Some classification results on biconservative hypersurfaces in pseudo-Euclidean spaces

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Acknowledgements

In this talk, we would like to present the results obtained in the following recent papers:

- **A. Upadhyay** and NcT, J. Math. Anal. Appl. (DOI:10.1016/j.jmaa.2016.07.053)
- **Y. Fu** and NcT, Int. J. Math. (2016).
- **F. Manfio**, NcT and **A. Upadhyay**, *On biconservative submanfolds in* $\mathbb{S}^n \times R$ (pre-print)

114F199

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Section 1:

Biharmonic Submanifolds

Biharmonic Mappings

Let $\phi : M \longrightarrow N$ be a mapping between (M^n, g) and $(N^m, \langle , \rangle)$ and $\tau(\phi) = \mathrm{trace}\nabla \mathrm{d}\phi$ the tension field of ϕ.

Biharmonic Mappings

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Biharmonic mappings

If ϕ is a critical point of the bienergy functional given by

$$E_2(\phi) = \frac{1}{2}\int_M |\tau(\phi)|^2 v_g,$$

then it is said to be a a biharmonic map.
If, in particular $\phi = f$ is an isometric immersion, then M is called a biharmonic submanifold of N^m.

Biconservative Mappings

For a biharmonic map, the bitension field τ_2 satisfies the following associated Euler-Lagrange equation

$$\tau_2(\phi) = -\Delta\tau(\phi) - \mathrm{trace}\widetilde{R}(d\phi, \tau(\phi))d\phi = 0,$$

where $\widetilde{R}$ is the curvature tensor of N.

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where $\widetilde{R}$ is the curvature tensor of N.

Biconservative mappings

Let $\phi : M \longrightarrow N$ be an isometric immersion satisfying

$$\langle \tau_2(\phi), d\phi \rangle = 0,$$

then ϕ is said to be a biconservative mapping.
If, in particular $\phi = f$ is an isometric immersion, then M is called a biconservative submanifold of N^m.

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Proposition

Let $f : M^m \to N^n$ be an isometric immersion between two Riemannian manifolds. Then f is biharmonic if and only if the tangent and normal components of $\tau_2(f)$ vanish, i.e.,

$$m\nabla(H^2) + 4\mathrm{tr}A_{\nabla^\perp_\cdot \mathbf{H}}(\cdot) + 4\mathrm{tr}\big(\tilde{R}(\cdot, \mathbf{H})\cdot\big)^T = 0$$

and

$$\mathrm{tr}\alpha_f(A_{\mathbf{H}}(\cdot), \cdot) - \Delta^\perp \mathbf{H} + 2\mathrm{tr}\big(\tilde{R}(\cdot, \mathbf{H})\cdot\big)^\perp = 0.$$

Equation of Biconservativity

It follows that an isometric immersion is biconservative if and only if

$$m\nabla(H^2) + 4\mathrm{tr}A_{\nabla^\perp_\cdot \mathbf{H}}(\cdot) + 4\mathrm{tr}\big(\tilde{R}(\cdot, \mathbf{H})\cdot\big)^T = 0.$$

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- If N is a Riemannian space form $R^n(c)$, then

$$m\nabla(H^2) + 4\mathrm{tr}A_{\nabla^\perp_\cdot \mathbf{H}}(\cdot) = 0.$$

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- If N is a Riemannian space form $R^n(c)$, then

$$m\nabla(H^2) + 4\mathrm{tr}A_{\nabla^\perp_\cdot \mathbf{H}}(\cdot) = 0.$$

  - If the codimension is 1, i.e., M is a hypersurface, then

$$S(\nabla H) = -\varepsilon\frac{nH}{2}\nabla H.$$

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  - If the codimension is 1, i.e., M is a hypersurface, then

$$S(\nabla H) = -\varepsilon\frac{nH}{2}\nabla H.$$

- If $N = \mathbb{S}^n \times R$, then the equation of biconservative becomes

$$\langle \mathbf{H}, \eta\rangle T = 0.$$

Section 2:

