0$

(5) Relations between some classes of manifolds, $n \geq 4$

Remark ([MADEO]). Let (M, g), $n \geq 4$, be a semi-Riemannian manifold satisfying $C \cdot R = L\, Q(g, R)$ on $\mathcal{U}_C \subset M$. From this we get on $\mathcal{U}_C$ $C \cdot S = L\, Q(g, S)$. Further, we have

$$
\begin{aligned}
C \cdot C &= C \cdot \left(R - \frac{1}{n-2}\, g \wedge S + \frac{\kappa}{(n-2)(n-1)}\, G \right) \\
&= C \cdot R - \frac{1}{n-2}\, g \wedge (C \cdot S) + \frac{\kappa}{(n-2)(n-1)}\, C \cdot G \\
&= L\, Q(g, R) - \frac{L}{n-2}\, g \wedge Q(g, S) \\
&= L\, Q(g, R) - \frac{L}{n-2}\, Q(g, g \wedge S) \;=\; L\, Q(g, R - \frac{1}{n-2}\, g \wedge S) \\
&= L\, Q(g, R - \frac{1}{n-2}\, g \wedge S + \frac{\kappa}{(n-2)(n-1)}\, G) \;=\; L\, Q(g, C).
\end{aligned}
$$

[MADEO] C. Murathan, K. Arslan, R. Deszcz, R. Ezentas and C. Özgür, On a certain class of hypersurfaces of semi-Euclidean spaces, Publ. Math. Debrecen 58 (2001), 587–604.