On evolution of invariant Riemannian metrics on special Wallach spaces

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This talk is devoted to a brief description of some results based on a joint work with Prof. Yu. Nikonorov and relating to evolution of invariant Riemannian metrics on generalized Wallach spaces under the normalized Ricci flow. As a main topic the special case $a = 1/4$ of generalized Wallach spaces is considered.

The normalized Ricci flow equation

The study of the normalized Ricci flow equation

$$\frac{\partial}{\partial t}\mathbf{g}(t) = -2\,\mathrm{Ric}_{\mathbf{g}} + 2\mathbf{g}(t)\frac{S_{\mathbf{g}}}{n} \tag{1}$$

for a 1-parameter family of Riemannian metrics $g(t)$ in a Riemannian manifold M^n was originally used by R. Hamilton and since then it has attracted the interest of many mathematicians. Recently, there is an increasing interest towards the study of the Ricci flow (normalized or not) on homogeneous spaces.

The Ricci flow and curvature positiveness problem

It is a natural type of problems to investigate whether or not the positiveness of the sectional curvature or positiveness of the Ricci curvature of Riemannian metrics is preserved under the Ricci flow. A recent survey on the evolution of positively curved Riemannian metrics under the Ricci flow could be found in [Ni].

[Ni] *Ni Lei. Ricci flow and manifolds with positive curvature.* In: R. Howe, M. Hunziker, J. F. Willenbring (eds.). Symmetry: representation theory and its applications. In honor of Nolan R. Wallach. Progress in Math., V. 257, NY: Birkhäuser/Springer, 2014, XXVIII+538 p., P. 491–504.

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Interesting results on the evolution of invariant Riemannian metrics could also be found in the papers [BoWi], [ChWa] and the references therein.

[BoWi] *Böhm C., Wilking B. Nonnegatively curved manifolds with finite fundamental groups admit metrics with positive Ricci curvature* // GAFA, Geometric and Functional Analysis (2007), V. 17, P. 665–681.
[ChWa] *Cheung M. W., Wallach N. R. Ricci flow and curvature on the variety of flags on the two dimensional projective space over the complexes, quaternions and octonions* // Proceedings of American Mathematical Society, (2015), V. 143, No. 1, P. 369–378.

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The Ricci flow and curvature positiveness problem on the Wallach spaces

In [AN] similar problems were studied for the Wallach spaces

$$\begin{array}{l} W_6 := SU(3)/T_{\max}, \\ W_{12} := Sp(3)/Sp(1) \times Sp(1) \times Sp(1), \\ W_{24} := F_4/Spin(8) \end{array} \tag{2}$$

and for generalized Wallach spaces (GWS).

[AN] *Abiev N. A., Nikonorov Yu. G. The evolution of positively curved invariant Riemannian metrics on the Wallach spaces under the Ricci flow //* Ann. Glob. Anal. Geom, 2016, V. 50, N 1. P. 65–84.

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Generalized Wallach spaces

The definition of GWS (or three-locally-symmetric spaces) one can find in the original papers [LNF], [Nik1], [NRS].

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[Nik1] *Nikonorov Yu. G. On a class of homogeneous compact Einstein manifolds* // Siberian Math. J. (2000), V. 41, No. 1, P. 168–172.
[NRS] *Nikonorov Yu. G., Rodionov E. D., Slavskii V. V.* Geometry of homogeneous Riemannian manifolds // Journal of Mathematical Sciences (New York), 146(7), 6313–6390 (2007).

According to these works GWS can be characterized by the triple of parameters (a_1, a_2, a_3), where $a_i \in [0, 1/2]$.

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The Wallach spaces as examples of generalized Wallach spaces

It should be noted that the values $a = 1/6$, $a = 1/8$, and $a = 1/9$ correspond to the Wallach spaces W_6, W_{12}, and W_{24} respectively.

Classification of generalized Wallach spaces

The classification of generalized Wallach spaces is obtained recently (independently) in the papers [Nik2] and [CKL]. It should also be noted that the classification of Yu. Nikonorov is complete whereas the authors of [CKL] assume G be simple.

[Nik2] *Nikonorov Yu. G. Classification of generalized Wallach spaces* // Geometriae Dedicata (2016), V. 181, P. 193–212.