Biconservative hypersurfaces in pseudo-Euclidean spaces

Section 2.1: [1]

Biconservative Hypersurfaces with diagonalizable shape operator in $\mathbb{E}^4_1$

[1]See [**Y. Fu** and NcT]

Hypersurfaces in $\mathbb{E}^4_1$

It is well-known that the shape operator of a hypersurface in $\mathbb{E}^4_1$ takes one of the following 4 forms for some smooth functions k_1, k_2, k_3, k_4 and ν.

$$\text{Case I. } S = \begin{pmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{pmatrix}, \quad \text{Case II. } S = \begin{pmatrix} k_1 & 1 & 0 \\ 0 & k_1 & 0 \\ 0 & 0 & k_3 \end{pmatrix},$$

$$\text{Case III. } S = \begin{pmatrix} k_1 & 0 & 0 \\ 0 & k_1 & 1 \\ -1 & 0 & k_1 \end{pmatrix}, \quad \text{Case IV. } S = \begin{pmatrix} k_1 & -\nu & 0 \\ \nu & k_1 & 0 \\ 0 & 0 & k_3 \end{pmatrix}$$

Shape operator of Biconservative hypersurfaces in $\mathbb{E}^4_1$

Equation of Biconservativity:

$$S(\nabla H) = -\varepsilon \frac{3H}{2} \nabla H \qquad \text{(BC)}$$

Case I. $S = \begin{pmatrix} -\varepsilon\frac{3H}{2} & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{pmatrix}$, Case II. $S = \begin{pmatrix} \frac{9H}{4} & 1 & 0 \\ 0 & \frac{9H}{4} & 0 \\ 0 & 0 & -\frac{3H}{2} \end{pmatrix}$,

Case III. $S = \begin{pmatrix} k_1 & 0 & 0 \\ 0 & k_1 & 1 \\ -1 & 0 & k_1 \end{pmatrix}$, Case IV. $S = \begin{pmatrix} \frac{9H}{4} & -\nu & 0 \\ \nu & \frac{9H}{4} & 0 \\ 0 & 0 & -\frac{3H}{2} \end{pmatrix}$

Biconservative hypersurfaces with diagonalizable shape operator I

Assume that M is a biconservative submanifold and its shape operator is diagonalizable. Then, we have

$$S(\nabla H) = -\varepsilon \frac{-3H}{2} \nabla H \qquad \text{(BC)}$$

Remark

If $\nabla H = 0$, then is satisfied trivially. Thus, we assume that ∇H does not vanish.

Biconservative hypersurfaces with diagonalizable shape operator II

Then, we have

$$e_2(k_1) = e_3(k_1) = 0, \quad e_1(k_1) \neq 0.$$

Remarks

- It is very easy to observe that multiplicity of k_1 is 1.
- If $k_2 = k_3$, then do Carmo and Dajczer's classical result shows that $M$is a rotational hypersurface[a].

[a] See [Trans. Amer. Math. Soc. 277(1983),685–709]

Hence, we assume that $k_1 - k_2$, $k_1 - k_3$ and $k_2 - k_3$ does not vanish.

Biconservative hypersurfaces with diagonalizable shape operator III

By a long computation, we obtain that

$$T_m M = \underbrace{D(m)}_{D=\mathrm{span}\{e_2,e_3\}} \oplus \underbrace{D^{\perp}(m)}_{D^{\perp}=\mathrm{span}\{e_1\}}$$

and further D and $D^{\perp}$ are involutive which yields

Proposition

There exists a local coordinate system (s, t, u) such that

$$e_1 = \frac{\partial}{\partial s}, \quad e_2 = \frac{1}{E_1}\frac{\partial}{\partial t}, \quad e_3 = \frac{1}{E_2}\frac{\partial}{\partial u}.$$

Further,

Proposition

If M has two distinct principal curvature, then it has a local parametrization

$$x(s,t,u) = \phi(s)\Theta(t,u) + \Gamma(s)$$

for some vector valued functions Θ, Γ and a function ϕ.
If M has three distinct principal curvature, then

$$x(s,t,u) = \phi_1(s)\Theta_1(t) + \phi_2(s)\Theta_2(u) + \Gamma(s)$$

for some vector valued functions $\Theta_1, \Theta_2, \Gamma$ and functions ϕ_1, ϕ_2.

