Lagrangian Submanifolds with Constant Angle Functions in the Nearly Kähler $\mathbb{S}^3 \times \mathbb{S}^3$

Burcu Bektaş
Istanbul Technical University, Istanbul, Turkey

Joint work with Marilena Moruz (Université de Valenciennes, France),
Joeri Van der Veken (KU Leuven, Belgium) and
Luc Vrancken (Université de Valenciennes, France)

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Basic Notions

- An even dimensional manifold is called sympletic manifold if it admits a symplectic form, which is a closed and nondegenerate 2-form.

Lagrangian Submanifold

A submanifold of a symplectic manifold is called Lagrangian submanifold if the symplectic form restricted to the manifold vanishes and if the dimension of the submanifold is half the dimension of the symplectic manifold.

- A local classification is trivial from the symplectic point of view. (Darboux Theorem)

- An almost Hermitian manifold (M, J, g) is a manifold M with metric g and almost complex structure J satisfying

$$g(JX, JY) = g(X, Y), \qquad X, Y \in TM. \tag{1}$$

Lagrangian Submanifold

A submanifold in an almost Hermitian manifold is called Lagrangian submanifold if the almost complex structure interchanges the tangent and the normal spaces and if the dimension is half the dimension.

- Lagrangian submanifold in Kähler manifold
  B.-Y.Chen and K. Oguie, *On totally real submanifolds*, Trans. Amer. Math. Soc. **193**(1974), 257–266.

A Nearly Kähler Manifold

A nearly Kähler manifold is almost Hermitian manifold (M, g, J) such that ∇J is skew symmetric, i.e.,

$$(\nabla_X J)Y + (\nabla_Y J)X = 0, \qquad X, Y \in TM. \tag{2}$$

- A. Gray, *Nearly Kähler manifolds*, J. Differential Geometry, **4**(1970), 283–309.
- P.-A. Nagy, *Nearly Kähler geometry and Riemannian folitaions*, Asian J. Math., **6**(2002), 481–504.

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The only homogeneous 6–dimensional nearly Kähler manifolds,

- the nearly Kähler $\mathbb{S}^6$
- $\mathbb{S}^3 \times \mathbb{S}^3$
- the complex projective space $\mathbb{CP}^3$
- the flag manifold $SU(3)/U(1) \times U(1)$

The importance of 6–dimensional nearly Kähler manifolds

- They serve as building block for arbitrary nearly Kähler manifolds
  P.-A. Nagy, *Nearly Kähler geometry and Riemannian folitaions*, Asian J. Math., **6**(2002), 481–504.
- Six dimensional nearly Kähler manifold is Einstein and there is a bijective correspondence between nearly Kähler structure and Killing spinors
  T. Friedrich, and R. Grunewald, *On the first eigenvalue of the Dirac operator on 6-dimensional manifolds*, Ann. Global Anal. Geom. **3** (1985), 265–273.

The nearly Kähler $\mathbb{S}^3 \times \mathbb{S}^3$

$$T_{(p,q)}(\mathbb{S}^3 \times \mathbb{S}^3) \cong T_p\mathbb{S}^3 \oplus T_q\mathbb{S}^3 \tag{3}$$

The tangent vector at (p,q): $Z(p,q) = (U(p,q), V(p,q))$

- The almost complex structure J on $\mathbb{S}^3 \times \mathbb{S}^3$

$$JZ(p,q) = \frac{1}{\sqrt{3}}(2pq^{-1}V - U, -2qp^{-1}U + V) \tag{4}$$

- The metric g on $\mathbb{S}^3 \times \mathbb{S}^3$

$$\begin{aligned} g(Z,Z') &= \frac{1}{2}(\langle Z,Z'\rangle + \langle JZ,JZ'\rangle) && (5) \\ &= \frac{4}{3}(\langle U,U'\rangle + \langle V,V'\rangle) - \frac{2}{3}(\langle p^{-1}U, q^{-1}V'\rangle + \langle p^{-1}U', q^{-1}V\rangle) && (6) \end{aligned}$$

where $\langle\cdot,\cdot\rangle$ is the product metric, tangent vectors $Z = (U,V)$ and $Z' = (U',V')$.

- The almost product structure P on $\mathbb{S}^3 \times \mathbb{S}^3$

$$PZ(p,q) = (pq^{-1}V, qp^{-1}U) \tag{7}$$

Properties of P

Denote $G(X,Y) = (\nabla_X J)Y$.