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Generic metrics

Let x_1, x_2 and x_3 be parameters of invariant Riemannian metric on GWS, where $x_i > 0$. Metrics with pairwise distinct x_i we call *generic*.

The case $a \in (0, 1/6) \cup (1/6, 1/4) \cup (1/4, 1/2)$

In [AN] we have the following

Theorem (Abiev, Nikonorov, 2016)

Let G/H be a generalized Wallach space with $a_1 = a_2 = a_3 =: a$, where $a \in (0, 1/2)$. Then the normalized Ricci flow evolves

- *all generic metrics with positive Ricci curvature into metrics with mixed Ricci curvature, if $a < 1/6$;*
- *all generic metrics into metrics with positive Ricci curvature, if $a \in (1/6, 1/4) \cup (1/4, 1/2)$.*

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Theorem 1 generalizes a result obtained by C. Böhm and B. Wilking (see Theorem 3.1 in [BoWi]):
on the Wallach space $W_{12} := Sp(3)/Sp(1) \times Sp(1) \times Sp(1)$ ($a = 1/8$) the Ricci flow evolves some positively curved metrics into metrics with mixed Ricci curvature.

The same assertion for the space $W_{24} := F_4/Spin(8)$ ($a = 1/9$) was obtained by M. W. Cheung and N. Wallach (see Theorem 3 in [ChWal]).

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The case $a = 1/6$

For $a = 1/6$ the following Theorem was proved in [AN]

Theorem (Abiev, Nikonorov, 2016)

Let G/H be a generalized Wallach space with $a_1 = a_2 = a_3 =: a = 1/6$. Suppose that it is supplied with the invariant Riemannian metric such that $x_k < x_i + x_j$ for all indices with $\{i, j, k\} = \{1, 2, 3\}$, then the normalized Ricci flow on G/H with this metric as the initial point, preserves the positivity of the Ricci curvature.

Theorem 2 generalizes Theorem 8 in [ChWal] proved for the space $W_6 := SU(3)/T_{\max}$.

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The case $a = 1/4$

At $a = 1/4$ we have a very special case of GWS. The main result is the following theorem in [Abi] that completes Theorem 1 above.

Theorem (Abiev, 2016)

Let G/H be a generalized Wallach space with $a_1 = a_2 = a_3 = 1/4$. Then the normalized Ricci flow evolves all generic metrics into metrics with positive Ricci curvature.

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A system of two ODEs in scale invariant variables (w_1, w_2)

In scale invariant variables

$$w_1 := \frac{x_3}{x_1}, \quad w_2 := \frac{x_3}{x_2}.$$

the NRF equation (1) can be reduced to the following system of ODEs for $w_1 > 0$ and $w_2 > 0$:

$$\begin{aligned} \tfrac{dw_1}{dt} &= (w_1 - 1)(w_1 - 2aw_1w_2 - 2aw_2), \\ \tfrac{dw_2}{dt} &= (w_2 - 1)(w_2 - 2aw_1w_2 - 2aw_1). \end{aligned} \qquad (3)$$

The domain R of invariant metrics on GWS with positive Ricci curvature

It is known that

$$\mathrm{Ric} = \mathbf{r_1}\ \mathrm{Id}|_{\mathfrak{p}_1} + \mathbf{r_2}\ \mathrm{Id}|_{\mathfrak{p}_2} + \mathbf{r_3}\ \mathrm{Id}|_{\mathfrak{p}_3},$$

where

$$\mathbf{r_i} := \frac{x_j x_k + a(x_i^2 - x_j^2 - x_k^2)}{2x_1x_2x_3}$$

— the principal values of the Ricci curvature of the invariant metric, $\{i, j, k\} = \{1, 2, 3\}$.