Biconservative hypersurfaces in $\mathbb{E}^4_1$

We have obtained the following families of biconservative hypersurfaces with diagonalizable shape operator.

Two distinct principal curvatures

- $x_1(s,t,u) = (f_1(s), s\cos t\sin u, s\sin t\sin u, s\cos u)$;
- $x_2(s,t,u) = (s\sinh u\sin t, s\cosh u\sin t, s\cos t, f_2(s))$;
- $x_3(s,t,u) = (s\cosh t, s\sinh t\sin u, \sinh t\cos u, f_3(s))$;
- $x_4(s,t,u) = \left(\frac{1}{2}s(t^2+u^2)+s+f_4(s), st, su, \frac{1}{2}s(t^2+u^2)+f_4(s)\right)$.

Biconservative hypersurfaces in $\mathbb{E}^4_1$

Zero Gauss-Kronecker Curvature

- A generalized cylinder $M^2_0 \times \mathbb{E}^1_1$ where M is a biconservative surface in $\mathbb{E}^3$;
- A generalized cylinder $M^2_0 \times \mathbb{E}^1$ where M is a biconservative Riemannian surface in $\mathbb{E}^3_1$;
- A generalized cylinder $M^2_1 \times \mathbb{E}^1$, where M is a biconservative Lorentzian surface in $\mathbb{E}^3_1$.

Biconservative hypersurfaces in $\mathbb{E}^4_1$

Three distinct principal curvatures

- $x_1(s,t,u) = (s\cosh t, s\sinh t, f_1(s)\cos u, f_1(s)\sin u)$;
- $x_2(s,t,u) = (s\sinh t, s\cosh t, f_2(s)\cos u, f_2(s)\sin u)$;
- A hypersurface in $\mathbb{E}^4_1$ given by

$$x_3(s,t,u) = \left(\frac{1}{2}s(t^2+u^2) + au^2 + s + \phi(s), st, (s+2a)u, \frac{1}{2}s(t^2+u^2) + au^2 + \phi(s)\right), \quad a \neq 0.$$

Section 2.2:[2]

Biconservative Hypersurfaces with index 2 in $\mathbb{E}^5_2$

[2]See [**A. Upadhyay** and NcT]

Shape operator of hypersurfaces in $\mathbb{E}^5_2$ I

Let $M^4_2 \hookrightarrow \mathbb{E}^5_2$. Then, by choosing an appropriated base field $\{e_1, e_2, e_3, e_4\}$ of the tangent bundle of M, the matrix representation of S can be assumed to be one of the following forms. Note that in each cases below, g denotes the induced metric tensor of M, i.e., $g_{ij} = \langle e_i, e_j \rangle$.

- $S = \begin{pmatrix} k_1 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 \\ 0 & 0 & k_3 & 0 \\ 0 & 0 & 0 & k_4 \end{pmatrix}, \qquad g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix};$
- $S = \begin{pmatrix} k_1 & 1 & 0 & 0 \\ 0 & k_1 & 0 & 0 \\ 0 & 0 & k_3 & 0 \\ 0 & 0 & 0 & k_4 \end{pmatrix}, \qquad g = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix};$
- $S = \begin{pmatrix} k_1 & 0 & 1 & 0 \\ 0 & k_1 & 0 & 0 \\ 0 & -1 & k_1 & 0 \\ 0 & 0 & 0 & k_4 \end{pmatrix}, \qquad g = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix};$
- $S = \begin{pmatrix} k_1 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 \\ 0 & 0 & k_3 & \beta_1 \\ 0 & 0 & -\beta_1 & k_3 \end{pmatrix}, \qquad g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix};$