- $P^2 = \mathrm{Id}$,
- $PJ = -JP$,
- $g(PZ, PZ') = g(Z, Z')$,
- $PG(X,Y) + G(PX, PY) = 0$,
- $G(X, PY) + PG(X,Y) = -2J(\tilde{\nabla}_X P)Y$.

The usual product structure $QZ = (-U, V)$.

Almost Product Structure on Lagrangian Submanifold

Let M be a Lagrangian submanifold in the nearly Kähler $\mathbb{S}^3 \times \mathbb{S}^3$.
The pull back of $T(\mathbb{S}^3 \times \mathbb{S}^3)$ to M splits into $TM \oplus JTM$.
There are two endomorphisms $A, B : TM \longrightarrow TM$ such that

$$PX = AX + JBX, \ \forall X \in TM. \tag{8}$$

- A and B are symmetric commuting endomorphisms that satisfy $A^2 + B^2 = I$.
- The covariant derivative of A and B

$$(\nabla_X A)Y = BS_{JX}Y - Jh(X, BY) + \frac{1}{2}(JG(X, AY) - AJG(X, Y)) \tag{9}$$

$$(\nabla_X B)Y = Jh(X, BY) - AS_{JX}Y + \frac{1}{2}(JG(X, BY) - BJG(X, Y)). \tag{10}$$

- A and B are symmetric operators whose Lie bracket is zero, i.e., $[A, B] = 0$.

A and B can be diagonalized simultaneously at a point of M and at each point p there is an orthonormal basis $e_1, e_2, e_3 \in T_pM$ such that

$$Pe_i = \cos 2\theta_i e_i + \sin 2\theta_i Je_i, \quad \forall i = 1, 2, 3. \tag{11}$$

- Gauss equation

$$\begin{aligned} R(X,Y)Z =& \frac{5}{12}(g(Y,Z)X - g(X,Z)Y) \\ &+ \frac{1}{3}(g(AY,Z)AX - g(AX,Z)AY + g(BY,Z)BX - g(BX,Z)BY) \\ &+ [S_{JX}, S_{JY}]Z \end{aligned} \tag{12}$$

- Codazzi equation

$$\begin{aligned} (\nabla h)(X,Y,Z) - (\nabla h)(Y,X,Z) =& \\ & \frac{1}{3}(g(AY,Z)JBX - g(AX,Z)JBY - g(BY,Z)JAX + g(BX,Z)JAY) \end{aligned} \tag{13}$$

For the Levi-Civita connection ∇ on M,

$$\nabla_{E_i} E_j = \omega_{ij}^k E_k \quad \text{with} \quad \omega_{ij}^k = -\omega_{ik}^j$$

$h_{ij}^k = g(h(E_i, E_k), JE_k)$, h_{ij}^k is totally symmetric.

Lemma (Y. Zhang, Z. Hu, B. Dioos, L. Vrancken, X. Wang, 2016)

Let M be a Lagrangian submanifold. Let $\{E_1, E_2, E_3\}$ be local orthonormal frame. Denote by h_{ij}^k and ω_{ij}^k the components of respectively the second fundamental form and the induced connection. Then we have

- $\theta_1 + \theta_2 + \theta_3$ is a multiple of π
- $E_i(\theta_j) = -h_{jj}^i$
- $h_{ij}^k \cos(\theta_j - \theta_k) = \left(\frac{\sqrt{3}}{6}\varepsilon_{ij}^k - \omega_{ij}^k\right)\sin(\theta_j - \theta_k), \quad j \neq k.$

Motivation

Lemma (B. Dioos, L. Vrancken, X. Wang, 2016)

If two of the angles are equal modulo π, then the Lagrangian submanifold is totally geodesic.

Corollary (B. Dioos, L. Vrancken, X. Wang, 2016)

Let M be a Lagrangian submanifold of the nearly Kähler $\mathbb{S}^3 \times \mathbb{S}^3$. If M is totally geodesic, then the angles θ_1, θ_2 and θ_3 are constant. Conversely, if the angles are constant and $h_{12}^3 = 0$, then M is totally geodesic.

- A. Moroianu and U. Semmelmann, *Generalized Killing spinors and Lagrangian graphs*, Differ. Geom. Appl. **37**(2014), 141–151.
- L. Schäfer and K. Smoczyk, *Decomposition and minimality of Lagrangian submanifolds in nearly Kähler manifolds*, Ann. Global Anal. Geom. **37** (2009), no. 3, 221–240.