The domain R of invariant metrics with positive Ricci curvature

Let us denote by R the set of invariant metrics with positive Ricci curvature. As shown in [AN] R is a connected domain

$$\mathbf{r_1} > 0, \quad \mathbf{r_2} > 0, \quad \mathbf{r_3} > 0$$

with a boundary consisting of the union of the curves r_1, r_2 and r_3 in the coordinates (w_1, w_2)

$$\begin{aligned} a w_1^2 w_2^2 + a w_1^2 - a w_2^2 - w_1^2 w_2 &= 0, \\ a w_1^2 w_2^2 - a w_1^2 + a w_2^2 - w_1 w_2^2 &= 0, \\ a w_1^2 w_2^2 - a w_1^2 - a w_2^2 + w_1 w_2 &= 0. \end{aligned} \tag{4}$$

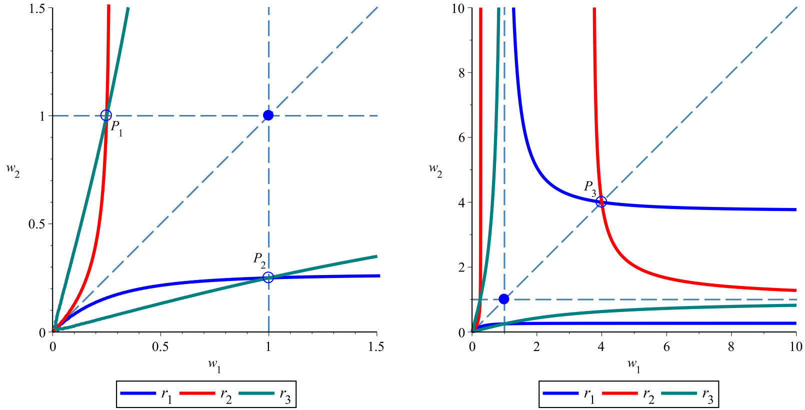

Fig. 0: $a = 1/4$: The curves r_1, r_2 and r_3

Using results of [AANS] we can prove the following Lemma

Lemma

At $a = 1/4$ the system (3) has an unique singular point $(1,1)$, which is a linear zero saddle with six hyperbolic sectors. Moreover, the lines $w_1 = 1$, $w_2 = 1$ and $w_2 = w_1$ are separatrices of this saddle.

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Recall that (w_1^0, w_2^0) is called a singular point of the linear zero type if the Jacobian matrix of the system equals to the zero matrix at this point.

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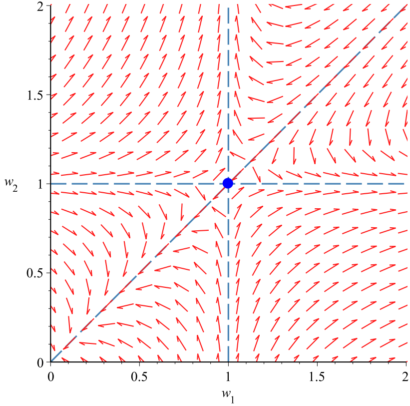

Fig. 1: $a = 1/4$: The linear zero saddle of the system (3) and its separatrices

Remark

At $a \neq 1/4$ the system (3) has exactly four nondegenerate singular points: 1 unstable node and 3 saddles. $a = 1/4$ is a bifurcation value when these four singular points split into one degenerate singular point $(1, 1)$. Hence, there is a qualitative difference between the cases $a \neq 1/4$ and $a = 1/4$.

Lemma

If $a = 1/4$ then every integral curve of the system (3) reaches the domain R in finite time.

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At $a = 1/4$ integral curves of (3) could intersect the boundary of the domain R at most once.

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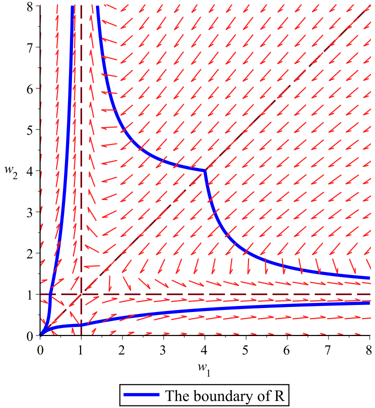

Fig. 2: The domain R and the phase portrait of the system (3) at $a = 1/4$

Remark

At $a \neq 1/4$ the Lemma above fails. There is a possibility when integral curves of (3) could intersect the boundary of the domain R twice (see [AN]).

The author is indebted to Prof. Yu. G. Nikonorov
for fruitful collaboration!

Thank You for Your attention!