Lagrangian Submanifolds with Constant Angle Functions in the Nearly Kähler $S^3 \times S^3$

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An even dimensional manifold is called **symplectic manifold** if it admits a symplectic form, which is a closed and nondegenerate 2-form.

**Lagrangian Submanifold**

A submanifold of a symplectic manifold is called **Lagrangian submanifold** if the symplectic form restricted to the manifold vanishes and if the dimension of the submanifold is half the dimension of the symplectic manifold.

A local classification is trivial from the symplectic point of view. (Darboux Theorem)
An almost Hermitian manifold \((M, J, g)\) is a manifold \(M\) with metric \(g\) and almost complex structure \(J\) satisfying
\[
g(JX, JY) = g(X, Y), \quad X, Y \in TM.
\] (1)

**Lagrangian Submanifold**

A submanifold in an almost Hermitian manifold is called Lagrangian submanifold if the almost complex structure interchanges the tangent and the normal spaces and if the dimension is half the dimension.

Lagrangian submanifold in Kähler manifold
A Nearly Kähler Manifold

A nearly Kähler manifold is almost Hermitian manifold \((M, g, J)\) such that \(\nabla J\) is skew symmetric, i.e.,

\[
(\nabla_X J)Y + (\nabla_Y J)X = 0, \quad X, Y \in TM.
\]  

(2)


The only homogeneous 6–dimensional nearly Kähler manifolds,

- the nearly Kähler $S^6$
- $S^3 \times S^3$
- the complex projective space $\mathbb{CP}^3$
- the flag manifold $SU(3)/U(1) \times U(1)$
The importance of 6–dimensional nearly Kähler manifolds

They serve as building block for arbitrary nearly Kähler manifolds

Six dimensional nearly Kähler manifold is Einstein and there is a bijective correspondence between nearly Kähler structure and Killing spinors
The nearly Kähler $S^3 \times S^3$

\[ T_{(p,q)}(S^3 \times S^3) \cong T_pS^3 \oplus T_qS^3 \]  

(3)

The tangent vector at $(p, q)$: $Z(p, q) = (U(p, q), V(p, q))$

- The almost complex structure $J$ on $S^3 \times S^3$

\[ JZ(p, q) = \frac{1}{\sqrt{3}} (2pq^{-1}V - U, -2qp^{-1}U + V) \]  

(4)

- The metric $g$ on $S^3 \times S^3$

\[ g(Z, Z') = \frac{1}{2} (\langle Z, Z' \rangle + \langle JZ, JZ' \rangle) \]  

\[ = \frac{4}{3} (\langle U, U' \rangle + \langle V, V' \rangle) - \frac{2}{3} (\langle p^{-1}U, q^{-1}V' \rangle + \langle p^{-1}U', q^{-1}V \rangle) \]  

(5)

(6)

where $\langle \cdot, \cdot \rangle$ is the product metric, tangent vectors $Z = (U, V)$ and $Z' = (U', V')$. 
The almost product structure $P$ on $S^3 \times S^3$

$$PZ(p, q) = (pq^{-1}V, qp^{-1}U)$$

(7)

Properties of $P$

Denote $G(X, Y) = (\nabla_x J)Y$.

- $P^2 = \text{Id}$,
- $PJ = -JP$,
- $g(PZ, PZ') = g(Z, Z')$,
- $PG(X, Y) + G(PX, PY) = 0$,
- $G(X, PY) + PG(X, Y) = -2J(\tilde{\nabla}_x P)Y$.

The usual product structure $QZ = (-U, V)$. 

Almost Product Structure on Lagrangian Submanifold

Let $M$ be a Lagrangian submanifold in the nearly Kähler $S^3 \times S^3$. The pull back of $T(S^3 \times S^3)$ to $M$ splits into $TM \oplus JTM$. There are two endomorphisms $A, B : TM \rightarrow TM$ such that

$$PX = AX + JBX, \quad \forall X \in TM. \quad (8)$$

- $A$ and $B$ are symmetric commuting endomorphisms that satisfy $A^2 + B^2 = I$.
- The covariant derivative of $A$ and $B$

$$\nabla_X A Y = BS_{JX} Y - Jh(X, BY) + \frac{1}{2}(JG(X, AY) - AJG(X, Y)) \quad (9)$$

$$\nabla_X B Y = Jh(X, BY) - AS_{JX} Y + \frac{1}{2}(JG(X, BY) - BJG(X, Y)). \quad (10)$$

- $A$ and $B$ are symmetric operators whose Lie bracket is zero, i.e., $[A, B] = 0$. 
$A$ and $B$ can be diagonalized simultaneously at a point of $M$ and at each point $p$ there is an orthonormal basis $e_1, e_2, e_3 \in T_p M$ such that

$$Pe_i = \cos 2\theta_i e_i + \sin 2\theta_i Je_i, \quad \forall i = 1, 2, 3. \quad (11)$$

