The geometry of left-invariant structures on nilpotent Lie groups

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Main goals:

- to construct left-invariant f*-structures* on special classes of nilpotent Lie groups using the theory of *canonical structures* on homogeneous k-symmetric spaces;
- to study the relation of these structures with the *generalized Hermitian geometry*.

Contents

1. Homogeneous k–symmetric spaces and canonical structures

Researchers who founded this theory: V.I.Vedernikov, N.A.Stepanov, A.Ledg
A.Gray, J.A.Wolf, A.S.Fedenko, O.Kowalski, L.V.Sabinin, V.Kac . . .

Definition 1. *Let G be a connected Lie group, Φ its (analytic) automorphism, G^{Φ} the subgroup of all fixed points of Φ, and G_o^{Φ} the identity component of G^{Φ}. Suppose a closed subgroup H of G satisfies the condition*

$$G_o^{\Phi} \subset H \subset G^{\Phi}.$$

Then G/H is called a homogeneous Φ-space.

Homogeneous Φ-spaces include homogeneous symmetric spaces $(\Phi^2 = id)$ and, more general, *homogeneous Φ-spaces of order k* $(\Phi^k = id)$ or, in the other terminology, *homogeneous k-symmetric spaces*

For any homogeneous Φ-space G/H one can define the mapping

$$S_o = D\colon\ G/H \to G/H,\ xH \to \Phi(x)H.$$

It is evident that in view of homogeneity the "symmetry" S_p can be defined at any point $p \in G/H$.

The class of homogeneous Φ–spaces is very large and contains even non-reductive homogeneous spaces. In our talk we dwell on homogeneous k-symmetric spaces G/H only.

Let $\mathfrak{g}$ and $\mathfrak{h}$ be the corresponding Lie algebras for G and H, $\varphi = d\Phi_e$ the automorphism of $\mathfrak{g}$, where $\varphi^k = id$. Consider the linear operator $A = \varphi - id$. It is known (N.A.Stepanov, 1967) that G/H is a reductive space for which the corresponding *canonical reductive decomposition* is of the form:

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, \;\; \mathfrak{m} = A\mathfrak{g}.$$

Besides, this decomposition is obviously φ–invariant. Denote by θ the restriction of φ to $\mathfrak{m}$. As usual, we identify $\mathfrak{m}$ with the tangent space $T_o(G/H)$ at the point $o = H$.

Definition 2 (VB, N.A.Stepanov, 1991)**.** *An invariant affinor structure F (i.e. a tensor field of type $(1,1)$) on a homogeneous k-symmetric space G/H is called canonical if its value at the point $o = H$ is a polynomial in θ.*

Denote by $\mathcal{A}(\theta)$ the set of all canonical affinor structures on G/H. It is easy to see that $\mathcal{A}(\theta)$ is a commutative subalgebra of the algebra $\mathcal{A}$ of all invariant affinor structures on G/H. It should be mentioned that all canonical structures are, in addition, invariant with respect to the "symmetries" $\{S_p\}$ of G/H.

Note that the algebra $\mathcal{A}(\theta)$ for any symmetric Φ-space ($\Phi^2 = id$) is trivial, i.e. it is isomorphic to $\mathbb{R}$.

The most remarkable example of canonical structures is the canonical almost complex structure $J = \frac{1}{\sqrt{3}}(\theta - \theta^2)$ on a homogeneous 3-symmetric space (N.A.Stepanov, J.Wolf, A.Gray, 1967-1968).

It turns out that for homogeneous k-symmetric spaces $(k \geq 3)$ the algebra $\mathcal{A}(\theta)$ contains a rich collection of classical structures. All these canonical structures on homogeneous k–symmetric spaces were completely described.

We will concentrate on the following affinor structures of classical types:
almost complex structures J ($J^2 = -1$);
almost product structures P ($P^2 = 1$);
f-structures ($f^3 + f = 0$) (K.Yano, 1963);
f-structures of hyperbolic type or, briefly, *h-structures* ($h^3 - h = 0$) (V.F.Kirichenko, 1983).
Clearly, f-structures and h-structures are generalizations of structures J and P respectively.

For future reference we indicate the general result for canonical f-structures only.

We use the notation: $s = [\frac{k-1}{2}]$ (integer part), $u = s$ (for odd k), and $u = s + 1$ (for even k).