Shape operator of hypersurfaces in $\mathbb{E}^5_2$ II

- $S = \begin{pmatrix} k_1 & 1 & 0 & 0 \\ 0 & k_1 & 0 & 0 \\ 0 & 0 & k_3 & 1 \\ 0 & 0 & 0 & k_3 \end{pmatrix}, \qquad g = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix};$
- $S = \begin{pmatrix} k_1 & 1 & 0 & 0 \\ 0 & k_1 & 0 & 0 \\ 0 & 0 & k_3 & \beta_1 \\ 0 & 0 & -\beta_1 & k_3 \end{pmatrix}, \qquad g = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix};$
- $S = \begin{pmatrix} k_1 & \beta_1 & 0 & 0 \\ -\beta_1 & k_1 & 0 & 0 \\ 0 & 0 & k_3 & \beta_2 \\ 0 & 0 & -\beta_2 & k_3 \end{pmatrix}, \qquad g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix};$
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- $S = \begin{pmatrix} k_1 & 0 & 1 & 0 \\ 0 & k_1 & 0 & 0 \\ 0 & 0 & k_1 & 1 \\ 0 & 1 & 0 & k_1 \end{pmatrix}, \qquad g = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

  for some smooth functions $k_1, k_2, k_3, k_4, \beta_1, \beta_2$.

Shape operator of biconservative hypersurfaces

Let M be a hypersurface of index 2 in $\mathbb{E}^5_2$ with H as its (first) mean curvature. If M is biconservative, then its shape operator S has one of the following forms:

$$\text{Case I. } S = \begin{pmatrix} -2H & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 \\ 0 & 0 & k_3 & 0 \\ 0 & 0 & 0 & k_4 \end{pmatrix}, \qquad \text{Case II. } S = \begin{pmatrix} -2H & 0 & 0 & 0 \\ 0 & k_2 & -\nu & 0 \\ 0 & \nu & k_2 & 0 \\ 0 & 0 & 0 & k_4 \end{pmatrix},$$

$$\text{Case III. } S = \begin{pmatrix} -2H & 0 & 0 & 0 \\ 0 & k_2 & 1 & 0 \\ 0 & 0 & k_2 & 0 \\ 0 & 0 & 0 & k_4 \end{pmatrix}, \qquad \text{Case IV. } S = \begin{pmatrix} -2H & 0 & 0 & 0 \\ 0 & 2H & 0 & 0 \\ 0 & 0 & 2H & -1 \\ 0 & 1 & 0 & 2H \end{pmatrix}$$

for some smooth functions k_1, k_2, k_3, k_4 and ν, where $e_1 = \frac{\nabla H}{\|\nabla H\|^2}$.

Classification Results I

Let M be biconservative hypersurface of index 2 in the pseudo-Euclidean space $\mathbb{E}^5_2$ and the shape operator S have the form

$$S = \mathrm{diag}(k_1, k_2, k_2, k_4), \quad k_4 \neq k_2$$

Then, M is one of the followings.