- **Gauss equation**

$$R(X, Y)Z = \frac{5}{12}(g(Y, Z)X - g(X, Z)Y)$$

$$+ \frac{1}{3}(g(AY, Z)AX - g(AX, Z)AY + g(BY, Z)BX - g(BX, Z)BY)$$

$$+ [S_{JX}, S_{JY}]Z \quad (12)$$

- **Codazzi equation**

$$(\nabla h)(X, Y, Z) - (\nabla h)(Y, X, Z) =$$

$$\frac{1}{3}(g(AY, Z)JBX - g(AX, Z)JBY - g(BY, Z)JAX + g(BX, Z)JAY)$$

$$\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad (13)$$
For the Levi-Civita connection $\nabla$ on $M$,

\[ \nabla_{E_i} E_j = \omega^k_{ij} E_k \quad \text{with} \quad \omega^k_{ij} = -\omega^j_{ik} \]

$s_k^i = g(h(E_i, E_k), J E_k)$, $s^k_i$ is totally symmetric.

**Lemma (Y. Zhang, Z. Hu, B. Dioos, L. Vrancken, X. Wang, 2016)**

Let $M$ be a Lagrangian submanifold. Let \{\(E_1, E_2, E_3\)\} be local orthonormal frame. Denote by $s^k_i$ and $\omega^k_{ij}$ the components of respectively the second fundamental form and the induced connection. Then we have

- $\theta_1 + \theta_2 + \theta_3$ is a multiple of $\pi$
- $E_i(\theta_j) = -s^i_j$
- $s^k_i \cos(\theta_j - \theta_k) = \left( \frac{\sqrt{3}}{6} \epsilon^k_{ij} - \omega^k_{ij} \right) \sin(\theta_j - \theta_k), \ j \neq k$. 
Motivation

Lemma (B. Dioos, L. Vrancken, X. Wang, 2016)
If two of the angles are equal modulo $\pi$, then the Lagrangian submanifold is totally geodesic.

Corollary (B. Dioos, L. Vrancken, X. Wang, 2016)
Let $M$ be a Lagrangian submanifold of the nearly Kähler $S^3 \times S^3$. If $M$ is totally geodesic, then the angles $\theta_1, \theta_2$ and $\theta_3$ are constant. Conversely, if the angles are constant and $h_{12}^3 = 0$, then $M$ is totally geodesic.
Theorem A (Y. Zhang, Z. Hu, B. Dioos, L. Vrancken, X. Wang, 2016)

Let $M$ be a totally geodesic Lagrangian submanifold in the nearly Kähler $S^3 \times S^3$. Then up to an isometry of the nearly Kähler $S^3 \times S^3$, $M$ is locally congruent with one of the following immersions:

1. $f: S^3 \to S^3 \times S^3 : u \mapsto (u, 1),$
2. $f: S^3 \to S^3 \times S^3 : u \mapsto (1, u),$
3. $f: S^3 \to S^3 \times S^3 : u \mapsto (u, u),$
4. $f: S^3 \to S^3 \times S^3 : u \mapsto (u, u\mathbf{i}),$
5. $f: S^3 \to S^3 \times S^3 : u \mapsto (u^{-1}, u\mathbf{i}u^{-1}),$
6. $f: S^3 \to S^3 \times S^3 : u \mapsto (u\mathbf{i}u^{-1}, u^{-1}).$
Round sphere \[
\begin{array}{ll}
(2\theta_1, 2\theta_2, 2\theta_3) &= (\frac{4\pi}{3}, \frac{4\pi}{3}, \frac{4\pi}{3}) \\
(2\theta_1, 2\theta_2, 2\theta_3) &= (\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3}) \\
(2\theta_1, 2\theta_2, 2\theta_3) &= (0, 0, 0)
\end{array}
\]

Berger sphere \[
\begin{array}{ll}
(2\theta_1, 2\theta_2, 2\theta_3) &= (\frac{4\pi}{3}, \frac{4\pi}{3}, \frac{4\pi}{3}) \\
(2\theta_1, 2\theta_2, 2\theta_3) &= (\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3}) \\
(2\theta_1, 2\theta_2, 2\theta_3) &= (0, 0, 0)
\end{array}
\]

Theorem B (B. Dioos, L. Vrancken, X. Wang, 2016)

Let $M$ be a Lagrangian submanifold of constant sectional curvature in the nearly Kähler $S^3 \times S^3$. Then, up to an isometry of the nearly Kähler $S^3 \times S^3$, $M$ is locally congruent with one of the following immersions:

1. $f: S^3 \to S^3 \times S^3 : u \mapsto (u, 1)$,
2. $f: S^3 \to S^3 \times S^3 : u \mapsto (1, u)$,
3. $f: S^3 \to S^3 \times S^3 : u \mapsto (u, u)$,
4. $f: S^3 \to S^3 \times S^3 : u \mapsto (uiu^{-1}, uju^{-1})$,
5. $f: \mathbb{R}^3 \to S^3 \times S^3 : (u, v, w) \mapsto (p(u, w), q(u, v))$, where $p$ and $q$ are constant mean curvature tori in $S^3$. 
Theorem 1

Let $f : M \to S^3 \times S^3$ be a Lagrangian immersion into a nearly Kähler manifold $S^3 \times S^3$ given by $f = (p, q)$ with the angle functions $\theta_i$. Then, $\tilde{f} : M \to S^3 \times S^3$ given by $\tilde{f} = (q, p)$ is also a Lagrangian immersion with the angle functions $\tilde{\theta}_i$ such that $\tilde{\theta}_i = \pi - \theta_i$.