Theorem A (Y. Zhang, Z. Hu, B. Dioos, L. Vrancken, X. Wang, 2016)

Let M be a totally geodesic Lagrangian submanifold in the nearly Kähler $\mathbb{S}^3 \times \mathbb{S}^3$. Then up to an isometry of the nearly Kähler $\mathbb{S}^3 \times \mathbb{S}^3$, M is locally congruent with one of the following immersions:

1. $f\colon S^3 \to \mathbb{S}^3 \times \mathbb{S}^3 : u \mapsto (u, 1)$,
2. $f\colon S^3 \to \mathbb{S}^3 \times \mathbb{S}^3 : u \mapsto (1, u)$,
3. $f\colon S^3 \to \mathbb{S}^3 \times \mathbb{S}^3 : u \mapsto (u, u)$,
4. $f\colon S^3 \to \mathbb{S}^3 \times \mathbb{S}^3 : u \mapsto (u, u\mathbf{i})$,
5. $f\colon S^3 \to \mathbb{S}^3 \times \mathbb{S}^3 : u \mapsto (u^{-1}, u\mathbf{i}u^{-1})$,
6. $f\colon S^3 \to \mathbb{S}^3 \times \mathbb{S}^3 : u \mapsto (u\mathbf{i}u^{-1}, u^{-1})$.

Round sphere
$$\begin{cases} (2\theta_1, 2\theta_2, 2\theta_3) & = & (\frac{4\pi}{3}, \frac{4\pi}{3}, \frac{4\pi}{3}) \\ (2\theta_1, 2\theta_2, 2\theta_3) & = & (\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3}) \\ (2\theta_1, 2\theta_2, 2\theta_3) & = & (0, 0, 0) \end{cases}$$

Berger sphere
$$\begin{cases} (2\theta_1, 2\theta_2, 2\theta_3) & = & (\frac{4\pi}{3}, \frac{4\pi}{3}, \frac{4\pi}{3}) \\ (2\theta_1, 2\theta_2, 2\theta_3) & = & (\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3}) \\ (2\theta_1, 2\theta_2, 2\theta_3) & = & (0, 0, 0) \end{cases}$$

Theorem B (B. Dioos, L. Vrancken, X. Wang, 2016)

Let M be a Lagrangian submanifold of constant sectional curvature in the nearly Kähler $\mathbb{S}^3 \times \mathbb{S}^3$. Then up to an isometry of the nearly Kähler $\mathbb{S}^3 \times \mathbb{S}^3$, M is locally congruent with one of the following immersions:

1. $f \colon \mathbb{S}^3 \to \mathbb{S}^3 \times \mathbb{S}^3 : u \mapsto (u, 1)$,
2. $f \colon \mathbb{S}^3 \to \mathbb{S}^3 \times \mathbb{S}^3 : u \mapsto (1, u)$,
3. $f \colon \mathbb{S}^3 \to \mathbb{S}^3 \times \mathbb{S}^3 : u \mapsto (u, u)$,
4. $f \colon \mathbb{S}^3 \to \mathbb{S}^3 \times \mathbb{S}^3 : u \mapsto (u\mathbf{i}u^{-1}, u\mathbf{j}u^{-1})$,
5. $f : \mathbb{R}^3 \to \mathbb{S}^3 \times \mathbb{S}^3 : (u, v, w) \mapsto (p(u, w), q(u, v))$, where p and q are constant mean curvature tori in $\mathbb{S}^3$.

Results

Theorem 1

Let $f : M \longrightarrow \mathbb{S}^3 \times \mathbb{S}^3$ be a Lagrangian immersion into a nearly Kähler manifold $\mathbb{S}^3 \times \mathbb{S}^3$ given by $f = (p, q)$ with the angle functions θ_i. Then, $\tilde{f} : M \longrightarrow \mathbb{S}^3 \times \mathbb{S}^3$ given by $\tilde{f} = (q, p)$ is also a Lagrangian immersion with the angle functions $\tilde{\theta}_i$ such that $\tilde{\theta}_i = \pi - \theta_i$.

Theorem 2

Let $f : M \longrightarrow \mathbb{S}^3 \times \mathbb{S}^3$ be a Lagrangian immersion defined by $f = (p, q)$ with the angle functions θ_i. Then, $f^\star : M \longrightarrow \mathbb{S}^3 \times \mathbb{S}^3$ defined by $f^\star = (\bar{p}, q\bar{p})$ is also a Lagrangian immersion with the angle functions $\theta_i^\star$ such that $\theta_i^\star = \frac{2\pi}{3} - \theta_i$.