Theorem 1 (VB, N.A.Stepanov,1991)**.** *Let G/H be a homogeneous k-symmetric space. All non-trivial canonical f-structures on G/H can be given by the operators*

$$f = \frac{2}{k} \sum_{m=1}^{u} \left(\sum_{j=1}^{u} \zeta_j \sin \frac{2\pi m j}{k} \right) \left(\theta^m - \theta^{k-m} \right),$$

where $\zeta_j \in \{-1; 0; 1\}$, $j = 1, 2, \ldots, u$, and not all coefficients ζ_j are zero. In particular, suppose that $-1 \notin spec\,\theta$. Then the polynomials f define canonical almost complex structures J iff all $\zeta_j \in \{-1; 1\}$.

We now particularize the results above mentioned for homogeneous Φ-spaces of orders 3, 4, 5, and 6 only.

Corollary 1. *Let G/H be a homogeneous 3-symmetric space. There are (up to sign) only the following canonical structures of classical type on G/H:*

$$J = \frac{1}{\sqrt{3}}(\theta - \theta^2), \; P = 1.$$

We noted that the existence of the structure J and its properties are well known (see N.A.Stepanov, J.Wolf, A.Gray, V.F.Kirichenko, . . .).

Corollary 2. *On a homogeneous 4-symmetric space there are (up to sign) the following canonical classical structures:*

$$P = \theta^2, \; f = \frac{1}{2}(\theta - \theta^3), \; h_1 = \frac{1}{2}(1 - \theta^2), \; h_2 = \frac{1}{2}(1 + \theta^2).$$

Corollary 3. *There exist (up to sign) only the following canonical structures of classical type on any homogeneous 5-symmetric space:*

$$P = \frac{1}{\sqrt{5}}(\theta - \theta^2 - \theta^3 + \theta^4);$$

$$J_1 = \alpha(\theta - \theta^4) - \beta(\theta^2 - \theta^3); \quad J_2 = \beta(\theta - \theta^4) + \alpha(\theta^2 - \theta^3);$$

$$f_1 = \gamma(\theta - \theta^4) + \delta(\theta^2 - \theta^3); \quad f_2 = \delta(\theta - \theta^4) - \gamma(\theta^2 - \theta^3);$$

$$h_1 = \frac{1}{2}(1 + P); \quad h_2 = \frac{1}{2}(1 - P);$$

where $\alpha = \frac{\sqrt{5+2\sqrt{5}}}{5}$; $\beta = \frac{\sqrt{5-2\sqrt{5}}}{5}$; $\gamma = \frac{\sqrt{10+2\sqrt{5}}}{10}$; $\delta = \frac{\sqrt{10-2\sqrt{5}}}{10}$.

Corollary 4. *There exist (up to sign) only the following canonical f-structures on any homogeneous 6-symmetric space:*

$$f_1 = \frac{\sqrt{3}}{6}(\theta + \theta^2 - \theta^4 - \theta^5), \ f_2 = \frac{\sqrt{3}}{6}(\theta - \theta^2 + \theta^4 - \theta^5),$$

$$f_3 = f_1 + f_2, \quad f_4 = f_1 - f_2,$$

where the structures f_1 and f_2 are the base canonical f–structures.

2. Canonical f-structures and the generalized Hermitian geometry.

2.1. Almost Hermitian structures

K	*Kähler structure*:	$\nabla J = 0$;
H	*Hermitian structure*:	$\nabla_X(J)Y - \nabla_{JX}(J)JY = 0$;
$\mathbf{G}_1$	AH-structure of class G_1, or G_1-*structure*:	$\nabla_X(J)X - \nabla_{JX}(J)JX = 0$;
QK	*quasi-Kähler structure*:	$\nabla_X(J)Y + \nabla_{JX}(J)JY = 0$;
AK	*almost Kähler structure*:	$d\,\Omega = 0$;
NK	*nearly Kähler structure*, or NK-*structure*:	$\nabla_X(J)X = 0$.

It is well known (see, for example, Gray-Hervella, 1980) that

$$\mathbf{K} \subset \mathbf{H} \subset \mathbf{G_1};\ \ \mathbf{K} \subset \mathbf{NK} \subset \mathbf{G_1};\ \ \mathbf{NK} = \mathbf{G_1} \cap \mathbf{QK};\ \ \mathbf{K} = \mathbf{H} \cap \mathbf{QK}.$$

As was already mentioned, the role of homogeneous almost Hermitian manifolds is particularly important "because they are the model spaces to which all other almost Hermitian manifolds can be compared" (A.Gray, 1983). We mention only one result closely related to our future consideration.