Theorem 2

Let $f : M \to S^3 \times S^3$ be a Lagrangian immersion defined by $f = (p, q)$ with the angle functions $\theta_i$. Then, $f^* : M \to S^3 \times S^3$ defined by $f^* = (\bar{p}, q\bar{p})$ is also a Lagrangian immersion with the angle functions $\theta_i^*$ such that $\theta_i^* = \frac{2\pi}{3} - \theta_i$. 
Lemma 3

Let $M$ be a Lagrangian submanifold of the nearly Kähler manifold $S^3 \times S^3$ with constant angle functions $\theta_i$.

i. If $M$ is a non-totally geodesic submanifold, then the nonzero components of $\omega^k_{ij}$ are given by

\[
\omega^3_{12} = \frac{\sqrt{3}}{6} - \frac{\cos(\theta_2 - \theta_3)}{\sin(\theta_2 - \theta_3)} h^3_{12},
\]

\[
\omega^1_{23} = \frac{\sqrt{3}}{6} + \frac{\cos(\theta_1 - \theta_3)}{\sin(\theta_1 - \theta_3)} h^3_{12},
\]

\[
\omega^2_{31} = \frac{\sqrt{3}}{6} - \frac{\cos(\theta_1 - \theta_2)}{\sin(\theta_1 - \theta_2)} h^3_{12}.
\]

ii. The Codazzi equations of the submanifold $M$ are as followings:

\[
E_i(h^3_{12}) = 0, \quad i = 1, 2, 3,
\]

\[
h^3_{12} \left(2(\omega^2_{13} + \omega^3_{21}) + \frac{1}{\sqrt{3}}\right) = \frac{1}{3} \sin(2(\theta_1 - \theta_2)),
\]

\[
h^3_{12} \left(2(\omega^3_{12} + \omega^2_{31}) - \frac{1}{\sqrt{3}}\right) = \frac{1}{3} \sin(2(\theta_1 - \theta_3)),
\]

\[
h^3_{12} \left(2(\omega^3_{21} + \omega^1_{32}) + \frac{1}{\sqrt{3}}\right) = \frac{1}{3} \sin(2(\theta_2 - \theta_3)).
\]
iii. The Gauss equations of the submanifold $M$ are given by

\[ \frac{5}{12} + \frac{1}{3} \cos(2(\theta_1 - \theta_2)) - (h_{12}^3)^2 = -\omega_{21}^3 \omega_{13}^2 + \omega_{12}^3 \omega_{31}^2 - \omega_{21}^3 \omega_{31}^2, \]  

(21)

\[ \frac{5}{12} + \frac{1}{3} \cos(2(\theta_1 - \theta_3)) - (h_{12}^3)^2 = -\omega_{31}^2 \omega_{12}^3 + \omega_{13}^2 \omega_{21}^3 - \omega_{31}^2 \omega_{21}^3, \]  

(22)

\[ \frac{5}{12} + \frac{1}{3} \cos(2(\theta_2 - \theta_3)) - (h_{12}^3)^2 = -\omega_{32}^1 \omega_{21}^3 + \omega_{23}^1 \omega_{12}^3 - \omega_{32}^1 \omega_{12}^3. \]  

(23)
Theorem 4

A Lagrangian submanifold of the nearly Kähler manifold $S^3 \times S^3$ with constant angle functions is either totally geodesic or has constant sectional curvature in $S^3 \times S^3$.

Case 1 $h^3_{12} = 0$, a totally geodesic Lagrangian submanifold of the nearly Kähler $S^3 \times S^3$

Case 2 $h^3_{12}$ is a nonzero constant.

i. $h^3_{12} = -\frac{1}{2}, (2\theta_1, 2\theta_2, 2\theta_3) = (0, \frac{2\pi}{3}, \frac{4\pi}{3})$, a flat Lagrangian submanifold.

ii. $h^3_{12} = -\frac{1}{4}, (2\theta_1, 2\theta_2, 2\theta_3) = (\frac{4\pi}{3}, \frac{2\pi}{3}, 0)$, Lagrangian submanifold with sectional curvature $\frac{3}{16}$.

iii. $h^3_{12} = \frac{1}{4}, (2\theta_1, 2\theta_2, 2\theta_3) = (0, \frac{2\pi}{3}, \frac{4\pi}{3})$, Lagrangian submanifold with sectional curvature $\frac{3}{16}$.

iv. $h^3_{12} = \frac{1}{2}, (2\theta_1, 2\theta_2, 2\theta_3) = (\frac{4\pi}{3}, \frac{2\pi}{3}, 0)$, a flat Lagrangian submanifold.

Corollary 5

A Lagrangian submanifold of the nearly Kähler manifold $S^3 \times S^3$ with constant angle functions is locally congruent to the immersions given in Theorem A and Theorem B.
References


THANK YOU FOR YOUR ATTENTION