On evolution of invariant Riemannian metrics on special Wallach spaces

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1 Introduction
   - The normalized Ricci flow equation
   - Evolution of Riemannian metrics under the Ricci flow
   - Generalized Wallach spaces

2 Results on evolution of Riemannian metrics on generalized Wallach spaces
   - The case $a \in (0, 1/2) \setminus \{1/4\}$: Theorems 1 and 2
   - The case $a = 1/4$: Theorem 3

3 The scheme of proving Theorem 3
   - Reduction of NRF equation on GWS to the system of ODEs
   - The domain $R$ of invariant metrics with positive Ricci curvature
   - Important Lemmas

4 Acknowledgments
This talk is devoted to a brief description of some results based on a joint work with Prof. Yu. Nikonorov and relating to evolution of invariant Riemannian metrics on generalized Wallach spaces under the normalized Ricci flow. As a main topic the special case $a = 1/4$ of generalized Wallach spaces is considered.
The study of the normalized Ricci flow equation

\[
\frac{\partial}{\partial t} g(t) = -2 \text{Ric}_g + 2g(t) \frac{S_g}{n}
\]  

(1)

for a 1-parameter family of Riemannian metrics \(g(t)\) in a Riemannian manifold \(M^n\) was originally used by \textit{R. Hamilton} and since then it has attracted the interest of many mathematicians. Recently, there is an increasing interest towards the study of the Ricci flow (normalized or not) on homogeneous spaces.
It is a natural type of problems to investigate whether or not the positiveness of the sectional curvature or positiveness of the Ricci curvature of Riemannian metrics is preserved under the Ricci flow. A recent survey on the evolution of positively curved Riemannian metrics under the Ricci flow could be found in [Ni].

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W_6 := SU(3)/T_{\text{max}}, \\
W_{12} := Sp(3)/Sp(1) \times Sp(1) \times Sp(1), \\
W_{24} := F_4/\text{Spin}(8)
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The definition of GWS (or three-locally-symmetric spaces) one can find in the original papers [LNF], [Nik1], [NRS].


According to these works GWS can be characterized by the triple of parameters \((a_1, a_2, a_3)\), where \(a_i \in [0, 1/2]\).
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The Wallach spaces as examples of generalized Wallach spaces

It should be noted that the values $a = 1/6$, $a = 1/8$, and $a = 1/9$ correspond to the Wallach spaces $W_6$, $W_{12}$, and $W_{24}$ respectively.
The classification of generalized Wallach spaces is obtained recently (independently) in the papers [Nik2] and [CKL]. It should also be noted that the classification of Yu. Nikonorov is complete whereas the authors of [CKL] assume $G$ be simple.


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Let $x_1, x_2$ and $x_3$ be parameters of invariant Riemannian metric on GWS, where $x_i > 0$. Metrics with pairwise distinct $x_i$ we call \textit{generic}.
The case \( a \in (0, 1/6) \cup (1/6, 1/4) \cup (1/4, 1/2) \)

In [AN] we have the following

**Theorem (Abiev, Nikonorov, 2016)**

Let \( G/H \) be a generalized Wallach space with \( a_1 = a_2 = a_3 =: a \), where \( a \in (0, 1/2) \). Then the normalized Ricci flow evolves

- All generic metrics with positive Ricci curvature into metrics with mixed Ricci curvature, if \( a < 1/6 \);
- All generic metrics into metrics with positive Ricci curvature, if \( a \in (1/6, 1/4) \cup (1/4, 1/2) \).

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- all generic metrics into metrics with positive Ricci curvature, if $a \in (1/6, 1/4) \cup (1/4, 1/2)$.

Theorem 1 generalizes a result obtained by C. Böhm and B. Wilking (see Theorem 3.1 in [BoWi]): on the Wallach space $W_{12} := Sp(3)/Sp(1) \times Sp(1) \times Sp(1) \ (a = 1/8)$ the Ricci flow evolves some positively curved metrics into metrics with mixed Ricci curvature.

The same assertion for the space $W_{24} := F_4/Spin(8) \ (a = 1/9)$ was obtained by M. W. Cheung and N. Wallach (see Theorem 3 in [ChWal]).
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For $a = 1/6$ the following Theorem was proved in [AN]

**Theorem (Abiev, Nikonorov, 2016)**

Let $G/H$ be a generalized Wallach space with $a_1 = a_2 = a_3 =: a = 1/6$. Suppose that it is supplied with the invariant Riemannian metric such that $x_k < x_i + x_j$ for all indices with $\{i, j, k\} = \{1, 2, 3\}$, then the normalized Ricci flow on $G/H$ with this metric as the initial point, preserves the positivity of the Ricci curvature.