Theorem 2. *(A.Gray, 1972) A homogeneous 3-symmetric space G/H with the canonical almost complex structure J and an invariant compatible metric g is a quasi-Kähler manifold. Moreover, $(G/H, J, g)$ belongs to the class* **NK** *if and only if g is naturally reductive.*

2.2. Metric f-structures

A fundamental role in the geometry of metric f-manifolds is played by the *composition tensor* T, which was explicitly evaluated (V.F.Kirichenko, 1986):

$$T(X,Y) = \frac{1}{4} f(\nabla_{fX}(f)fY - \nabla_{f^2X}(f)f^2Y), \tag{1}$$

where ∇ is the Levi-Civita connection of a (pseudo)Riemannian manifold (M,g), $X, Y \in \mathfrak{X}(M)$.Using this tensor T, the algebraic structure of a so-called *adjoint Q-algebra* in $\mathfrak{X}(M)$ can be defined by the formula: $X * Y = T(X,Y)$. It gives the opportunity to introduce some classes of metric f-structures in terms of natural properties of the adjoint Q-algebra. We enumerate below the main classes of metric f-structures together with their defining properties:

Kf	*Kähler f–structure*:	$\nabla f = 0$;
Hf	*Hermitian f–structure*:	$T(X,Y) = 0$, i.e. $\mathfrak{X}(M)$ is an abelian Q-algebra;
$\mathbf{G_1}$f	f-structure of class G_1, or *$G_1 f$-structure*:	$T(X,X) = 0$, i.e. $\mathfrak{X}(M)$ is an anticommutative Q-algebra
QKf	*quasi-Kähler f–structure*:	$\nabla_X f + T_X f = 0$;
Kill f	*Killing f-structure*:	$\nabla_X(f)X = 0$;
NKf	*nearly Kähler f-structure,* or *NKf-structure*:	$\nabla_{fX}(f)fX = 0$.

The following relationships between the classes mentioned are evident:

$$\mathbf{Kf} = \mathbf{Hf} \cap \mathbf{QKf}; \quad \mathbf{Kf} \subset \mathbf{Hf} \subset \mathbf{G_1 f}; \quad \mathbf{Kf} \subset \mathbf{Kill\ f} \subset \mathbf{NKf} \subset \mathbf{G_1 f}.$$

It is important to note that in the special case $f = J$ we obtain the corresponding classes of almost Hermitian structures (16 Gray-Hervella classes).

In particular, for $f = J$ the classes **Kill f** and **NKf** coincide with the well-known class **NK** of *nearly Kähler structures*.

2.3. Canonical metric f-structures on homogeneous k-symmetr spaces

Recall that $(G/H, g)$ is *naturally reductive* with respect to a reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ if

$$g([X,Y]_{\mathfrak{m}}, Z) = g(X, [Y,Z]_{\mathfrak{m}})$$

for all $X, Y, Z \in \mathfrak{m}$. Here the subscript $\mathfrak{m}$ denotes the projection of $\mathfrak{g}$ onto $\mathfrak{m}$ with respect to the reductive decomposition.

Any invariant metric f-structure on a reductive homogeneous space G/H determines the orthogonal decomposition $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$ such that $\mathfrak{m}_1 = Im\ f$, $\mathfrak{m}_2 = Ker\ f$.

We stress the particular role of canonical structures on homogeneous 4- and 5-symmetric spaces.

Theorem 3. *The canonical f-structure $f = \frac{1}{2}(\theta - \theta^3)$ on any naturally reductive 4-symmetric space $(G/H, g)$ is both a Hermitian f-structure and a nearly Kähler f-structure. Moreover, the following conditions are equivalent:*

1) f is a Kähler f-structure; 2) f is a Killing f-structure; 3) f is a quasi-Kähler f-structure; 4) f is an integrable f-structure; 5) $[\mathfrak{m}_1, \mathfrak{m}_1] \subset \mathfrak{h}$; 6) $[\mathfrak{m}_1, \mathfrak{m}_2] = 0$; 7) G/H is a locally symmetric space: $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$.