- $x(s,t,u,v) = (\phi_2 \mathrm{sinh} v, \phi_1 \mathrm{cosh} t, \phi_1 \mathrm{sinh} t \cos u, \phi_1 \mathrm{sinh} t \sin u, \phi_2 \mathrm{cosh} v)$,
- $x(s,t,u,v) = (\phi_2 \cos v, \phi_2 \sin v, \phi_1 \cos t, \phi_1 \sin t \cos u, \phi_1 \sin t \sin u)$,
- $x(s,t,u,v) = (\phi_1 \mathrm{cosh} t \sin u, \phi_1 \mathrm{cosh} t \cos u, \phi_1 \mathrm{sinh} t, \phi_2 \cos v, \phi_2 \sin v)$,
- $x(s,t,u,v) = (\phi_2 \mathrm{sinh} v, \phi_1 \mathrm{sinh} t, \phi_1 \mathrm{cosh} t \cos u, \phi_1 \mathrm{cosh} t \sin u, \phi_2 \mathrm{cosh} v)$,
- $x(s,t,u,v) = (\phi_2 \mathrm{cosh} v, \phi_1 \mathrm{sinh} t, \phi_1 \mathrm{cosh} t \cos u, \phi_1 \mathrm{cosh} t \sin u, \phi_2 \mathrm{sinh} v)$,
- $x(s,t,u,v) = (\phi_1 \mathrm{sinh} t cosu, \phi_1 \mathrm{sinh} t \sin u, \phi_1 \mathrm{cosh} u, \phi_2 \cos v, \phi_2 \sin v)$,

Classification Results II

- A hypersurface given by

$$x(s,t,u,v) = \Big(\frac{s}{2}\left(t^2+u^2-v^2\right) - av^2 + \psi, v(2a+s), st, su, \frac{s}{2}\left(t^2+u^2-v^2\right) - av^2 + \psi - s \Big) \quad (1)$$

for a non-zero constants a and a smooth function $\psi = \psi(s)$ such that $1 - 2\psi' < 0$;

- A hypersurface given by

$$x(s,t,u,v) = \left(\frac{s\left(t^2-u^2-v^2\right)}{2} + av^2 + \psi, st, su, v(s-2a), \frac{s\left(t^2-u^2-v^2\right)}{2} + av^2 + \psi + s \right) \quad (2)$$

for a non-zero constants a and a smooth function $\psi = \psi(\tilde{s})$ such that $1 + 2\psi' < 0$.

A further note

Consider the hypersurface given by

$$x(s, t_1, t_2, \dots t_{n-1}) = \Big(-a_1 t_1^2 + a_2 t_2^2 + \cdots + a_{n-1} t_{n-1}^2 + \frac{s\|t\|^2}{2} + \psi,$$
$$t_1(s + 2a_1), t_2(s + 2a_2), \dots, t_{n-1}(s + 2a_2),$$
$$-a_1 t_1^2 + a_2 t_2^2 + \cdots + a_{n-1} t_{n-1}^2 + \frac{s\|t\|^2}{2} + \psi - s\Big),$$

where $\|t\|^2 = t_2^2 + t_3^2 + \cdots + t_{n-1}^2 - t_1^2$.
This provides an example of biconservative hypersurface for a particularly chosen smooth function ψ. Moreover, if all constants $a_1, a_2, \dots a_{n-1}$ are distinct, then M has n **distinct** principal curvatures.

Section 3:

Biconservative Submanifolds in $\mathbb{S}^n \times \mathbb{R}$

Section 3.1:

Submanifolds in $\mathbb{S}^n \times \mathbb{R}$

Basic facts

Given an isometric immersion $f : M^m \to \mathbb{S}^n \times \mathbb{R}$, let ∂_t be a unit vector field tangent to the second factor. Then, a tangent vector field T on M^m and a normal vector field η along f are defined by

$$\partial_t = f_* T + \eta.$$

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$$\partial_t = f_* T + \eta.$$

The class $\mathcal{A}$

We will denote by $\mathcal{A}$ the class of isometric immersions $f : M^m \to \mathbb{S}^n \times \mathbb{R}$ with the property that T is an eigenvector of all shape operators of f.

The class $\mathcal{A}$

- The class $\mathcal{A}$ was introduced in [3] for hypersurfaces

[3] R. Tojeiro, *On a class of hypersurfaces in* $\mathbb{S}^n \times \mathbb{R}$ *and* $\mathbb{H}^n \times \mathbb{R}$, Bull. Braz. Math. Soc. (N. S.) **41**, no. 2, 199–209, (2010).

