On classification problems in theory of differential equations: algebra + geometry

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A Problem

Hermann Weyl asserted that the soul of every mathematician is a field of battle between the angel of pure geometry and the devil of abstract algebra. We are curious to see what will come out of the meeting of these two in a peaceful environment, namely in theory of differential equations.

I. Newton : "6accdae13eff7i3l9n4o4qrr4s8t12ux"

("It is useful to solve differential equations")

S. Lie : "It is useful to classify differential equations".

Classification with respect to the action of two groups:

- (Point group): $(x, y) \to (X(x, y), Y(x, y))$;
- (Contact group): $(x, y, p := y') \to (X(x, y, p), Y(x, y, p), P(x, y, p))$.

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History

Consider *ordinary differential equation of order* n:

$$y^{(n)} = F(x, y, y', \dots, y^{(n-1)}). \qquad (1)$$

- ($n = 1$, S. Lie) (1) $\sim_{Point}$ $y' = 0$.
- ($n = 2$, S. Lie) (1) $\sim_{Cont}$ $y'' = 0$.
- ($n = 2$, point classification) A. Tresse (sub-invariants, 1896) B. Kruglikov (invariants, 2009).
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Degenerated case

Consider *ordinary differential equation of order 2*:

$$y'' = a_3(x,y)(y')^3 + a_2(x,y)(y')^2 + a_1(x,y)(y') + a_0(x,y). \quad (2)$$

Equation (2) is closely connected with projective geometry. It was studied by S. Lie, A. Tresse, J. Liuville, E. Cartan, etc. Final solution: V. Yumaguzhin (2008).

The next degeneration: equations

$$y'' = F(x,y). \quad (3)$$

Problem (S. Lie)

Obtain point classification of second order ODEs (3).

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Approach

- Calculate point symmetry group G for the class of ODEs $y'' = F(x,y)$ (i.e. such transformations $(x,y) \to (X(x,y), Y(x,y))$ that preserve class (3)).
- Calculate algebra of differential invariants for the action of group G on the right parts F of ODEs $y'' = F(x,y)$ (i.e. algebra of G-invariant functions which depend on F and its derivatives).
- Calculate the dependencies between differential invariants, which uniquely define the classes of G-equivalence for ODEs (3).

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Point symmetry pseudogroup G

Theorem (S. Lie)

- *Connected component of the identity of point symmetry pseudogroup G, which preserves class of ODEs $y'' = F(x, y)$, consists of transformations*

$$x \mapsto A(x), \qquad y \mapsto C \cdot \sqrt{A'(x)}y + B(x),$$

where A, $B \in C^\infty(\mathbb{R})$ are smooth functions and $C \in \mathbb{R}_+$ is a real constant.

- *Lie algebra $\mathfrak{g}$ of group G consists of vector fields*

$$X := a(x)\partial_x + \Big((a'(x)/2 + c)y + b(x)\Big)\partial_y,$$

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Differential invariants

Let us consider the action of the symmetry group G on the right parts F of ODE's.

Denote by $[F]_a^k$ the k-jet of function F at point a (i.e. the segment of the Taylor series of F in point a up to the members of order k). Let $J^k\mathbb{R}^2$ be the k-jet space of functions F with coordinates (x, y, u, u_{ij}), i.e. $u_{ij}([F]_a^k) = \frac{\partial^{i+j} F}{\partial^i x \partial^j y}(a)$.

Differential invariant of group G of order k is G-invariant function J on k-jet space $J^k\mathbb{R}^2$.

Invariant derivative is vector field ∇ such that $[\nabla, \xi] = 0$ for all $\xi \in \mathfrak{g}$.

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Describe the differential invariant field of point symmetry group G.

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Differential invariant algebra

Theorem

1. Algebra of differential invariants is generated by invariants

$$J := \frac{u_{02}u_{04}}{u_{03}^2} \quad \text{and}$$

$$K := \frac{u_{03}}{u_{02}^2}\left((u_{22} + 5u_{01}u_{02} + uu_{02}) + 6u_{12} \cdot \frac{2u_{13}u_{03} - u_{04}u_{12}}{5u_{04}u_{02} - 6u_{03}^2}\right)$$

of order 4 and invariant derivations

$$\nabla_1 := \frac{u_{02}}{u_{03}} \cdot \frac{d}{dy} \quad \text{and}$$

$$\nabla_2 := \frac{\sqrt{u_{03}}}{u_{02}}\left(\frac{d}{dx} - \frac{5u_{13}u_{02} - 6u_{12}u_{03}}{5u_{04}u_{02} - 6u_{03}^2} \cdot \frac{d}{dy}\right).$$

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Differential invariant algebra

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1. Algebra of differential invariants is generated by invariants

$$J := \frac{u_{02}u_{04}}{u_{03}^2} \quad \textit{and}$$

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Classification theorem

For a given function F consider the map $\pi_F \colon \mathbb{R}^2 \to \mathbb{R}^6$,

$$\pi_F(a) = (J([F]^4_a), K([F]^4_a), J_{10}([F]^4_a), \dots, K_{01}([F]^4_a))$$

(here $L_{ij} := \nabla_1^i \nabla_2^j L$). Put $S_F := \mathrm{Im}(\pi_F)$.
One can express invariants J_{10}, J_{01}, K_{10}, K_{01} through invariants J and K:

$$J_{10}(F) = \mathscr{J}_{10}(J(F), K(F)), \;\; J_{01}(F) = \mathscr{J}_{01}(J(F), K(F)),$$
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Functions F and $\widetilde{F}$ are locally G-equivalent $\Leftrightarrow S_F = S_{\widetilde{F}} \Leftrightarrow$ sets (4) for F and $\widetilde{F}$ coincide.

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Generalizations

In general case the dependencies (4) can not be calculated by computer. But if function F is rational, then our classification can be modified.

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Let function F be algebraic, for example,

$$a(x,y)F^2 + b(x,y)F + c(x,y) = 0,$$

where a, b and c are polynomials.
Then we obtain implicit differential equation

$$a(x,y)(y'')^2 + b(x,y)y'' + c(x,y) = 0.$$

Problem

Classify implicit differential equations in 2-jet space with respect to the action of the symmetry group G.

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THANK YOU FOR THE ATTENTION!