Theorem 4. *Let* $(G/H, g)$ *be a naturally reductive* 5*-symmetric space,* f_1 *and* f_2*,* J_1 *and* J_2 *the canonical structures on this space. Then* f_1 *and* f_2 *belong to both classes* **Hf** *and* **NKf***. Moreover, the following conditions are equivalent:*

1) f_1 *is a Kähler* f*-structure; 2)* f_2 *is a Kähler* f*-structure; 3)* f_1 *is a Killing* f*-structure; 4)* f_2 *is a Killing* f*-structure; 5)* f_1 *is a quasi-Kähler* f*-structure; 6)* f_2 *is a quasi-Kähler* f*-structure; 7)* f_1 *is an integrable* f*-structure; 8)* f_2 *is an integrable* f*-structure; 9)* J_1 *and* J_2 *are* NK*-structures; 10)* $[\mathfrak{m}_1, \mathfrak{m}_2] = 0$ *(here* $\mathfrak{m}_1 = Im\, f_1 = Ker\, f_2, \mathfrak{m}_2 = Im\, f_2 = Ker\, f_1$*); 11)* G/H *is a locally symmetric space:* $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$*.*

Remark. Now there are general results for canonical f-structures on Riemannian homogeneous k-symmetric spaces for any k in the case of naturally reductive as well as "diagonal" metrics (VB, A.Samsonov, 2010-2011).

We should mention other geometric structures on homogeneous k-symmetric spaces, which are of contemporary interest in geometry and topology:

- symplectic structures on k-symmetric spaces compatible with the corresponding "symmetries" of order k (A.Tralle, M.Bocheński);
- topology of homogeneous k-symmetric spaces, in particular, geometric formality (D. Kotschick, S. Terzić, Jelena Grbić);
- geometry of elliptic integrable systems (I.Khemar).

3. Lie groups as Riemannian homogeneous k-symmetric spaces

Important particular case: $M = G/G^{\Phi}$, where $\Phi^k = id$ and $G^{\Phi} = \{e\}$ trivial. As a result, $M = G/G^{\Phi} = G/\{e\} = G$ is a homogeneous k-symmetric space.

We start with several important examples.

Example 1.

$(G \equiv \mathbb{R}^3(a, b, c), g)$ the group of hyperbolic motions of the plane $\mathbb{R}^2$, solvable, not nilpotent Lie group. (just Sol-geometry)

This is a Riemannian homogeneous 4-symmetric space (O.Kowalski, 1980).

The left-invariant canonical f-structure on this group is a Hermitian f-structure, not nearly Kähler, non-integrable (VB, 2001).

Example 2.

The generalization of the previous group:
$(G_n \equiv \mathbb{R}^{2n+1}, g)$ solvable Lie group.
This is a Riemannian homogeneous $(2n+2)$-symmetric space (M.Bozek, 1980).
Using our technique, a construction of generalized Hermitian structure $(g, f_1, \ldots, f_n, T)$ of rank n was realized, i.e. $T = 0$. (VB, D.Vylegzhanin, 2004).

Example 3.

The 6–dimensional generalized Heisenberg group (N, g) (A.Kaplan, 1981).
(N, g) can be represented as a Riemannian 3- and 4-symmetric space (F.Tricerri, L.Vanhecke, 1983).

$k = 3$: The canonical almost Hermitian structure J is not nearly Kähler (g is not naturally reductive).

$k = 4$: The canonical f-structure is both a nearly Kähler and Hermitian f-structure (VB, 1994).

Moreover, (N, g) is also a Riemannian 6-symmetric space. Then we obtain 4 canonical f-structures. We proved that the base f-structures f_1 and f_2 are non-integrable, nearly Kähler and Hermitian f-structures. Moreover, $J = f_3 = f_1 + f_2$ is a classical almost Hermitian structure of strictly class G_1 (i.e. neither nearly Kähler nor Hermitian structure).

It should be mentioned that G_1–structures of such a kind have interesting applications in *heterotic strings* (P.Ivanov, S.Ivanov, 2005).

Example 4. The 5-dimensional Heisenberg group $H(2, 1)$ as a Riemannian homogeneous 6–symmetric space. It is proved that all the canonical

f-structures f_i, $i = 1, \dots, 4$ are Hermitian f-structures. Besides, the base f-structures f_1 and f_2 are integrable, but the other f-structures f_3 and f_4 are not integrable.

We notice that the group $H(2, 1)$ is used in constructing the 6-dimensional nilmanifold connected with the *heterotic equations* of motion in *string theory* (M.Fernandez, S.Ivanov, L.Ugarte, R.Villacampa, 2009).