[4] B. Mendonça, R. Tojeiro, *Umbilical submanifolds of* $\mathbb{S}^n \times \mathbb{R}$, Canad. J. Math. **66**, no. 2, 400–428, (2014).

The class $\mathcal{A}$

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- It extended to submanifolds of $\mathbb{S}^n \times \mathbb{R}$ in [4].

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The class $\mathcal{A}$

- The class $\mathcal{A}$ was introduced in [3] for hypersurfaces
- It extended to submanifolds of $\mathbb{S}^n \times \mathbb{R}$ in [4].
- Trivial examples are
  - $(T = 0)$ Slices $\mathbb{S}^n \times \{t_0\}$,
  - $(|T| = 1)$The vertical cylinders $N^{m-1} \times \mathbb{R}$, where N^{m-1} is a submanifold of $\mathbb{S}^n$.

[3] R. Tojeiro, *On a class of hypersurfaces in $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$*, Bull. Braz. Math. Soc. (N. S.) **41**, no. 2, 199–209, (2010).

[4] B. Mendonça, R. Tojeiro, *Umbilical submanifolds of $\mathbb{S}^n \times \mathbb{R}$*, Canad. J. Math. **66**, no. 2, 400–428, (2014).

Biconservative Submanifolds

Note that the curvature tensor of $\mathbb{S}^n \times \mathbb{R}$ is

$$\tilde{R}(X, Y)Z = \{\langle Y, Z\rangle X - \langle X, Z\rangle Y\} + (-\langle Y, T\rangle + \langle X, T\rangle)\langle Z, T\rangle T + (\langle X, Z\rangle\langle Y, T\rangle - \langle Y, Z\rangle\langle X, T\rangle)\, T.$$

Biconservative Submanifolds

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Therefore, the equation of biconservativity become

$$m\nabla\|H\|^2 + 4\operatorname{tr}A_{\nabla^\perp_{\cdot} H}(\cdot) + 4\underbrace{n\langle H,\eta\rangle T}_{\operatorname{tr}\left(\tilde{R}(\cdot,H)\cdot\right)^T} = 0$$

for an isometric immersion $f : M^m \to \mathbb{S}^n \times \mathbb{R}$.

Special Cases

$$m\nabla\|H\|^2 + 4\mathrm{tr}A_{\nabla^\perp_\cdot H}(\cdot) + 4n\langle H,\eta\rangle T = 0$$

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Special Cases

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- If M is a **hypersurface**, then we have

$$+S(\nabla H) + \frac{nH}{2}\nabla\|H\| + n\langle H,\eta\rangle T = 0$$

- If M has parallel mean curvature vector, then $\langle H,\eta\rangle T = 0$.

Section 3.2:

Biconservative submanifolds with parallel mean curvature vector

Previous works

Biconservative surfaces in $\mathbb{S}^n \times \mathbb{R}$ with parallel mean curvature studied in [5].
Note that by a Hopf theorem given in [6], M lies $\mathbb{S}^4 \times \mathbb{R}$.

[5] See D. Fetcu, C. Oniciuc and A. L. Pinheiro, J. Math.Anal.Appl. 425(2015), 588–609

[6] H. Alencar, M. do Carmo and R. Tribuzy, J. Differential Geom. 84 (2010) 1–17

Biconservative submanifolds in $\mathbb{S}^n \times \mathbb{R}$

Consider a biconservative submanifolds with parallel non-zero mean curvature vector field and codimension 2. Then, we have

$$\langle H, \eta \rangle T = 0.$$

If $T = 0$, i.e., ∂_t is normal to M. In this case, M is an open part of the slice $\mathbb{S}^n \times \{t_0\}$. Thus, consider

$$T \neq 0, \quad \langle H, \eta \rangle = 0.$$

Biconservative submanifolds in $\mathbb{S}^n \times \mathbb{R}$

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$$T \neq 0, \quad \langle H, \eta \rangle = 0.$$

Note that a simple computation considering $\langle H, \eta \rangle = 0$ yields $A_H(T) = 0$. Therefore, dimension of the distribution

$$E_0(H) = \{X \in TM | A_H(X) = 0\}$$

is k and $1 \leq k < n$.