Theorem 2 generalizes Theorem 8 in [ChWal] proved for the space $W_6 := SU(3)/T_{\text{max}}$. 
The case $a = 1/6$

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The case $a = 1/4$

At $a = 1/4$ we have a very special case of GWS. The main result is the following theorem in [Abi] that completes Theorem 1 above.

**Theorem (Abiev, 2016)**

*Let $G/H$ be a generalized Wallach space with $a_1 = a_2 = a_3 = 1/4$. Then the normalized Ricci flow evolves all generic metrics into metrics with positive Ricci curvature.*

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A system of two ODEs in scale invariant variables \((w_1, w_2)\)

In scale invariant variables

\[
\begin{align*}
  w_1 &:= \frac{x_3}{x_1}, \\
  w_2 &:= \frac{x_3}{x_2}.
\end{align*}
\]

the NRF equation (1) can be reduced to the following system of ODEs for \(w_1 > 0\) and \(w_2 > 0\):

\[
\begin{align*}
  \frac{dw_1}{dt} &= (w_1 - 1)(w_1 - 2aw_1w_2 - 2aw_2), \\
  \frac{dw_2}{dt} &= (w_2 - 1)(w_2 - 2aw_1w_2 - 2aw_1).
\end{align*}
\]
It is known that

\[ \text{Ric} = r_1 \text{Id}|_{p_1} + r_2 \text{Id}|_{p_2} + r_3 \text{Id}|_{p_3}, \]

where

\[ r_i := \frac{x_j x_k + a(x_i^2 - x_j^2 - x_k^2)}{2x_1 x_2 x_3} \]

— the principal values of the Ricci curvature of the invariant metric, \( \{i, j, k\} = \{1, 2, 3\} \).
Let us denote by $R$ the set of invariant metrics with positive Ricci curvature. As shown in [AN] $R$ is a connected domain

$$r_1 > 0, \quad r_2 > 0, \quad r_3 > 0$$

with a boundary consisting of the union of the curves $r_1$, $r_2$ and $r_3$ in the coordinates $(w_1, w_2)$

$$aw_1^2 w_2^2 + aw_1^2 - aw_2^2 - w_1^2 w_2 = 0,$$

$$aw_1^2 w_2^2 - aw_1^2 + aw_2^2 - w_1 w_2^2 = 0,$$

$$aw_1^2 w_2^2 - aw_1^2 - aw_2^2 + w_1 w_2 = 0.$$ (4)
**Introduction**

Results on evolution of Riemannian metrics on generalized Wallach spaces

The scheme of proving Theorem 3

Acknowledgments

Reduction of NRF equation on GWS to the system of ODEs

The domain $R$ of invariant metrics with positive Ricci curvature

Important Lemmas

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**Fig. 0**: $a = 1/4$: The curves $r_1$, $r_2$ and $r_3$
Using results of [AANS] we can prove the following Lemma

**Lemma**

At $a = 1/4$ the system (3) has an unique singular point $(1, 1)$, which is a linear zero saddle with six hyperbolic sectors. Moreover, the lines $w_1 = 1$, $w_2 = 1$ and $w_2 = w_1$ are separatrices of this saddle.


Recall that $(w_1^0, w_2^0)$ is called a singular point of the linear zero type if the Jacobian matrix of the system equals to the zero matrix at this point.
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**Fig. 1**: $a = 1/4$: The linear zero saddle of the system (3) and its separatrices
Remark

At $\alpha \neq 1/4$ the system (3) has exactly four nondegenerate singular points: 1 unstable node and 3 saddles. $\alpha = 1/4$ is a bifurcation value when these four singular points split into one degenerate singular point $(1,1)$. Hence, there is a qualitative difference between the cases $\alpha \neq 1/4$ and $\alpha = 1/4$. 
Lemma

If $a = \frac{1}{4}$ then every integral curve of the system (3) reaches the domain $R$ in finite time.

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At $a = \frac{1}{4}$ integral curves of (3) could intersect the boundary of the domain $R$ at most once.
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Fig. 2: The domain $R$ and the phase portrait of the system (3) at $a = 1/4$
Remark

At $a \neq 1/4$ the Lemma above fails. There is a possibility when integral curves of (3) could intersect the boundary of the domain $\mathcal{R}$ twice (see [AN]).
The author is indebted to Prof. Yu. G. Nikonorov for fruitful collaboration!
Thank You for Your attention!