4. Left-invariant f-structures on 2-step nilpotent and filiform Lie groups

Many results of this section were obtained jointly P.A.Dubovik.

4.1. General approach. Let G be a 2-step nilpotent Lie group, $\mathfrak{g}$ its Lie algebra, $Z(\mathfrak{g})$ the center of $\mathfrak{g}$. Consider a left-invariant metric f-structure on G with respect to a left-invariant Riemannian metric g.

Theorem 5 (VB, P.Dubovik, 2013)**.** (*i*) *If* $Z(\mathfrak{g}) \subset Ker\, f$ *then* f *is a Hermitian* f*-structure, but it is not a Kähler* f*-structure.*
(*ii*) *If* $Im\, f \subset Z(\mathfrak{g})$ *then* f *is both a Hermitian and a nearly Kähler* f*-structure, but it is not a Kähler* f*-structure.*

Example 5. Let $H(n, 1)$ be a $(2n + 1)$-dimensional matrix Heisenberg group. We can consider $H(n, 1)$ as a Riemannian homogeneous k–symmetric space, where k is even.
As an application of a previous theorem, we obtain

Theorem 6 (VB, P.Dubovik, 2013)**.** *Any left-invariant canonical* f*-structure on a* $(2n + 1)$*-dimensional matrix Heisenberg group* $H(n, 1)$ *is a Hermitian* f*-structure, but it is not a Kähler* f*-structure.*

4.2. Left-invariant f-structures on Lie groups.

Consider more general approach.

Let G be a connected Lie group, $\mathfrak{g}$ its Lie algebra. Denote by $\mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}]$ and $\mathfrak{g}^{(2)} = [\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}]$ the first and the second ideal of the derived series. Consider a left-invariant Riemannian metric g on G determined by the Euclidean inner product on $\mathfrak{g}$.

Theorem 7 (P.Dubovik, 2013)**.** *Let f be a left-invariant metric f-structure on G satisfying any of the following conditions:*

(*i*) $\mathfrak{g}^{(1)} \subset Ker\, f$;

(*ii*) $Im\, f \subset \mathfrak{g}^{(1)}$, $\mathfrak{g}^{(2)} \subset Ker\, f$;

(*iii*) $Im\, f \subset Z(\mathfrak{g}) \subset \mathfrak{g}^{(1)}$.

Then f is a Hermitian f-structure. Moreover, the condition (*iii*) *implies that f is a nearly Kähler f-structure. In addition, under the* *condition* (i) f *is a nearly Kähler* f*-structure if and only if* $[fX, f^2X] = 0$ *for any* $X \in \mathfrak{g}$.

Note that, for example, the 6-dimensional generalized Heisenberg group and the 5-dimensional Heisenberg group $H(2,1)$ admit f-structures mentioned in the above theorem.

4.3. Filiform Lie groups

Let $\mathfrak{g}$ be a nilpotent Lie algebra of dimension m. Let

$$C^0\mathfrak{g} \supset C^1\mathfrak{g} \supset \cdots \supset C^{m-2}\mathfrak{g} \supset C^{m-1}\mathfrak{g} = 0$$

be the descending central series of $\mathfrak{g}$, where

$$C^0\mathfrak{g} = \mathfrak{g}, C^i\mathfrak{g} = [\mathfrak{g}, C^{i-1}\mathfrak{g}], \quad 1 \leq i \leq m-1.$$

A Lie algebra $\mathfrak{g}$ is called *filiform* if $dim\, C^k\mathfrak{g} = m - k - 1$ for $k = 1, \ldots, m-1$. A Lie group G is called *filiform* if its Lie algebra is filiform.

Note that the filiform Lie algebras have the maximal possible nilindex, that is $m-1$.

Basic examples of $(n+1)$-dimensional filiform Lie algebras:

1. The Lie algebra L_n:

$[X_0, X_i] = X_{i+1},\ i = 1, \ldots, n-1.$

2. The Lie algebra $Q_n (n = 2k+1)$:

$[X_0, X_i] = X_{i+1},\ i = 1, \ldots, n-1,$

$[X_i, X_{n-i}] = (-1)^i X_n,\ i = 1, \ldots, k.$

Filiform Lie algebras and Lie groups are intensively studied in many directions:

- The Riemannian geometry (sectional curvatures, Ricci curvature etc): M. Kerr, Tr. Payne (2010);
- Graded filiform Lie algebras: D. Millionshchikov (2004-present);

- Totally geodesic subalgebras of filiform Lie algebras: Yu.Nikolayevsky et al (2013-present);
- Solvable extensions of filiform Lie groups: Yu.Nikolayevsky, Yu.Nikonorov (2015).