A further direct computation also yields $A_\eta(T) \in E_0(H)$. Hence, we have

Theorem

Every biconservative submanifolds in $\mathbb{S}^n \times \mathbb{R}$ with codimension 2 and parallel mean curvature vector belongs to the class $\mathcal{A}$ if $\dim E_0(H) = 1$.

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Note that, in particular if M is a surface in $\mathbb{S}^n \times \mathbb{R}$, then we have $\dim E_0(H) = 1$. Hence,

Corollary

Every biconservative surface in $\mathbb{S}^n \times \mathbb{R}$ with parallel mean curvature vector belongs to the class $\mathcal{A}$.

Remark. Compare the result obtained in [7].

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The case $\dim E_0(H) = k > 1$

Lemma

An isometric immersion $f : M^n \to \mathbb{S}^{n+1} \times \mathbb{R}$ is biconservative if and only if there exists a local orthonormal frame field $\{e_1, e_2, \ldots, e_n; e_{n+1}, e_{n+2}\}$ such that

(1) $e_1 = \frac{T}{|T|}$, $e_{n+1} = \frac{H}{|H|}$, $e_{n+2} = \frac{\eta}{|\eta|}$,

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(2) Shape operators along e_{n+1} and e_{n+2} have matrix representations given by

$$A_{n+1} = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \underset{\sim}{S}_1 \end{array}\right) \quad \text{and} \quad A_{n+2} = \left(\begin{array}{c|c} \underset{\sim}{S}_2 & 0 \\ \hline 0 & \underset{\sim}{B} \end{array}\right)$$

for some diagonalized matrices $\underset{\sim}{S}_1$, $\underset{\sim}{B}$ and a symmetric matrix $\underset{\sim}{S}_2$ such that $\operatorname{tr}(\underset{\sim}{S}_1) = \text{const} \neq 0$, $\operatorname{tr}(\underset{\sim}{S}_2) + \operatorname{tr}(\underset{\sim}{B}) = 0$ and

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(3) $\nabla_X Y \in E_0(H)$ whenever $X, Y \in E_0(H)$.

An example of biconservative submanifold in $\mathbb{S}^4 \times \mathbb{R}$

We put

$$\mathbb{S}^n(a^{-2})\times\mathbb{R} = \{(x_1, x_2, \ldots, x_{n+2}) \in \mathbb{R}^{n+2} | x_1^2+x_2^2+\cdots+x_{n+1}^2 = a^2\}, \quad n > 1$$

which implies $\partial_t = (\underbrace{0, 0, \ldots, 0}_{(n+1)\text{-times}}, 1)$.

Example

Let $\phi = (\phi_1, \phi_2, \phi_3, \phi_4) : \tilde{M}^2 \to \mathbb{S}^2(a^{-2}) \times \mathbb{R}$ be an oriented minimal immersion and $a^2 + b^2 = 1$. Then the isometric immersion $f : M \to \mathbb{S}^4 \times \mathbb{R}$ given by

$$f(s, u_1, u_2) = \left(b \cos\frac{s}{b}, b \sin\frac{s}{b}, \phi_1(u_1, u_2), \phi_2(u_1, u_2), \phi_3(u_1, u_2), \phi_4(u_1, u_2)\right)$$

is a biconservative immersion with parallel mean curvature vector field and $\dim E_0(H) = 2$.

Classification Result

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described above .

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Referances II

N.C. Turgay *H-hypersurfaces with 3 distinct principal curvatures in the Euclidean spaces* (accepted 4 days ago)

THANK YOU