Lemma 3

Let M be a Lagrangian submanifold of the nearly Kähler manifold $\mathbb{S}^3 \times \mathbb{S}^3$ with constant angle functions θ_i.

i. If M is a non-totally geodesic submanifold, then the nonzero components of ω_{ij}^k are given by

$$\omega_{12}^3 = \frac{\sqrt{3}}{6} - \frac{\cos(\theta_2 - \theta_3)}{\sin(\theta_2 - \theta_3)} h_{12}^3, \tag{14}$$

$$\omega_{23}^1 = \frac{\sqrt{3}}{6} + \frac{\cos(\theta_1 - \theta_3)}{\sin(\theta_1 - \theta_3)} h_{12}^3, \tag{15}$$

$$\omega_{31}^2 = \frac{\sqrt{3}}{6} - \frac{\cos(\theta_1 - \theta_2)}{\sin(\theta_1 - \theta_2)} h_{12}^3. \tag{16}$$

ii. The Codazzi equations of the submanifold M are as followings:

$$E_i(h_{12}^3) = 0, \quad i = 1, 2, 3, \tag{17}$$

$$h_{12}^3 \left(2(\omega_{13}^2 + \omega_{21}^3) + \frac{1}{\sqrt{3}} \right) = \frac{1}{3} \sin(2(\theta_1 - \theta_2)), \tag{18}$$

$$h_{12}^3 \left(2(\omega_{12}^3 + \omega_{31}^2) - \frac{1}{\sqrt{3}} \right) = \frac{1}{3} \sin(2(\theta_1 - \theta_3)), \tag{19}$$

$$h_{12}^3 \left(2(\omega_{21}^3 + \omega_{32}^1) + \frac{1}{\sqrt{3}} \right) = \frac{1}{3} \sin(2(\theta_2 - \theta_3)). \tag{20}$$

iii. The Gauss equations of the submanifold M are given by

$$\frac{5}{12} + \frac{1}{3}\cos(2(\theta_1 - \theta_2)) - (h^3_{12})^2 = -\omega^3_{21}\omega^2_{13} + \omega^3_{12}\omega^2_{31} - \omega^3_{21}\omega^2_{31}, \tag{21}$$

$$\frac{5}{12} + \frac{1}{3}\cos(2(\theta_1 - \theta_3)) - (h^3_{12})^2 = -\omega^2_{31}\omega^3_{12} + \omega^2_{13}\omega^3_{21} - \omega^2_{31}\omega^3_{21}, \tag{22}$$

$$\frac{5}{12} + \frac{1}{3}\cos(2(\theta_2 - \theta_3)) - (h^3_{12})^2 = -\omega^1_{32}\omega^3_{21} + \omega^1_{23}\omega^3_{12} - \omega^1_{32}\omega^3_{12}. \tag{23}$$

Theorem 4

A Lagrangian submanifold of the nearly Kähler manifold $\mathbb{S}^3 \times \mathbb{S}^3$ with constant angle functions is either totally geodesic or has constant sectional curvature in $\mathbb{S}^3 \times \mathbb{S}^3$.

Case 1 $h_{12}^3 = 0$, a totally geodesic Lagrangian submanifold of the nearly Kähler $\mathbb{S}^3 \times \mathbb{S}^3$

Case 2 h_{12}^3 is a nonzero constant.

i. $h_{12}^3 = -\frac{1}{2}$, $(2\theta_1, 2\theta_2, 2\theta_3) = (0, \frac{2\pi}{3}, \frac{4\pi}{3})$,
a flat Lagrangian submanifold.

ii. $h_{12}^3 = -\frac{1}{4}$, $(2\theta_1, 2\theta_2, 2\theta_3) = (\frac{4\pi}{3}, \frac{2\pi}{3}, 0)$,
Lagrangian submanifold with sectional curvature $\frac{3}{16}$.

iii. $h_{12}^3 = \frac{1}{4}$, $(2\theta_1, 2\theta_2, 2\theta_3) = (0, \frac{2\pi}{3}, \frac{4\pi}{3})$,
Lagrangian submanifold with sectional curvature $\frac{3}{16}$.

iv. $h_{12}^3 = \frac{1}{2}$, $(2\theta_1, 2\theta_2, 2\theta_3) = (\frac{4\pi}{3}, \frac{2\pi}{3}, 0)$,
a flat Lagrangian submanifold.

Corollary 5

A Lagrangian submanifold of the nearly Kähler manifold $\mathbb{S}^3 \times \mathbb{S}^3$ with constant angle functions is locally congruent to the immersions given in Theorem A and Theorem B.

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THANK YOU FOR YOUR ATTENTION