4.4. Left-invariant f-structures on 6-dimensional filiform Lie groups.

The classification of 6-dimensional nilpotent Lie algebras was obtained by V.V.Morozov (1958), there exist 32 types of such algebras.

We select from this list 5 filiform Lie algebras:

(1) The Lie algebra $\mathfrak{g} = L_5$:

$[e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_1, e_4] = e_5, [e_1, e_5] = e_6.$

Proposition 1. If $e_1 \in Ker\, f$, then f is a Hermitian f-structure.

For example, the following f-structure satisfies the above condition:

$f(e_1) = f(e_2) = 0, \ f(e_3) = -e_4, \ f(e_4) = e_3,$

$f(e_5) = e_6, \ f(e_6) = -e_5.$

(2) The Lie algebra $\mathfrak{g} = Q_5$:

$[e_1, e_2] = e_3, [e_1, e_5] = e_6, [e_2, e_3] = e_4, [e_2, e_4] = e_5, [e_3, e_4] = e_6.$

Proposition 2. Suppose any of the following conditions is satisfied: $e_1, e_4 \in Ker\, f, \quad e_3, e_5 \in Ker\, f, \quad e_2, e_6 \in Ker\, f$. Then f is a Hermitian f-structure.

For example, the following f-structure satisfies the above condition:
$f(e_1) = f(e_4) = 0,\ f(e_2) = -e_3,\ f(e_3) = e_2,$
$f(e_5) = e_6,\ f(e_6) = -e_5.$

On analogy, the other three filiform Lie algebras were studied.

Remark. All the above f-structures are Hermitian, but not nearly Kähler. The natural question: are there nearly Kähler f-structures in the case?

5. Left-invariant f-structures on the groups $H(p,r)$.

The following groups were introduced by M.Goze and Y.Haraguchi (1982):

$$H(p,r) = M_{1p} \times M_{pr} \times M_{1r},$$

where matrices M_{ij} have dimensions $1 \times p, p \times r, 1 \times r$ respectively.

The multiplication in $H(p,r)$:

$$(x,y,z)\,(x',y',z') = (x+x', y+y', z+z'+\frac{1}{2}(xy'-x'y)).$$

$H(p,r)$ is a $(rp+r+p)$-dimensional 2-step nilpotent Lie group, which can be equipped with the left-invariant Riemannian metric g. The particular case $H(p,1)$ (i.e. $r=1$) is exactly the matrix Heisenberg group.

Theorem 8. *(P.Piu, M.Goze, 1993)* $(H(p,r), g)$ *is naturally reductive if and only if* $H(p,r)$ *is a Heisenberg group (i.e.* $r=1$*).*

Denote by $\mathfrak{h}(p,r)$ the corresponding Lie algebra.

Question. Are there canonical f-structures on the groups $H(p,r)$?

Example. Consider the case $p = r = 2$, i.e. the 8-dimensional group $H(2,2)$. Lie brackets for the orthonormal basis in $\mathfrak{h}(2,2)$ are:

$$[e_1, e_5] = [e_2, e_7] = e_3, \quad [e_1, e_6] = [e_2, e_8] = e_4.$$

We construct two metric automorphisms of order 4 of the Lie algebra $\mathfrak{h}(2,2)$. As a result, $H(2,2)$ is a Riemannian 4-symmetric space in two ways. So, we can compute the corresponding canonical f-structure for both cases and study their properties (the work is in progress).

6. Recent interesting information

The only homogeneous 6-dimensional nearly Kähler manifolds:

$$S^6,\ S^3 \times S^3,\ \mathbb{C}P^3,\ SU(3)/T_{max}$$

(A.Gray, S.Salamon, Nagy, J.Butruille). All of these spaces are 3-symmetric.

Lorenzo Foscolo, Mark Haskins, *New G_2-holonomy cones and exotic nearly Kähler structure on S^6 and $S^3 \times S^3$* // Annals of Mathematics (Accepted: 27 July 2016)

From the Abstract:

"...We prove the existence of the first complete inhomogeneous nearly Kähler 6-manifolds by proving the existence of at least one cohomogeneity one nearly Kähler structure on the 6-sphere and on the product of a pair of 3-spheres..."

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THANK YOU FOR YOUR ATTENTION!