Liouville type theorems for Riemannian twisted and warped products

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The *doubly twisted product* ${}_{\lambda_1}M_1 \times_{\lambda_2} M_2$ is the product manifold $M_1 \times M_2$ furnished with the Riemannian metric $g = \lambda_1^2 \pi_1^* g_1 + \lambda_2^2 \pi_2^* g_2$ where $\lambda_i : M_1 \times M_2 \to \mathbb{R}$ is a positive function and $\pi_i : M_1 \times M_2 \to M_i$ is the canonical projection for an arbitrary $i = 1, 2$ (see [1]). Then we can formulate a Liouville type theorem for doubly twisted products.

Theorem 1. *Let (M, g) be a doubly twisted product ${}_{\lambda_1}M_1 \times_{\lambda_2} M_2$ of some Riemannian manifolds (M_1, g_1) and (M_2, g_2). If (M, g) is a complete, noncompact and oriented Riemannian manifold (M, g) with $\|\pi_{2*}(\nabla log\lambda_1) + \pi_{1*}(\nabla log\lambda_2)\| \in L^1(M, g)$ and nonpositive mixed scalar curvature s_{mix}, then λ_1 and λ_2 are constants C_1 and C_2, respectively, and (M, g) is the direct product $M_1 \times M_2$ of $(M_1, \overline{g}_1)$ and $(M_2, \overline{g}_2)$ for $\overline{g}_1 = C_1^2 g_1$ and $\overline{g}_2 = C_2^2 g_2$.*

Let $f : (M, g) \to (\overline{M}, \overline{g})$ be a *projective submersion* with geodesically complete fibres, then (M, g) is isometric to a twisted product $M_1 \times_\lambda M_2$ (see [2]). In this case, the following corollary is true.

Corollary. *Let (M, g) be an n-dimensional complete, noncompact and simply connected Riemannian manifold and $f : (M, g) \to (\overline{M}, \overline{g})$ be a projective submersion onto another m-dimensional $(m < n)$ Riemannian manifold $(\overline{M}, \overline{g})$ with geodesically complete fibres. If the mixed sectional curvature of (M, g) is nonnegative then it is isometric to a direct product $M_1 \times M_2$.*

The *doubly warped product* ${}_{\lambda_1}M_1 \times_{\lambda_2} M_2$ is the twisted product ${}_{\lambda_1}M_1 \times_{\lambda_2} M_2$ for the case when $\lambda_1 : M_2 \to \mathbb{R}$ and $\lambda_2 : M_1 \to \mathbb{R}$ (see [1]). As a corollary of Theorem 1 we formulate the following Liouville-type theorem.

Theorem 2. *Let (M, g) be a doubly warped product ${}_{\lambda_1}M_1 \times_{\lambda_2} M_2$ of complete Riemannian manifolds (M_1, g_1) and (M_2, g_2) such that $inf\lambda_1 > 0$ or $inf\lambda_2 > 0$ and the mixed scalar curvature s_{mix} is nonpositive. If the gradient of $log(\lambda_1\lambda_2)$ has integrable norm, then λ_1 and λ_2 are constants C_1 and C_2, respectively, and (M, g) is the direct product of $(M_1, \overline{g}_1)$ and $(M_2, \overline{g}_2)$ for $\overline{g}_1 = C_1^2 g_1$ and $\overline{g}_2 = C_2^2 g_2$.*

[1] Kazan S, Sahin B. Characterization of twisted product manifolds to be warped product manifolds. Acta Mathematica Universitatis Comenianae. 2013; 82(2): 253-263.
[2] Stepanov SE. On the global theory of projective mappings. Mathematical Notes. 1995; 58(1): 752-756.